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# Quadrature Methods and Their Splitting Extrapolations for Parallel Computation of Axisymmetric Stokes Fluid Flow

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***Abstract*—By ring potential theory, the double integral equations of axisymmetric Stokes equations can be converted into the single integral equations. The splitting extrapolation methods (SEMs) are applied to the boundary integral equations (BIEs) of axisymmetric Dirichlet's problem governed by Stokes equations by the mechanical quadrature methods (MQMs). An asymptoic expansion with odd powers of error is presented, which posseses high accuracy order $O(h_{\max}^3)$. Using $h^3$-Richardson splitting extrapolation algorithms, the accuracy order of the approximation can be greatly improved, and an a posteriori error estimate can be obtained for constructing a self-adaptive parallel algorithm. The efficiency of the parallel algorithm is illustrated by examples.**

***Index Terms*—Axisymmetric Stokes Problems, Splitting Extrapolation Methods, Boundary Integral Equations, Parallel Computation.**

## I. INTRODUCTION

Consider the Stokes interior problem:

$$\begin{cases} p_{,i} = \mu \sum_{j=1}^{3} u_{i,jj}(x), x=(x_1,x_2,x_3)\in V, \\ \sum_{j=1}^{3} u_{i,i}(x) = 0, x=(x_1,x_2,x_3)\in V, \\ u_i(x) = u_{i0}(x), x=(x_1,x_2,x_3)\in \partial V, \\ \int_S \sum_{j=1}^{3} u_i(x) n_i(x) ds = 0, x=(x_1,x_2,x_3)\in V, \end{cases} \quad (1.1a)$$

and the Stokes exterior problem:

$$\begin{cases} p_{,i} = \mu \sum_{j=1}^{3} u_{i,jj}(x), x=(x_1,x_2,x_3)\in V^c, \\ \sum_{j=1}^{3} u_{i,i}(x) = 0, x=(x_1,x_2,x_3)\in V^c, \\ u_i(x) = u_{i0}(x), x=(x_1,x_2,x_3)\in \partial V, \\ u_i(x)=0,\ p(x)=0, |x| = \sqrt{x_1^2+x_2^2+x_3^2} \to \infty, \end{cases} \quad (1.1b)$$

Where $V$ is an axisymmetric bounded domain of space $\Re^3$. $V$ is formed by rotating a two-dimensional bounded region $\Omega$ with the boundary $\Gamma = \sum_{m=1}^{d} \Gamma_m$, called a generatrix line of $V$, around $z$-axis. In (1.1), $\partial V$ is the surface of $V$, and $\bar{V} = V \cup \partial V$, and $V^c = \Re^3 \backslash \bar{V}$, and $n=(n_1,n_2,n_3)$ is the outward unit normal vector to $\partial V$, and $u=(u_1,u_2,u_3)$ is a fluid velocity, and $p=(p_1,p_2,p_3)$ is a pressure; and $u_{i,j} = \partial u_i / \partial u_j (i,j=1,2,3)$ is the dynamic viscosity of the flow; and $\mu$ is a constant, and the repeated subscripts imply the summation 1 to 3. If the rotation axis, i.e., $z$-axis, is include by $V$, then $\Gamma = \Gamma_o$ is open arcs, and if the rotation $z$-axis, is include by $V$, then $\Gamma = \Gamma_c$ is close arcs. In addition, we assume that $Q_m$ is the corner points of the generatrix line $\Gamma$.

There exist many difficulties in problems of complicated geometries and infinite domains for solving (1.1) by the finite element method (FEM) and the finite difference method (FDM) [1], [3]-[5], [24], [27]. The finite volume method (FVM) too relies on discretizing space (into small volumes) and so has the obvious drawback for exterior creeping flow problems. The boundary integral method is the method of choice when solving external problems and those involving complex regions [7], [10], [13], [14]. The dimension of the problem is reduced and integration along complicated curves can be accurately evaluated.

Based on Ladyzhenskaya's theory and the single-layer potential theory [9], [18], (1.1) can be converted into the following boundary integral equations (see, e.g., [9], [15], [16], [18], [23]-[26])

$$\begin{cases} u_k(x) = \int_{\partial V} \sum_{i=1}^{3} u_{ki}^{**}(x,y)\sigma_i(y) ds_y, x\in V\cup V^c, k=1,2,3; \\ p(x) = \int_{\partial V} \sum_{i=1}^{3} p_i^{**}(x,y)\sigma_i(y) ds_y, x\in V\cup V^c, \end{cases} \quad (1.2a)$$

Where

$$\begin{cases} u_{ki}^{**}(x,y) = \frac{1}{8\pi\mu}\left[\frac{\delta_{ki}}{|x-y|} + \frac{(x_k-y_k)(x_i-y_i)}{|x-y|^3}\right], \\ p_i^{**}(x,y) = \frac{x_i-y_i}{4\pi|x-y|^3}, k,i=1,2,3, \end{cases} \quad (1.2b)$$

are the fundamental solutions of (1.1); $|x-y|$ shows the Euclidean distance with $y=(y_1,y_2,y_3)$; and $\delta_{ki}$ is Kronecker sign.

In [9], [18], it has been proved that if the boundary $\partial V$ is a Liapunov surface, then the single-layer potential $u_k(x)$ is a continuous function in $V\cup V^c \cup \partial V$; and the dense functions $\sigma = (\sigma_1(y), \sigma_2(y), \sigma_3(y))$ satisfy the

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following BIEs of the first kind

$$u_{k0}(x)=\int_{\partial V}\sum_{i=1}^{3}u_{ki}^{**}(x,y)\sigma_i(y)ds_y, x\in\partial V, k=1,2,3. \quad (1.3a)$$

Based on [9], [18], [26], the solutions $\sigma=(\sigma_1(y),\sigma_2(y),\sigma_3(y))$ of (1.3a) are not unique, because $\sigma+n$ all are the solutions of (1.3a). It is known that (1.3a) is solvable if and only if it satisfies the following compatibility condition [5]

$$(\sigma,n)=\int_{\partial V}\sum_{i=1}^{3}\sigma_i(y)ds_y=0. \quad (1.3b)$$

Once the single-layer dense functions are solved from (1.3a) and (1.3b), the fluid velocity $u$ and pressure $p$ can be obtained by (1.2a).

So far numerical methods for solving (1.3) are Galerkin and collocation methods [1], [2], [4], [5], [13], [17], [18], [24]. However, the discrete matrix is full and each element has to calculate the double weakly singular integral for collocation methods or the four-fold weak singular integral for Galerkin methods, which imply that CPU time expanded by calculating discrete matrix is so more as to exceed greatly to solve discrete equations, and the accuracy order of solutions is very low. Mechanical quadrature methods (MQMs) have been applied to solve the boundary integrals well (see, e.g., [8], [9]). Obviously, we can construct MQMs for solving Stokes BIEs and prove that the methods are convergent by using some quadrature rules to deal with the singular integrals similar to [8], [9], then the calculation of the discrete matrix becomes very simple and the most of work can be saved. In this paper, first, by ring potential theory, the double integral equations of axisymmetric Stokes Dirichlet's problems can be converted into the single integral equations. Secondly, by periodic transformations [22] to eliminate singularities of solutions and kernel integrals at discontinuous points, we propose high accuracy MQMs for solving BIEs (1.3) of axisymmetric Stokes Dirichlet's problems by using quadrature rules [21]. Thirdly, based on multivariate asymptotic expansions of errors with odd power $O(h_i^3)(i=1,\dots,d)$, we establish the splitting extrapolation algorithms [8], [11], [28] with highly accuracy approximations and get a posterior error estimate as adaptive algorithms by solving discrete equations on some coarse mesh partitions in parallel. Finally, numerical results show further that the methods are very effective.

## II. Quadrature Methods

If $V\subset\Re^3$ is an axisymmetric bounded domain, and is translating in the direction of its symmetric axis, then the flow field is also axisymmetric, and the single-layer potential dense functions $\sigma=(\sigma_1(y),\sigma_2(y),\sigma_3(y))$ is independent of the azimuthally angle in a cylindrical polar coordinate system $(r,\theta,z)$, that is, we have

$$\sigma=(\sigma_r\cos\theta,\sigma_r\sin\theta,\sigma_z),\theta\in[0,2\pi], \quad (2.1)$$

Where $\sigma_r$ and $\sigma_z$ are the components of $\sigma$ in the radial $r$ and axial $z$ directions respectively. Without loss of generality, let the boundary have the parametric representation

$$x=(r(t)\cos\theta,r(t)\sin\theta,z(t)),0\le t\le 1,0\le\theta\le 2\pi,$$

and choosing $x$ to lie on the $\theta=0$ plane results in the following from for $x_0=(r_0,0,z_0)$ and $y=(r(t)\cos\theta,r(t)\sin\theta,z(t))$, we have

$$|x_0-y|=\sqrt{4rr_0(1-k^2\cos^2(\theta/2))/k^2} \quad (2.2)$$

with $k^2=4rr_0/[(r+r_0)^2+(z+z_0)^2]$. For $r>0$ and $r_0>0$, by the three-dimensional fundamental solution of axisymmetric Stokes' problem, substituting (2.1) and (2.2) into (1.2b), we obtain

$$u_{rr}^*(r,z;r_0,z_0)=\int_0^{2\pi}(u_{11}^{**}\cos\theta+u_{12}^{**}\sin\theta)d\theta=\frac{1}{4\pi\mu rr_0c}\{[(r^2+r_0^2)+2(z-z_0)^2]K(k)-[c^2+\frac{(z-z_0)^2(r^2+r_0^2+(z-z_0)^2)}{c_0^2}]E(k)\ \}, \quad (2.3a)$$

$$u_{rz}^*(r,z;r_0,z_0)=\int_0^{2\pi}u_{13}^{**}d\theta=\frac{z-z_0}{4\pi\mu r_0c}\{-K(k)+\frac{(r^2-r_0^2+(z-z_0)^2)}{c_0^2}E(k)\ \}, \quad (2.3b)$$

$$u_{zr}^*(r,z;r_0,z_0)=\int_0^{2\pi}(u_{31}^{**}\cos\theta+u_{32}^{**}\sin\theta)d\theta=\frac{z-z_0}{4\pi\mu rc}\{\ K(k)+\frac{(r^2-r_0^2-(z-z_0)^2)}{c_0^2}E(k)\ \}, \quad (2.3c)$$

$$u_{zz}^*(r,z;r_0,z_0)=\int_0^{2\pi}u_{33}^{**}d\theta=\frac{1}{2\pi\mu c}\{K(k)+\frac{(z-z_0)^2}{c_0^2}E(k)\}, \quad (2.3d)$$

and

$$p_z^*(r,z;r_0,z_0)=\int_0^{2\pi}p_3^{**}d\theta=\frac{(z-z_0)E(k)}{\pi c^{3/2}(1-k^2)}, \quad (2.3e)$$

$$p_r^*(r,z;r_0,z_0)=\int_0^{2\pi}(p_1^{**}\cos\theta+p_2^{**}\sin\theta)d\theta=\frac{1}{\pi c^{3/2}}\{-\frac{2r_0}{k^2}K(k)+\frac{2r_0-(r+r_0)k^2}{k^2(1-k^2)}E(k)\}, \quad (2.3f)$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first kind and second kind respectively, and $c^2=(r+r_0)^2+(z-z_0)^2$, $c_0{}^2=(r-r_0)^2+(z-z_0)^2$.

Based on (1.3a) and (2.3), we obtain the following BIEs of axisymmetric Stokes' equations:

$$u_{r0}(r_q,z_q)=\sum_{m=1}^{d}\int_{\Gamma_m}r_m[u_{rr}^*(r_m,z_m;r_q,z_q)\sigma_r(r_m,z_m)+u_{rz}^*(r_m,z_m;r_q,z_q)\sigma_z(r_m,z_m)d\Gamma, \quad (2.4a)$$

and

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$$u_{z0}(r_q,z_q)=\sum_{m=1}^{d}\int_{\Gamma_m} r_m[u^*_{zr}(r_m,z_m;r_q,z_q)\sigma_r(r_m,z_m) \quad (2.4b)$$
$$+u^*_{zz}(r_m,z_m;r_q,z_q)\sigma_z(r_m,z_m)d\Gamma,$$

where $(r_q,z_q)\in\Gamma_q(q=1,...,d)$. The compatibility condition (1.3b) is reduced into

$$(\sigma,n)=2\pi\sum_{m=1}^{d}\int_{\Gamma_m} r_m[n_r(r_m,z_m)\sigma_r(r_m,z_m) \quad (2.4c)$$
$$+n_z(r_m,z_m)\sigma_z(r_m,z_m)d\Gamma=0.$$

Assume that the piece wise boundary can be described by the parameter mapping

$$x_m(s)=(x_{m1}(s),x_{m2}(s))\in C^{\gamma}[0,1](\gamma\in N):[0,1]\to\Gamma_m$$

with

$$|x_m^{'}(s)|^2=|x_{m1}^{'}(s)|^2+|x_{m2}^{'}(s)|^2>0.$$

Since the solutions and the integral kernels at the corner points have the singularities, by Sidi's transformation [22] we take

$$s=s(\theta)=\varphi_\alpha(\theta):[0,1]\to[0,1],\alpha\in N, \quad (2.5)$$

to eliminate the singularities from $\sigma_i(r,z)(i=r,z)$ and integral kernels at the corner points, where $\varphi_\alpha(\theta)=\vartheta_\alpha(\theta)/_\alpha(1),\ \vartheta_\alpha(\theta)=\int_0^\theta(\sin\pi\beta)^\alpha d\beta.$

Define the boundary integral operators on $[0,1](m=1,...,d)$:

$$(\bar{B}_{ij}^{qm}v)(t)=\int_0^1 r_m(\tau)u_{ij}^{*qm}(t,\tau)v(\tau)\varphi_\alpha^{'}(\tau)|x_m^{'}(\tau)|d\tau,(q,m=1,...,d),$$

where

$$u_{ij}^{*qm}(t,\tau)=u_{ij}^*(x_m(\varphi_\alpha(\tau)),x_q(\varphi_\alpha(\tau)))(i,j=r,z),$$

$$r_m(\tau)=r_m(x_m(\varphi_\alpha(\tau))),$$

and the linear functional

$$l^m(n_i^m,\cdot)v=2\pi\int_0^1 r_m(\tau)n_i^m(\tau)v(\tau)\varphi_\alpha^{'}(\tau)|x_m^{'}(\tau)|d\tau,$$

(2.6)
where

$$n_i^m(\tau)=n_i^m(x_m(\varphi_\alpha(\tau))).$$

Using Lagrange multiplier method, we have the operaor equations

$$\sum_{m=1}^{d}\begin{bmatrix}\bar{B}_{rr}^{qm} & \bar{B}_{rz}^{qm} & n_r^m\\ \bar{B}_{zr}^{qm} & \bar{B}_{zz}^{qm} & n_z^m\\ l^m(n_r,\cdot) & l^m(n_z,\cdot) & 0\end{bmatrix}\begin{bmatrix}\sigma_r^m(\tau)\\ \sigma_z^m(\tau)\\ \lambda\end{bmatrix}=\begin{bmatrix}u_{r0}^q(t)\\ u_{z0}^q(t)\\ 0\end{bmatrix}(q=1,...,d),$$

(2.7)

where $\lambda$ is a Lagrange multiplier, $\sigma_i^m(\tau)=\sigma_i^m(x_m(\varphi_\alpha(\tau))\varphi_\alpha^{'}(\tau)|x_\alpha^{'}(\tau)|,$ and $u_{i0}^q(t)=u_{i0}(x_q(\varphi_\alpha(t)))\varphi_\alpha^{'}(\tau)(i=r,z).$ Let $\bar{b}_{ij}^{qm}(t,\tau)$ be the integral kernels of integral operators $\bar{B}_{ij}^{qm}(i,j=r,z).$ Since the kernels of operators $\bar{B}_{rr}^{qq}$ and $\bar{B}_{zz}^{qq}$ have the logarithmic singularities, we have the following decomposition

$$\bar{B}_{rr}^{qq}=A_{00}^{qq}+B_{rr}^{qq},q=1,...,d, \quad (2.8)$$

and

$$\bar{B}_{zz}^{qq}=A_{00}^{qq}+B_{zz}^{qq},q=1,...,d. \quad (2.9)$$

If the end of curve $\Gamma_q$ is not at the axis of symmetry, we have

$$a_{00}^{qq}(t,\tau)=-\frac{r_q(\tau)}{4\pi r_q(t)}\ln|2e^{-1/2}\sin(\pi(t-\tau))|,$$

where $a_{00}^{qq}(t,\tau)$ is the kernel of $A_{00}^{qq}$ . If the end of curve is in the axis of symmetry, we have

$$a_{00}^{qq}(0,\tau)=-\frac{1}{4\pi}\ln|2e^{-1/2}\sin(\pi\tau)|,\ for\quad t=0,$$

and

$$a_{00}^{qq}(1,\tau)=-\frac{1}{4\pi}\ln|2e^{-1/2}\sin(\pi(1-\tau))|\ for\quad t=1.$$

Thus, the operator equations (2.7) become

$$(\tilde{A}+\tilde{B})\sigma=u_0, \quad (2.10a)$$

where

$$\tilde{A}=diag(A^{11},...,A^{dd},1),$$

with

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$$A^{qq} = diag(A_{00}^{qq}, A_{00}^{qq}),$$

and

$$\overset{\square}{B} = \begin{bmatrix} B^{11} & \dots & B^{1d} & n^1 \\ \dots & \dots & \dots & \dots \\ B^{d1} & \dots & B^{dd} & n^d \\ l^{1*} & \dots & l^{d*} & -1 \end{bmatrix}$$

with

$$B^{qm} = \begin{bmatrix} B_{rr}^{qm} & B_{rz}^{qm} \\ B_{zr}^{qm} & B_{zz}^{qm} \end{bmatrix}, \qquad (2.10b)$$

and

$$l^{q*} = (l^q(n_r,\square), l^q(n_z,\square)),\ n^q = (n_r^q, n_z^q)^T,$$

with

$$\sigma = (\sigma^1, \dots, \sigma^d, \lambda)^T,\ \sigma^q = (\sigma_r^q, \sigma_z^q)^T,$$

and $u_0 = (u^1, \dots, u^d, 0)^T$ with $u^q = (u_{r0}^q, u_{z0}^q)^T$.

The kernels $b_{ij}^{qm}(t,\tau)(i,j = r,z)$ of operators $B_{ij}^{qm}$ are

$$b_{rr}^{qm}(t,\tau) = \begin{cases} \bar{b}_{rr}^{qq}(t,\tau) - a_{00}^{qq}(t,\tau),\ for\quad q = m \\ \bar{b}_{rr}^{qm}(t,\tau),\ for\quad q \neq m, \end{cases} \qquad (2.10c)$$

and

$$b_{zz}^{qm}(t,\tau) = \begin{cases} \bar{b}_{zz}^{qq}(t,\tau) - a_{00}^{qq}(t,\tau),\ for\quad q = m \\ \bar{b}_{zz}^{qm}(t,\tau),\ for\quad q \neq m, \end{cases} \qquad (2.10d)$$

and $b_{rz}^{qm}(t,\tau) = \bar{b}_{rz}^{qm}(t,\tau),\ b_{zr}^{qm}(t,\tau) = \bar{b}_{zr}^{qm}(t,\tau).$

Since $A_{00}^{qq} : H^m(0,1) \to H^{m+1}(0,1)$ is a bitmapping operator [1], $\overset{\square}{A} : (H^m(0,1))^{2d} \times R \to (H^{m+1}(0,1))^{2d} \times R$ also is a bitmapping operator. Hence, (2.10a) is converted into

$$(E + \overset{\square}{A}^{-1}\overset{\square}{B})\sigma = \overset{\square}{A}^{-1}u_0, \qquad (2.11)$$

Where $E$ is the identity operator? Let $\{\tau_{m\rho} = (\rho - 1/2)h_m, \rho = 1, \dots, n_m, h = 1/n_m, m = 1, \dots, d\}$ be the mesh point set.

Since $b_{ij}^{qm}(t,\tau)(i,j = r,z)$ are the smooth functions on $[0,1]$ for $q \neq m$, using the quadrature rules [6], we construct its nystrom approximation $B_{ij,h}^{qm}$ of the integral operators $B_{ij}^{qm}$,

$$(B_{ij,h}^{qm} v_m)(t) = h_m \sum_{\beta=1}^{n_m} b_{ij}^{qm}(t, \tau_{m\beta}) v_m(\tau_{m\beta}), q, m = 1, \dots, d, \qquad (2.12a)$$

and the error

$$(B_{ij,h}^{qm} v_m)(t) - (B_{ij}^{qm} v_m)(t) = O(h_m^{2\gamma}), \gamma \in N. \qquad (2.12b)$$

Since $B_{ii}^{qq} = A_{00}^{qq} + \bar{B}_{ii}^{qq}(i = r,z)$ have the logarithmic singularities on $[0,1]$ for $q = m$, by the quadrature formula [21], we get the following approximations $A_{00,h}^{qq}$ of the integral operators $A_{00}^{qq}$,

$$\begin{aligned}(A_{00,h}^{qq} v_q)(t_{q\rho}) &= h_q \sum_{\beta=1, t_{q\rho} \neq t_{q\beta}}^{n_q} a_{00}^{qq}(t_{q\rho}, \tau_{q\beta}) v_q(\tau_{q\rho}) - \frac{h_q}{4\pi} \ln(e^{-1/2} h_q) v_q(\tau_{q\rho}) \\ &= \sum_{\beta=1, t_{q\rho} \neq t_{q\beta}}^{n_m} \frac{-h_q r(\tau_{q\beta})}{4\pi r(\tau_{q\rho})} \ln|2e^{-1/2} \sin(\pi(t_{q\rho} - \tau_{q\beta})) v_q(\tau_{q\beta}) \\ &\quad - \frac{h_q}{4\pi} \ln(e^{-1/2} h_q) v_q(\tau_{q\rho}),\end{aligned} \qquad (2.13a)$$

and the errors

$$\begin{aligned}&(A_{00,h}^{qq} v_q)(t_{q\rho}) - (A_{00}^{qq} v_q)(t_{q\rho}) \\ &= \frac{-1}{\pi r(t_{q\rho})} \sum_{\mu=1}^{2\gamma-1} \frac{\varsigma'(-2\mu) d^{2\mu}(r(t) v_q(t))}{(2\mu)! dt^{2\mu}} |_{t=t_{q\rho}} h_q^{2\mu+1} + O(h_q^{2r}),\end{aligned} \qquad (2.13b)$$

where $t_{q\rho} = (\rho - 1/2)h_q$ and $\varsigma'(t)$ is the derivative of Riemann zeta function, and

$$\begin{aligned}(B_{ii,h}^{qq} v_q)(t_{q\rho}) &= h_q \sum_{\beta=1, t_{q\rho} \neq t_{q\beta}}^{n_q} b_{ii}^{qq}(t_{q\rho}, \tau_{q\beta}) v_q(\tau_{q\beta}) + \frac{h_q}{4\pi}[\ln|8r(t_{q\rho}) \\ &- \ln|\frac{e^{1/2} x_q'(t_{q\rho}) \varphi_\alpha'(t_{q\rho})}{2\pi}|](h_q) v_q(\tau_{q\rho}), q = 1, \dots, d, \beta - 1, \dots, n_q.\end{aligned} \qquad (2.14)$$

Hence, the approximations $\bar{B}_{ii,h}^{qq}$ of the integral operators $\bar{B}_{ii}^{qq}$ are defined by

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$$(\bar{B}_{ii,h}^{qq}v_q)(t_{q\rho})=[A_{00,h}^{qq}v_q+B_{ii,h}^{qq}v_q](t_{q\rho})=h_q\sum_{\beta=1,t_{q\rho}\neq t_{q\beta}}^{n_q}\bar{b}_{ii}^{qq}(t_{q\rho},\tau_{q\beta})v_q(\tau_{q\beta})$$
$$+\frac{h_q}{4\pi}[\ln|8r(t_{q\rho})-\ln|\frac{e^{1/2}x_q'(t_{q\rho})\varphi_\alpha'(t_{q\rho})}{2\pi}|-\ln(e^{-1/2}h_q)](h_q)v_q(\tau_{q\rho}).$$

(2.15)

Thus we obtain approximate equations of (2.10),

$$(\hat{A}_h+\hat{B}_h)\sigma^h=u_0^h, \quad (2.16)$$

where

$$\hat{A}_h=diag(A_h^{11},...,A_h^{dd}),\ A_h^{qq}=diag(A_{00,h}^{qq},A_{00,h}^{qq}),$$

and

$$\hat{B}_h=\begin{bmatrix} B_h^{11} & \cdots & B_h^{1d} & n_h^1 \\ \cdots & \cdots & \cdots & \cdots \\ B_h^{d1} & \cdots & B_h^{dd} & n_h^d \\ l_h^{1*} & \cdots & l_h^{d*} & -1 \end{bmatrix}$$

with

$$B_h^{qm}=\begin{bmatrix} B_{rr,h}^{qm} & B_{rz,h}^{qm} \\ B_{zr,h}^{qm} & B_{zz,h}^{qm} \end{bmatrix},$$

and

$$l_h^{q*}=(l_h^q(n_r,\cdot),l_h^q(n_z,\cdot)),\ n_h^q=(n_{r,h}^q,n_{z,h}^q)^T,$$
$$\sigma^h=(\sigma_h^1,...,\sigma_h^d,\lambda)^T$$

with

$$\sigma_h^q=(\sigma_{r,h}^q,\sigma_{z,h}^q)^T,$$

and

$$\sigma_{i,h}^q=(\sigma_{i,h}^q(\tau_{q1}),...,\sigma_{i,h}^q(\tau_{qn_q}))^T(i=r,z),$$
$$u_0^h=(u_h^1,...,u_h^d,0)^T$$

with

$$u_h^q=(u_{r0,h}^q,u_{z0,h}^q)^T,$$

and

$$u_{i0,h}^q=(u_{i0,h}^q(t_{q1}),...,u_{i0,h}^q(t_{qn_q})^T(i=r,z).$$

Once $\sigma^h$ is solved by (2.16), using the quadrature rule [6], we obtain

$$\begin{cases} u_k(x)=\sum_{i=1}^{3}\sum_{m=1}^{d}h_m\sum_{j=0}^{n-1}\left(\int_0^{2\pi}u_{kj}(x-y(\tau_j))d\theta\right)\sigma_{mi}^h(\tau_j)|x'(\tau_j)|\varphi_\alpha'(\tau_j), \\ p(x)=\sum_{i=1}^{3}\sum_{m=1}^{d}h_m\sum_{j=0}^{n-1}\left(\int_0^{2\pi}p_i(x-y(\tau_j))d\theta\right)\sigma_{mi}^h(\tau_j)|x'(\tau_j)|\varphi_\alpha'(\tau_j), \end{cases}$$

(2.17)

Where $y(\tau_j)=(r(\varphi_\alpha(\tau_j))\cos\theta,r(\varphi_\alpha(\tau_j))\sin\theta,z(t))\in\Re^2\setminus T.$

## III. SPLITTING EXTRAPOLATIONS

Based on [8], [9], we can obtain the following important result.

Let $\Gamma_m$ be the smooth curves, $u_{i0}=u_{i0}|_{\Gamma_m}\in C^4(\Gamma_m)$, and $x_{mi}(t)\in C^5[0,1], m=1,...,d, i=1,2.$ then the approximate errors possess the following multivariate asymptotic expansions

$$\sigma-\sigma^h=\sum_{\mu=1}^{\gamma-1}diag(h_1^3,...,h_l^3)\Phi_\mu+O(h_0^3),h_0^3=\max_{1\le m\le d}h_m,$$

(3.1)

Where $\Phi_\mu=(\Phi_{\mu1},...,\Phi_{\mu l})^T$ and $\Phi_{\mu m}\in C[0,1]$ is independent of $h_m$.

The multi-parameter asymptotic expansion (3.1) means that the SEMs can be applied to solve (1.3), that is, a higher order accuracy $O(h_0^3)$ at the coarse grid points can be obtained by solving some discrete equations in parallel. The process of the SEMs is as follows [7], [8], [11].

**Step 1.** Take $h^{(0)}=(h_1,...,h_d)$ and $h^{(m)}=(h_1,...,h_m/2,...,h_d)$. Then we can solve the problem (2.16) in parallel according to the mesh parameter $h^{(m)}$. Let $\sigma^{h^m}(t_\rho)$ be their solutions.

**Step 2.** Implement the following SEMs on the coarse grid points

$$\sigma^*(t_\rho)=\frac{8}{7}\left[\sum_{m=1}^{d}\sigma^{h^{(m)}}(t_\rho)-(d-\frac{7}{8})\sigma^{h^{(0)}}(t_\rho)\right].$$

(3.2)

**Step 3.** Compute the approximations of $u_i(y)$ and $p(y)$ from (2.17) using $\sigma^*(t_\rho)$. By the inequality

$$|\sigma(t_\rho)-\frac{1}{d}\sum_{m=1}^{d}\sigma^{h^{(m)}}(t_\rho)|\le|\sigma(t_\rho)-\frac{8}{7}\left[\sum_{m=1}^{d}\sigma^{h^{(m)}}(t_\rho)-(d-\frac{7}{8})\sigma^{h^{(0)}}(t_\rho)\right]|+$$
$$\frac{8d-7}{7}|\frac{1}{d}\sum_{m=1}^{d}\sigma^{h^{(m)}}(t_\rho)-\sigma^{h^{(0)}}(t_\rho)|\le\frac{8d-7}{7}|\frac{1}{d}\sum_{m=1}^{d}\sigma^{h^{(m)}}(t_\rho)-\sigma^{h^{(0)}}(t_\rho)|+O(h_0^5),$$

we get a posteriori estimate of the error, which can be applied to construct some adaptive parallel algorithms.

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## IV. NUMERICAL RESULTS

In this section, we study unbounded axisymmetric fluid flow past an array of rigid balls. Due to the great application in the sedimentation study, the problem of the unbounded axisymmetric fluid flow past an array of rigid balls has attracted much attention from researches. Assume the line of centers of the spheres coincides with $z$ -axis. The radii $R$ of the balls are not necessary same. Figure 1 presents a particular case, where the radii $R$ of all balls is equal to $a$ , the distance between the centers of two consecutive balls is $2d$ . At present, several analytic methods have been presented for solving such this problem. For example, Happer and Brenner concluded the application of the method of reflection in [7] and obtained some analytic solutions for certain cases. The relative error is under 1% when $d/a > 3$. However, the relative error shoots up to 11% when $d/a = 1$ as two balls tangent to each other. In 1971, Glukman, Pfeffer and Weinbaum presented the multipole method [13], the relative error of which is within 2.5% for any value of $d/a$ when same balls are evenly distributed. While, the application of such method to more complicated cases is not straightforward, for which the computational cost is too expensive. The method presented in this paper yields good numerical results. First, we consider a particular case in [13], the flow past two same balls with radii $a$ in the direction of the negative $z$ -axis direction with the uniform speed $U$ , i.e., the fluid velocity is $V = -U$ , and the distance between the centers $2d$ . The resistances $F_z$ on the balls are same. We use the resistance factor $\chi = -\dfrac{F_z}{8\pi\mu U}$ to test our method; the exact solution was given by Stimson and Jeffery [20], that is,

$$\chi = \frac{4}{3}\sinh\alpha \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \left[ 1 - \frac{4\sinh^2(n+\frac{1}{2})\alpha - (2n+1)^2\sinh^2\alpha}{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha} \right].$$

Fig. 2-5 shows the computational results using $(2^k, 2^k)(k = 4,\dots,7)$ boundary nodes by transformation $\varphi_3$. From Table 1 we can see ratio $\approx 8$, to agree with our theory, where $n_1$ and $n_2$ are the node numbers on the first and second balls respectively, and $\dfrac{d}{\alpha}$ is equivalent to 1.0001, 1.01, 2 and 10 respectively.

Secondly, we consider balls with different size. Suppose three stationary balls with centers at (0,0), (0,5) and (0,11) and radii $R_1 = 1$, $R_2 = 4$ and $R_3 = 2$. The inflow velocity $V = -1$. The numerical solutions of resistances on three balls $F = (F_{Z1}, F_{Z2}, F_{Z3})^T$ are listed in Table 2, which shows the numerical result has the precision of at least 4 decimal places, although the exact solutions cannot obtain.

At last, the flow field yielded by the motion of the array of rigid balls with different size $(R_1, R_2, R_3)$ and speed $(v_1, v_2, v_3)$ in the unbounded flow (the inflow velocity $V$ ) are plotted in Figure 6-8.

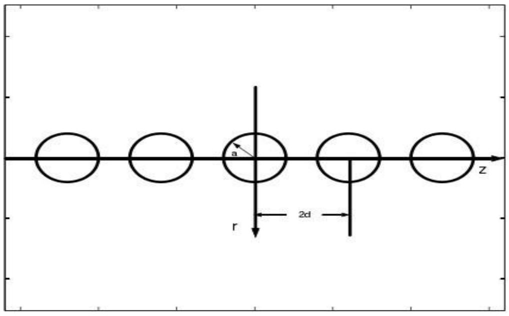


**Fig. 1. Multi-sphere**

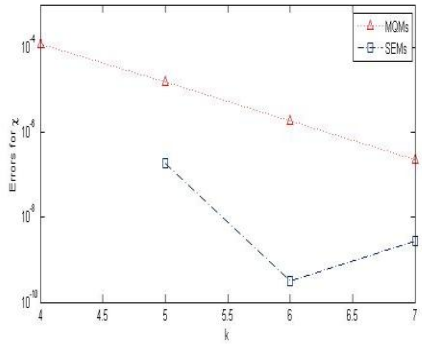


**Fig. 2. Errors of $\chi$ by MQMs and SEMs ($\dfrac{d}{\alpha} = 1.0001$)**

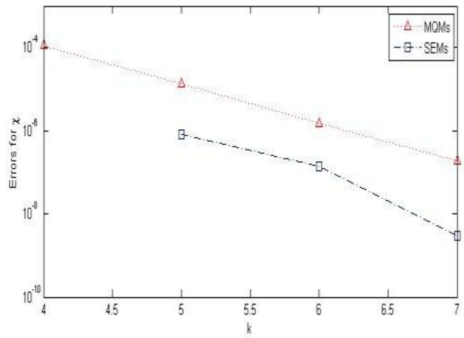


**Fig. 3. Errors of $\chi$ by MQMs and SEMs ($\dfrac{d}{\alpha} = 1.01$)**

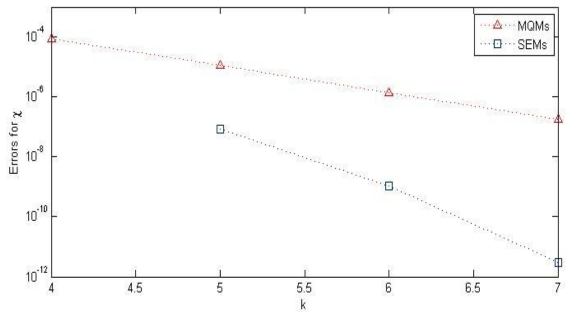


**Fig. 4. Errors of $\chi$ by MQMs and SEMs ($\dfrac{d}{\alpha} = 2$ )**

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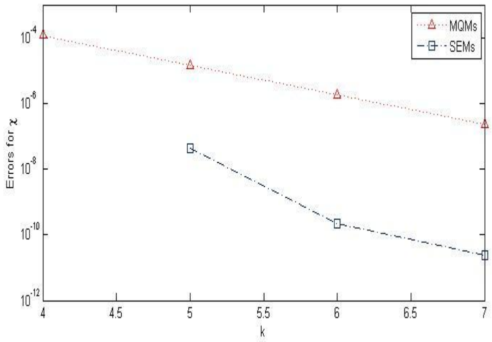


**Fig. 5. Errors of $\chi$ by MQMs and SEMs ($\frac{d}{\alpha}=10$)**

**Table 1. Errors of ratio for MQMs**

| $n=(n_1,n_2)$ | $ratio_\chi^{1.0001}$ | $ratio_\chi^{1.01}$ | $ratio_\chi^{2}$ | $ratio_\chi^{10}$ |
|---|---|---|---|---|
| (16,16) | - | - | - | - |
| (32,32) | 7.915 | 8.429 | 7.947 | 7.980 |
| (64,64) | 8.001 | 8.648 | 7.995 | 8.001 |
| (128,128) | 8.087 | 8.111 | 7.999 | 8.000 |

**Table 2. Numerical Solutions of Resistances on Three Balls**

| $n=(n_1,n_2,n_3)$ | $F_z=(F_{z1},F_{z2},F_{z3})^T$ | $h^3-SEMs$ |
|---|---|---|
| (16,16,16) | $\begin{pmatrix}8.835199053\\60.003418899\\22.494547512\end{pmatrix}$ | |
| (32,32,32) | $\begin{pmatrix}8.836047974\\60.010437194\\22.496613832\end{pmatrix}$ | $\begin{pmatrix}8.836169249\\60.011439807\\22.496909021\end{pmatrix}$ |
| (64,64,64) | $\begin{pmatrix}8.836154779\\60.011319218\\22.496874821\end{pmatrix}$ | $\begin{pmatrix}8.836170037\\60.011445221\\22.496912105\end{pmatrix}$ |
| (128,128,128) | $\begin{pmatrix}8.836168125\\60.011429526\\22.496907470\end{pmatrix}$ | $\begin{pmatrix}8.836170031\\60.011445285\\22.496912135\end{pmatrix}$ |

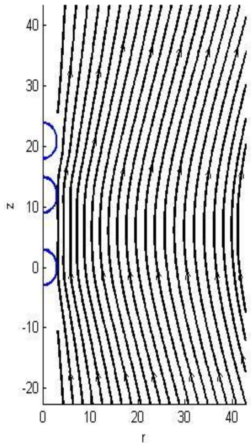


**Fig. 6. Streamline 1,**

$$V=0, v_1=v_2=2, v_3=1, R_1=R_2=R_3=3$$

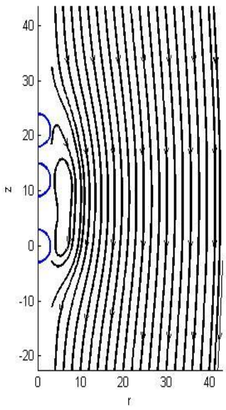


**Fig. 7. Streamline 2,**

$$V=-3, v_1=v_2=2, v_3=1, R_1=R_2=R_3=3$$

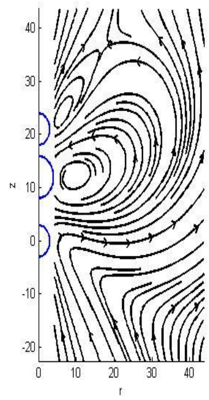


**Fig. 8. Streamline 3,**

$$V=0, v_1=2, v_2=2, v_3=1, R_1=3, R_2=4, R_3=3$$

## V. CONCLUSION

The following conclusions can be drawn concerning the mechanical quadrature method:

(a) Computing entry of discrete matrices is simple and straightforward, without any singular integrals. The mechanical quadrature method involves a high accuracy algorithm with convergent rate $O(h_i^3)(i=1,\ldots,d)$.

(b) The larger the scales of the problem, the more precise are the results that can be obtained according to the numerical results. The extrapolation algorithm is not very complex, but it is very effective.

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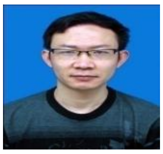

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