ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 2, Issue 9, March 2013

Algorithms to Realize All Possible Vertex Magic Total Labeling and Super Vertex Magic Total Labeling Of Complete Graphs

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***Abstract:** A vertex magic total labeling of a graph G = (V,E) is a bijection f : V U E → {1,2,3……….|V| + |E|} such that for every vertex w; the sum f(w) + Σ $_{uw\ \varepsilon E}$ f(uw) is a constant. It is well known that all complete graphs Kn admit a vertex magic total labeling. In this paper we present a new algorithm to obtain any desired magic constant within the allowable range using the concepts of subset construction and magic square. The Super Vertex Magic Total Labeling of a graph is the Vertex Magic Labeling with the condition that all the vertices of the takes the labels 1,2,3,….,v. We use the magic square of order n to construct Super Vertex Magic Total Labeling for Kn.*

Keywords: **Complete Graphs, Vertex Magic Total Labeling, Super Vertex Magic Total Labeling, Magic Square**

I. INTRODUCTION

Let G = (V,E) be a finite, simple, and undirected. We denote m = |E| and n = |V |. The labeling of a graph is a map that takes graph elements V or E or V U E to numbers (usually positive or non-negative integers). A comprehensive survey of graph labeling is given in Gallian [1]. The notion of vertex magic total labeling was introduced by Gray et al. [3, 4]. A vertex magic total labeling of a graph G = (V;E) is a bijection f from V U E to the set of integers 1, 2, 3,……….., n + m such that for every vertex v, f(v) + Σ $_{uw\ \varepsilon E}$ f(uw) = k; where k is a constant. Miller, Macdougall, Slamin, and Wallis [7] gave a vertex magic total labeling of complete graphs Kn for n ≡ 2 (mod 4) by making use of vertex magic total labeling of Kn/2 where n/2 is odd. Lin and Miller [6] gave a vertex magic total labeling of complete graphs Kn for n ≡ 4 (mod 8) by making use of vertex magic total labeling of Kn/4 where n=4 is odd. H.K.Krishnappa, N.K.Srinath and P.Ramakanth Kumar [5] gave a vertex magic total labeling of complete graphs Kn, for all n values. Labellings for Kn for n odd, are presented by Macdougall, Miller, Slamin, and Wallis [8]. A simpler proof of the fact that complete graphs have a vertex magic total labeling was given by Gray et. al [4]. Focusing on Kn, n odd, McQuillan and Smith [9], presented a new technique to arrive at a vertex magic total labeling of complete graphs of odd order for all values of k between $[n(n^2+3)]/4$ and $[n(n+1)^2]/4$. More recently, Gomez [2] has shown that Kn has a super vertex magic labeling, a vertex magic labeling where the smallest labels are given to the vertices. A recent result of Gray shows that all regular graphs have a vertex magic total labeling [3]. Thus, it is clearly established that vertex magic total labeling exists for all complete graphs. *In this paper we present a new algorithm to obtain any desired magic constant within the allowable range using the concepts of subset construction and magic square. While our result is not new, the techniques are new and may be of independent interest.*

$$= \frac{n^2+n}{2}$$

$$\text{So, } K_{min} = \frac{\left[\frac{n^4-2n^3+3n^2-2n}{4} + \frac{n^2+n}{2}\right]}{n}$$

$$= \frac{\left[\frac{n^4-2n^3+3n^2-2n+2n^2+2n}{4}\right]}{n}$$

$$= \frac{n^4+3n^2}{4n} = \frac{n^3+3n}{4} = \frac{n(n^2+3)}{4}$$

If we choose {1, 2,3, … n} to label the vertices of $K_n$and n+1, n+2, ….. + (n+n(n-1)/2) to label the edges of K_n, we will get the maximum magic constant.

$$K_{max} = \frac{n(n+1)^2}{4}$$

$$= \frac{\left[\sum_{i=1}^{n} i + 2\sum_{j=1}^{m}(n+j)\right]}{n}$$

$$= \frac{\left[\frac{n(n+1)}{2} + 2\left(mn + \frac{m(m+1)}{2}\right)\right]}{n}$$

Substitute for m=(n(n-1))/2 and upon simplification, we get

$$= \frac{\left[\frac{n(n+1)}{2} + 2\left(\frac{n^4+2n^3-n^2-2n}{8}\right)\right]}{n}$$

$$= \frac{\left[\frac{n^4+2n^3-n^2-2n+2n^2+2n}{4}\right]}{n}$$

$$= \frac{n^4+2n^3+n^2}{4n} = \frac{n(n+1)^2}{4}$$

1. Algorithm to obtain the desired magic constant.

Algorithm VMTL (n, k)

//This algorithm gives one possible labeling for the given complete graph Kn and the //required magic constant k. This uses subset construction method SUBSET( ), searching //method SEARCH( ) and insert an element into the linked list INSERT_END( ). All data //used in this algorithm are integers.

Step 1: [Initialization]

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
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m←n(n-1)/2;
Repeat **for** i ←1 to n+m do
N[i-1]=I;

Step 2: [Compute all possible magic Constants for the specified Kn and store these values in an array MC[ ]]
X←$n(n^2+3)/4$;
Y←$n(n+1)^2/4$;
l←0;
Repeat **for** j← X to Y do
MC[l]←j;
l←l+1;
l←l-1;

Step 3: [Check if k lies within the feasible range]
if((k<MC[0]) || (k>MC[l]))
then exit(0);
Repeat **for** i←0 to l/2 do **if**(MC[i]=k)
then Compute a subset of V of size n from N such that the sum of these n numbers equals to
C←[nm+n(n+1)/2] – ni
SUBSET(N,V,C,n);

Step 4: [Find the set E from N by eliminating V i.e. E=N-V]
j←0;
Repeat **for** i←0 to n+m-1 do
if(SEARCH(V, N[i])=FALSE)
then E[j]←N[i];
j←j+1;

Step 5: [From E compute n number of subsets of length n-1 each such that the sum of each set equals to k-V[i], where i=0,1,2,3,….n-1. While finding subsets make sure that between any two sets there is exactly one common number. Finally insert the numbers of each set into the linked list]
Repeat **for** i←0 to n-1 do
SUBSET(E, temp, k-V[i], n-1);
Repeat **for** j←0 to n-2 do
INSERT_END(L, temp[j]);

Step 6: [Add the numbers of n subsets stored in L into the matrix such that the common number between the set i and i+j fills at (i, i+j) position in the matrix where i=1,2,3…,n and j=i+1,…….,n]

Step 7: [At this stage the upper triangular matrix is filled completely. By using the symmetrical property copy the upper triangular matrix into the lower triangular matrix]
Repeat **for** i←1 to n do
Repeat **for** j←i+1 to n do
M[j,i]←M[i,j];

Step 8: [Fill the first row and first column with the vertex labels V[ ] and the diagonal (top left to bottom right)entries to 0]
Repeat **for** i←1 to n do
M[i,0]←V[i-1];
M[0,i]←V[i-1];

Repeat **for** i←1 to n do
M[i,i]←0;

Step 9: [Take the sum of each column or row from 1 to n and verify that these sums equals to the magic constant k]

Step 10: Stop

Working of this algorithm is illustrated with the following examples.

Example 1: Realization of all possible magic constants for K_3.
In this case n=3 and m=3(3-1)/2=3. Set of labels is {1, 2, 3, 4, 5, 6}.
Magic constants are $3(3^2+3)/4 \leq k \leq 3(3+1)^2/4$ i.e. k={9,10,11,12}

Table 1: Labelings of complete graphs on 3 vertices.

-	4	5	6
4	-	3	2
5	3	-	1
6	2	1	-
k=9			

-	2	4	6
2	-	5	3
4	5	-	1
6	3	1	-
k=10			

-	5	3	1
5	-	2	4
3	2	-	6
1	4	6	-

-	3	2	1
3	-	4	5
2	4	-	6
1	5	6	-

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 2, Issue 9, March 2013

k=11 k=12

Example 2: Realization of all possible magic constants for K_4. In this case n=4 and m=4(4-1)/2=6. Set of labels is {1,2,3,4,5,6,7,8,9,10}.

Magic constants are $4(4^2+3)/4 \leq k \leq 4(4+1)^2/4$ i.e. k={19,20,21,22,23,24,25}

Due to some law of small numbers in operation the constants 19, 22 and 25 are not realizable. The constants 20 and 21 are realizable and their corresponding duals 24 and 23 are listed in table 2 below.

Table 2: Labelings of complete graphs on 4 vertices.

-	3	8	9	10
3	-	6	4	7
8	6	-	5	1
9	4	5	-	2
10	7	1	2	-

k=20

-	2	7	8	9
2	-	10	4	5
7	10	-	3	1
8	4	3	-	6
9	5	1	6	-

k=21

-	9	4	3	2
9	-	1	7	6
4	1	-	8	10
3	7	8	-	5
2	6	10	5	-

k=23

-	8	3	2	1
8	-	5	7	4
3	5	-	6	10
2	7	6	-	9
1	4	10	9	-

k=24

Example 3: Realization of all possible magic constants for K_5.

In this case n=5 and m=5(5-1)/2=10. Set of labels is {1,2,3,……..,14,15}.

Magic constants are $5(5^2+3)/4 \leq k \leq 5(5+1)^2/4$ i.e. k={35,36,37,38,39,40, 41,42,43,44,45}. Constants 45,44,43,42,41 are duals of 35,36,37,38,39 respectively. All the labelings are listed in table 3 below.

Table 3: Labelings of complete graphs on 5 vertices.

-	11	12	13	14	15
11	-	10	9	1	4
12	10	-	2	8	3
13	9	2	-	5	6
14	1	8	5	-	7
15	4	3	6	7	-

k=35

-	9	11	12	13	15
9	-	4	14	8	1
11	4	-	5	6	10
12	14	5	-	2	3
13	8	6	2	-	7
15	1	10	3	7	-

k=36

-	9	10	11	12	13
9	-	15	8	2	3
10	15	-	7	4	1
11	8	7	-	5	6
12	2	4	5	-	14
13	3	1	6	14	-

k=37

-	5	8	10	12	15
5	-	14	11	7	1
8	14	-	9	4	3
10	11	9	-	2	6
12	7	4	2	-	13
15	1	3	6	13	-

k=38

-	2	3	12	13	15
2	-	14	7	11	5
3	14	-	10	8	4
12	7	10	-	1	9
13	11	8	1	-	6
15	5	4	9	6	-

k=39

-	2	4	6	13	15
2	-	14	12	11	1
4	14	-	10	5	7
6	12	10	-	3	9
13	11	5	3	-	8
15	1	7	9	8	-

k=40

The dual of the above labelings gives the remaining magic constants.

Example 4: Realization of all possible magic constants for K_6.

In this case n=6 and m=6(6-1)/2=15. Set of labels are {1,2,3,……..,20,21}.

Magic constants are $6(6^2+3)/4 \leq k \leq 6(6+1)^2/4$ i.e. k={59,60,61,62,63,64,65, 66,67,68,69,70,71,72,73}. Constants 73,72,71,70,69,68,67,66 are duals of 59,60,61,62,63, 64,65,66 respectively. All the labelings are listed in table 4 below.

Table 4: Labelings of complete graphs on 6 vertices.

-	13	17	18	19	20	21
13	-	16	15	7	3	5
17	16	-	4	2	6	14
18	15	4	-	12	9	1
19	7	2	12	-	11	8
20	3	6	9	11	-	10
21	5	14	1	8	10	-

k=59

-	7	17	18	19	20	21
7	-	16	15	14	5	3
17	16	-	2	1	13	11
18	15	2	-	12	4	9
19	14	1	12	-	8	6
20	5	13	4	8	-	10
21	3	11	9	6	10	-

k=60

-	6	12	18	19	20	21
6	-	17	16	15	2	5
12	17	-	1	9	8	14
18	16	1	-	3	13	10
19	15	9	3	-	11	4
20	2	8	13	11	-	7
21	5	14	10	4	7	-
k=61						

-	4	8	18	19	20	21
4	-	17	15	16	1	9
8	17	-	3	14	13	7
18	15	3	-	5	10	11
19	16	14	5	-	6	2
20	1	13	10	6	-	12
21	9	7	11	2	12	-
k=62						

-	3	4	17	19	20	21
3	-	18	16	15	6	5
4	18	-	8	7	12	14
17	16	8	-	11	10	1
19	15	7	11	-	2	9
20	6	12	10	2	-	13
21	5	14	1	9	13	-
k=63						

-	2	4	12	19	20	21

II. ALGORITHMS TO CONSTRUCT THE SUPER VERTEX MAGIC LABELING OF COMPLETE GRAPHS.

The process of constructing SVMTL of complete graph is carried out as follows:

- K_n , where n is odd.
- K_n , where $n \equiv 0 \bmod 4$ and n >4.
- K_n , where $n \equiv 2 \bmod 4$ are not SVMTL.

1.1 Algorithm to construct SVMTL for Kn, where n is odd.

Algorithm **SVMTL_Kn(n)**

/* This algorithm takes an integer parameter n, the number of vertices of the complete graph Kn. This algorithm constructs an n x n matrix, the construction is similar to the construction of magic square of order n. Let M denotes such a n x n matrix and i, j are row and column indices respectively. The array S contains numbers from 1 to n+m, where m=n(n-1)/2. */

Step 1: [Initialization]

i←1; j←1; m←n(n-1)/2;
Repeat for k←1 to n+m do
S[k]←k;

2	-	18	16	17	1	10
4	18	-	6	13	15	8
12	16	6	-	7	14	9
19	17	13	7	-	3	5
20	1	15	14	3	-	11
21	10	8	9	5	11	-
k=64						

-	3	4	5	19	20	21
3	-	18	16	2	14	12
4	18	-	10	17	9	7
5	16	10	-	15	13	6
19	2	17	15	-	1	11
20	14	9	13	1	-	8
21	12	7	6	11	8	-
k=65						

-	1	2	3	19	20	21
1	-	18	17	16	10	4
2	18	-	13	6	12	15
3	17	13	-	11	8	14
19	16	6	11	-	9	5
20	10	12	8	9	-	7
21	4	15	14	5	7	-
k=66						

The dual of the above labelings gives the remaining magic constants.

Step 2: Fill the matrix M starting with [i, j] in such a way that the filling process proceeds in down-right direction using the numbers of S. Assume that the rows and columns are cyclically around. If the present position is already filled then get back to the previous position and start filling from its down position. Repeat this until all the numbers from S are filled in M.

Step 3: At this stage n+n(n-1)/2 cells of M are filled and the remaining n(n-1)/2 cells are empty. Fill these empty cells by symmetry i.e M[i,j]=M[j,i].

Step 4: Now use the entries of M to label the vertices and edges of Kn
as follows. The entries across the diagonal are used to label the vertices and the left out entries to label the edges of the Complete graph Kn.

Step 5: Stop.

Applying this algorithm to K_3 and K_5 are illustrated bellow.

Example 5: K_3 Here n=3 and m=3(3-1)/2=3 and so S={1, 2, 3, 4, 5, 6}.

Table 5: SVMTL of Complete Graphs on 3 Vertices.

1		

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
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1		
	2	

1		
	2	
		3

1		
	2	4
		3

1		
	2	4
5		3

1	6	
	2	4
5		3

By symmetry fill the remaining cells.

1	6	5
6	2	4
5	4	3

The magic constant is 12= $[3(3+1)^2]/4$.

Example 6: K_5 Here n=5 and m=5(5-1)/2=10 and so S= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}.

Table 6: SVMTL of complete graph on 5 vertices.

1				
	2			
		3		
			4	
				5

1			10	
	2			6
7		3		
	8		4	
		9		5

1	14		10	
	2	15		6
7		3	11	
	8		4	12
13		9		5

By symmetry fill the remaining cells.

1	14	7	10	13
14	2	15	8	6
7	15	3	11	9
10	8	11	4	12
13	6	9	12	5

The magic constant is 45= $[5(5+1)^2]/4$.

5.2 There exists no SVMTL for K_n, where $n \equiv 2 \bmod 4$.

From theorem 2.1 it is known that the maximum magic constant for K_n is $[n(n+1)^2]/4$ for any positive integer n. If n is even and $n \equiv 2 \bmod 4$, the equation $[n(n+1)^2]/4$ evaluates to a non integer, hence it is not possible to obtain SVMTL for K_n, where $n \equiv 2 \bmod 4$.

Example 7: if n=6, then $[n(n+1)^2]/4$ becomes

$$[6(6+1)^2]/4 = (6 \times 49)/4 = 73.5.$$

5.3 There exists SVMTL for K_n, where $n \equiv 0 \bmod 4$.

In [12] it is known that it is possible to realize all possible magic constants for Kn.

From theorem 2.1 it is known that the maximum magic constant for K_n is $[n(n+1)^2]/4$ for any positive integer n. If n is even and $n \equiv 0 \bmod 4$, the equation $[n(n+1)^2]/4$ evaluates to an integer, hence it is possible to obtain SVMTL for K_n, where $n \equiv 0 \bmod 4$.

Example 8: For K_8, here n=8 and the magic constant is $[8(8+1)^2]/4 = 162$.

Using the algorithm in [12], the 8 x 8 matrix which realizes the SVMTL for K_8 is as shown below.

Table 7: SVMTL of complete graph on 8 vertices.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	-	15	19	16	24	35	31	21
2	15	-	23	11	12	30	33	36
3	19	23	-	20	27	10	28	32
4	16	11	20	-	34	29	22	26
5	24	12	27	34	-	25	18	17
6	35	30	10	29	25	-	14	13
7	31	33	28	22	18	14	-	9
8	21	36	32	26	17	13	9	-
k	**162**	**162**	**162**	**162**	**162**	**162**	**162**	**162**

III. CONCLUSION

Making use of our algorithm for any complete graph Kn where n=3 and n>4, any allowable constant k can be realizable and the realization of super vertex magic total labeling of complete graphs can be achieved.

IV. CHALLENGES

1. The labeling obtained for the specified constant k in the complete graph Kn is not unique. So, how many distinct solutions are there for the constant k in the given graph Kn?
2. The algorithm proposed above is by the method of brute force. Hence the total time taken to realize the

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 2, Issue 9, March 2013

given constant is more. Is there any method by which we can reduce the overall time required to realize the given constant.

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