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Application of Cubic Spline Collocation Method for the Numerical Solution of Parabolic Partial Differential Equations

O. A. Taiwo, O.S. Odetunde

Abstract— ***This work deals with the computational techniques for solving initial boundary value problems in parabolic partial differential equations. The whole idea of the techniques is based on the replacement of the time derivative by the finite difference approximation and the space derivative by the cubic spline recursive relation. The resulting equations were than perturbed by Chebyshev polynomials which increased the number of unknown constants with extra computational efforts. Numerical examples are given to illustrate the effectiveness of the methods.***

Index Terms – **Parabolic partial differential equations, Chebyshev polynomials, Collocation, Perturbed, Standard, Maximum errors, Schemes.**

I. Introduction

Problems involving time t as one independent variable lead usually to parabolic equation. The simplest parabolic equation $\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$ derives form the theory of heat conduction and its solution gives, for example, the temperature u at a distance x unit of length from one end of a thermally insulted bar after t seconds of heat conduction. In such a problem the temperatures at the ends of a bar length l are often known for all time. To specify a unique solution to this type of problem, additional conditions must be imposed upon its solution function. Typically these conditions occur in the form of boundary values that are prescribed on all or part of the perimeter of the region in which the solution is sought. Therefore, the nature of the boundary and boundary values are usually the determining factors in setting up an appropriate numerical scheme for obtaining the approximate solution.

1. Standard Explicit and Implicit Schemes

We shall consider two dimensional second-order equation

$$a(x,t)u_{xx} + b(x,t)u_{xx} + c(x,t)u_{u} + d(x,t)u_t + e(x,t)u_t + f(x,t)u + g = 0 \quad (2.1)$$

If we assume that substitution has been made such that

$e(x,t) = -1,\ b(x,t) = c(x,t) = g(x,t) = 0$, then, equation (2.1) reduces to the form:

$$u_t = a(x,t)u_{xx} + d(x,t)u_x + f(x,t)u \quad (2.2)$$

Equation (2.2) is subject to the following initial and boundary conditions respectively.

$$m(x,t) = m_{xx} + at = 0 \ \forall\ x_0 \le x \le x_N \quad (2.3)$$

$$u^* = \begin{cases} P_1(t) & at\ x = x_0 \\ P_2(t) & at\ x = x_N \end{cases} \quad (2.4)$$

Where $m(x)$, $p_1(t)$ and $p_2(t)$ are known functions when a function u and its derivatives are single-valued, finite and continuous function of x, then by Taylor's theorem,

$$u(x+h) = u(x) + hu'(x) + {}^1/_2 h^2 u''(x) + {}^1/_6 h^3 u'''(x) + 0(h^4) \quad (2.5)$$

$$u(x-h) = u(x) - hu'(x) + {}^1/_2 h^2 u''(x) - {}^1/_6 h^3 u'''(x) + 0(h^4) \quad (2.6)$$

Addition and subtraction of equations (2.5) and (2.6) give the equation of the respective forms:

$$u''(x) \cong {}^1/_{h^2}[u(x+h) - 2u(x) + u(x-h)] \quad (2.7)$$

$$u'(x) \cong {}^1/_{2h}[u(x+h) - u(x-h)] \quad (2.8)$$

For simplicity, denote

$$\left.\begin{matrix} u(x+h) = u_{i+1,j}, & u(x-h) = u_{i-1,j}, & u(t+k) = u_{i,j+1}, & u(x) = u_{i,j} \\ a(x,t) = a_{i,j}, & d(x,t) = d_{i,j}, & \text{and} & f(x,t) = f_{i,j} \end{matrix}\right\} \quad (2.9)$$

Thus substituting equations (2.7) and (2.8) in form of equation (2.9) into equation (2.2), we have:

$$\left[\frac{1}{k(u]_{i,j-1}} - u_{i,j}\right) = a_{ij}\left[\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}\right] + d_{i,j}\left[\frac{u_{i+1,j} - u_{i-1,j}}{2h}\right] + f_{ij}u_{ij} \quad (2.10)$$

Letting $r_1 = {}^k/_{h^2}$ and $r_2 = {}^k/_{2h}$ (2.11)

Putting equation (2.11) in equation (2.10) and after simplification, we have

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$$u_{i,j+1} = (r_1 a_{i,j} + r_2 d_{i,j})u_{i+1,j} + (1 - 2r_1 a_{i,j} + kf_{i,j})u_{i,j} + (r_1 a_{i,j} - r_2 d_{i,j})u_{i-1,j} \quad (2.12)$$

The formula (2.12) is a general explicit representation to (2.2)

A more general implicit scheme is that due to Crank-Nicolson and is given by

$$■([1/k(u] \downarrow(i, j+1) - u_\downarrow(i,j)) = a_\downarrow(i,j)/2.2h\ [(u_\downarrow(1-i,j) - [2u] \downarrow(i,j) + u_\downarrow(i+1,j)) + [(u] \downarrow(i-i,j+$$

(2.13)

Simplifying equation (2.13), we obtain

$$■(@■((r_\downarrow 2\ d_\downarrow(i,j) - r_\downarrow 1\ a_\downarrow(i,j))\ u_\downarrow(i,j+1) + (2 + 2r_\downarrow 1\ a_\downarrow(i,j) - [kf] \downarrow(i,j))\ u_\downarrow(i,j+1,) + (-r_\downarrow 1\ a_\downarrow(i,j) - r_\downarrow 2\ d_\downarrow(i,j))\ u_\downarrow(i+i,j+1)@ = (r_\downarrow 1\ d_\downarrow(i,$$

(2.14)

2. *Standard Implicit Method By Cubic Spline*

With reference to the set of data points (x_j, u_j), $j = 0, 1, \ldots, n$

Where $x_0 < x_1 < x_2 < \cdots\cdots\cdots < x_n$, we define a cubic spline $S(x)$ such that

$$├■("(i)\ "\ S(x) = U_\downarrow(j,) \quad j = 0, 1, \ldots\ldots\ldots, "\ "n \qquad @("ii")\ S(x)"$$

By assuming equal internal, $S(x)$ is given as:

$$S(x) = \frac{1}{h3!}\left[[(x]_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j\right] + \frac{1}{h}\left[u_{j-1} - \frac{h^3}{3!}M_{j-1}\right]$$

$$[(x]_j - x) + \frac{1}{h}\left[u_j - \frac{h^2 M_j}{3!}\right](x - x_{j-1}) \qquad (3.2)$$

Where $h = x - x_{j-1}$ and $M_j = S''(x_j)$

Differentiating equation (3.2) with respect to x and substituting $x = x_j$, we get

$$[S^1(x]_j -) = {}^h/_3 M_j + {}^h/_6 M_{j+1} + {}^1/_h (u_j - u_{j+1}), \quad j = 1, 2, \ldots\ldots\ldots, n-1 \qquad (3.3)$$

Similarly,

$$[S^1(x]_j +) = {}^h/_3 M_j + {}^h/_6 M_{j+1} + {}^1/_h (u_{j+1} - u_j) \qquad (3.4)$$

In order to ensure continuity of $[S'(x]_j +)$ at $x = x_j$, equations (3.3) and (3.4) must be same. This equality gives the recurrence relation for M_j.

$$M_{j+1} + 4M_j + M_{j+1} = {}^6/_{h^2}(u_{j-1} - 2u_j + u_{j+1}), \quad j = 1, 2, \ldots\ldots\ldots, n-1 \qquad (3.5)$$

Thus equation (3.5) constitutes (n – 1) equation in (n+1) unknowns.

In a similar manner, combination of equations (3.3) and (3.4) in a suitable manner yield a recurrence relation for $M_{i,j}$ which is given as:

$$m_{i,j} + 4m_j + m_{j+1} = {}^3/_h(u_{j+1} - u_{j-1}), \quad j = 1, 2, \ldots\ldots\ldots, n-1 \qquad (3.6)$$

Analogously with equations (3.5) and (3.6), the following relations hold

$$\left.\begin{array}{l} M_{i-i,j} + 4M_{i,j} + M_{i+i,j} = {}^6/_{h^3}(u_{i-i,j} - 2u_{i,j} + u_{i+i,j}) \\ M_{i-i,j} + 4M_{i,j+1} + M_{i+i,j+1} = {}^6/_{h^3}(u_{i-i,j} - 2u_{i,j} + u_{i+i,j+1}) \end{array}\right\} \quad (3.7)$$

And

$$\left.\begin{array}{l} M_{i-i,j} + 4M_{i,j} + M_{i+i,j} = {}^3/_h(u_{i-i,j} - u_{i+i,j}) \\ M_{i-i,j+1} + 4M_{i,j+1} + M_{i+i,j+1} = {}^3/_h(u_{i+i,j+1} - u_{i-i,j+1}) \end{array}\right\} \quad (3.8)$$

Substituting equation (3.7) and (3.8) into equation (2.2), we have

$$4 + 12r_1$$

Thus, equation (3.9) is referred to as the standard implicit formula by cubic spline.

3. *Perturbed Implicit Collocation method*

When equation (2.2) is slightly perturbed, it gives

$$U_t = a(x,t)u_{xx} + d(x,t)u_x + f(x,t)u + H_N(x)$$

(4.1)

Where

$$H_N(x) = \tau_1 T_N(x) + \tau_2 T_{N-1}(x)$$

(4.2)

$T_N(x)$ is the shifted Chebyshev polynomial of degree N defined by

$$T_N(x) = Cos\left[NCos^{-1}\left\{\frac{2x - x_0 - x_N}{x_N - x_0}\right\}\right]$$

(4.3)

Substituting equation (4.2) in equation (4.1), we obtain

$$u_t = a(x,t)u_{xx} + d(x,t)u_x + f(x,t)u + \tau_1 T_N(x) + \tau_2 T_{N-1}(x) \quad (4.4)$$

Using the recurrence relation for $M_{i,j}$ and $m_{i,j}$ in equation (4.4) after simplification we obtain

$$(1 - 6r_1 a_{i,j}\theta - 6r_2 d_{1,j}\theta)a_{i-i,j+1} + (4 + 12r_1 a_{i,j}\theta - kf_{i,j})u_{i,j+1} +$$

$$(1 - 6r_1 a_{i,j}\theta + 6r_2 d_{i,j}\theta)u_{i+i,j+1}$$
$$[1 - 6r_1 a_{i,j}(1-\theta) + 6r_2 d_{1,j}(1-\theta)]u_{i,j-1} + [4 + 12r_1 a_{i,j}\theta - kf_{i,j}]u_{i,j} +$$
$$[1 - 6r_1 a_{i,j}(1-\theta) + 6r_2 d_{1,j}(1-\theta)]u_{i,j+1} + \tau_1 T_N(x) + \tau_2 T_{N-1}(x) \quad (4.5)$$

Collocating equation (4.5) at point $x = x_i$,where $x_i = x_0 + \frac{(x_n - x_0)i}{N}$, $i \geq 0$ and substituting the values of

$$T_N(x_t) = C_0^N + C_1^N x_i + C_2^N x_i^2 + \dots\dots\dots C_N^N x_i^N$$

and

$$T_{N-1}(x_t) = C_0^{N-1} + C_1^{N-1} x_i^1 + C_2^{N-1} x_i^2 + \dots\dots\dots C_{N-1}^{N-1} x_i^N \quad (4.6)$$

into equations (4.5), we get

$$(1 - 6r_1 a_{i,j}\theta - 6r_2 d_{1,j}\theta)a_{i-i,j+1} + (4 + 12r_1 a_{i,j}\theta - kf_{i,j})u_{i,j+1} +$$
$$(1 - 6r_1 a_{i,j}\theta + 6r_2 d_{i,j}\theta)u_{i+1,j+1} - \{C_0^N + C_1^N x_i + C_2^N x_i^2 + \dots\dots\dots C_N^N x_i^N\}\tau_1 - \{C_0^{N-1} + C_1^{N-1} x_i^1 + C_2^{N-1} x_i^2 + \dots\dots$$
$$[1 + 6r_1 a_{i,j}(1-\theta) + 6r_2 d_{i,j}(1-\theta)]u_{i-1,j} +$$
$$[4 + 12r_1 a_{i,j}\theta - kf_{i,j}]u_{i,j} + [1 + 6r_1 a_{i,j}(1-\theta) - 6r_2 d_{i,j}(1-\theta)]u_{i+i,j} \quad (4.7)$$

The following two equations shall be added to equation (4.7) to make it complete.
The equations are as follows:

$$\left.\begin{aligned} &[(1 + 2r]_1)u_{1,1} - r_1 u_{1,2} = r_1 u_{0,0} + [(1 - 2r]_1)u_{1,0} + r_1 u_{2,0} \\ &-r_1 u_{1,1} + (1 + 2r_1)u_{1,2} - r_1 u_{1,2} = r_1 u_{1,0} + [(1 - 2r]_1)u_{2,0} + r_1 u_{3,0} \end{aligned}\right\} \quad (4.8)$$

Altogether, equation (4.7) with equation (4.8) comprise a set of (N + 2) algebraic equation in (N + 2) unknowns.
Thus, the (N + 2) equations can be put in matrix form as

AY = B (4.9)

Where

$$Y = [u_{1,1}, u_{1,2}, u_1, m_1, \tau_1, \tau_2]^T$$

A =

$$[\blacksquare(\blacksquare((4 + 12r_1 1\ a_1(i,j)\ \theta - kf_1(i,j))\&(1 - 6r_1 1\ a_1(i,j)\ \theta + 6r_1 2\ d_1(i,j)\ \theta)\&@(1 - 6r_1 1\ a_1(i,j)\ \theta + 6r_1 2\ d_1(i,j)\ \theta)\&(4 + 12r_1 1$$

$$B = [\blacksquare((1 + 6r_1 1\ a_1(i,j)\ (1-\theta) - 6\ [r_1 2\ d]_1(i,j)\ (1-\theta)]\ u_1(0,j) + [4 + 12r_1 1\ a_1(i,j)(1-\theta) + kf_1(i,j)]\ u_1(1,j) + [1 + 6r_1 1\ a_1(i,j)$$

Thus, the above systems of equations can be solved using Gaussian elimination method.

II. NUMERICAL EXAMPLE

A. Example 1:

$$u_t = u_{xx}, \quad 0 \leq x \leq \pi, \quad t \geq 0$$
$$u(0,1) = u(\pi,t) = 0 \text{ and } u(x,0) = \sin x$$

Taking

$$r_1 = 1/\sqrt{20},\ h = 1/20 \ \text{with}\ \theta = 1/2 + 1/6r$$

Analytical solution is $u(x,t) = e^{-t}\sin x$

In all cases, we have defined our errors as:

$$e_N(x) = \max_{x_0 \leq x \leq x_N} |u(x) - u_N(x)|$$

TABLE I: Errors for example 1 at the point of symmetry

No of time step	Spline Solution	Perturbed Solution		
		Case N = 2	Case N = 4	Case N = 6
1	1.2 x 10^{-3}	8.50 x 10^{-4}	7.76 x 10^{-4}	3.99 x 10^{-4}
2	2.3 x 10^{-5}	1.70 x 10^{-3}	1.638 x 10^{-3}	3.6 x 10^{-4}
3	3.4 x 10^{-5}	2.57 x 10^{-3}	2.552 x 10^{-3}	3.6 x 10^{-4}

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4	4.4×10^{-5}	3.44×10^{-3}	3.497×10^{-3}	5.3×10^{-3}
5	5.5×10^{-5}	4.463×10^{-3}	4.463×10^{-3}	1.107×10^{-3}
6	5.6×10^{-5}	5.23×10^{-3}	5.442×10^{-3}	1.735×10^{-3}
7	7.7×10^{-5}	6.12×10^{-3}	6.431×10^{-3}	2.393×10^{-3}
8	8.6×10^{-5}	7.570×10^{-3}	7.426×10^{-3}	3.069×10^{-3}

B. *Example 2:*

$$u_t = u_{xx}, \qquad 0 \le x \le 1$$

$$u(0,1) = u(1,t) = 0 \ \ and \ \ u = \sin \pi x$$

Taking $r_1 = 1, h = 0.1 \ and \ \theta = {}^1/_2 + {}^1/_{6\pi}$

Analytical solution is $u(x,t) = e^{-\pi^2} \sin \pi x$

TABLE II: Errors for example 2 at the point of symmetry

No of time step	Spline Solution	Perturbed Solution	
		Case N = 8	Case N = 9
1	5.64×10^{-4}	2.4179×10^{-2}	2.4913×10^{-2}
2	1.211×10^{-3}	4.8493×10^{-2}	2.0483×10^{-2}
3	1.665×10^{-3}	6.2676×10^{-2}	1.8993×10^{-2}
4	2.038×10^{-3}	7.4811×10^{-2}	1.7351×10^{-2}
5	2.335×10^{-3}	8.4872×10^{-2}	1.6035×10^{-2}
6	2.562×10^{-3}	9.3792×10^{-2}	1.4893×10^{-2}
7	2.727×10^{-3}	1.01776×10^{-1}	1.3908×10^{-2}
8	2.839×10^{-3}	1.09003×10^{-1}	1.3056×10^{-2}

III. CONCLUSION

The computation techniques in parabolic equations have been described. The methods were shown to be accurate, efficient and general in application for solving heat model problems. In particular, we observed that as the values of N increases, the solutions of the perturbed method approaches analytical solution. The methods were good for the examples considered and the extra work done in solving matrices equations were compensated for in terms of the maximum errors value obtained as these can be seen from the tables of results shown.

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