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Approximate Solution of Generalized Riccati Differential Equations by Iterative Decomposition Algorithm

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Abstract- per, solution of general Riccati equation is studied by using the iterative decomposition method. The equation under consideration includes one with variable coefficient and one in matrix form. Comparison with exact solution and some of the existing methods shows that the iterative decomposition method is a powerful method that gives an accurate result with a fewer terms. Numerical examples are given to illustrate the accuracy and efficiency of this method.

Keywords: **Riccati equation, Variation Iteration Method, Runge-kutta method, Decomposition, Accuracy, Efficiency.**

I. INTRODUCTION

The Riccati differential equation is a famous differential equation. But, the general solution of the equation can not be expressed by either elementary functions or integrations of elementary functions. This paper deals with the following general Riccati differential equation.

$$\frac{du}{dt} = r(t)u^2(t) + q(t)u(t) + p(t) \quad (1.1)$$

$$u(0) = g(t) \quad (1.2)$$

Where $p(t), q(t), r(t)$ and $g(t)$ are scalar functions. The conditions on equation (1.1) are such that existence and uniqueness of solution is guaranteed. [8], [10].About Riccati differential equation, many scholars have studied the solutions of the equation (1.1) named after the Italian noble man called Jacopo Francesco Riccati (1676-1754). El-Tawil et al [3] presented the usage of Adomian decomposition to solve the non linear Riccati differential equation in the analytic form .Tan and Abbasbandy [13] employed the analytic technique called Homotopy analysis method (HAM) to solve a quadratic Riccati equation. The likes of Polyamin and Zaitsevi, Bulut and Evans , Wazwaz, Adomian, He, Adomian and Rach,and most recently Bathia et al had all work previously on equation (1.1) using different methods to obtain the solution of equation((1.1) (see Ref. [1,2,4,5,9,10,11,13,15])just to mention a few. For the fundamental theories of Riccati equation and its numerous applications [5], recently, its usefulness has been extended to financial mathematics.

The basic motivation for this paper is the need for a solution technique which can be applied with relative ease, requiring minimal mathematical details without Jettisoning its and efficiency accuracy. Numerical comparison between Iterative Decomposition Method and some of the existing methods are given. The paper is organized as follows. In section 2, the iterative decomposition method and its adaptation to Riccati Equation are briefly presented. In section 3, numerical results of applying the algorithm are shown. Finally, the conclusions of our results are presented in section 4.

II. SOLUTION TECHNIQUES TO NON-LINEAR PROBLEMS

A.).Consider a non-linear initial value problem $u'(t) + q(t)u(t) + N(u(t)) = g(t);\ a \le t \le b$ (2.1)

$$u(a) = \alpha \quad (2.2)$$

α *is constant,* Where $N(u(t))$ is the non-linear term. Equation (2.1) can be re-written in operator form as

$$Lu = -q(t)u(t) - N(u(t)) + g(t) \quad (2.3)$$

Where the differential operator L is given by

$$L(\cdot) = \frac{d}{dt}(\cdot) \quad (2.4)$$

The inverse operator L^{-1} *is thus a definite integral operator defined by*

$$L^{-1}(\cdot) = \int_a^t (\cdot)dt$$

Operating the inverse operator (2.5) *on equation* (2.1), it follows that

$$u(t) = \alpha + L^{-1}g(t) + L^{-1}(-q(t)u(t) - N((u(t))$$

The iterative decomposition method assumes that the unknown functions $u(t)$ can be expressed in terms of an infinite series of the form

$$u(t) = \sum_{n=0}^{\infty} u_n(t)$$

So that the component $U_n(t)$ can be determine recursively. For the sake of convenience, we complete the idea behind the method [6, 12], it is obvious that (2.6) is of the form

$$u(t) = N(u) + f$$

This equation (2.8) is equivalent to

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$$\sum_{n=0}^{\infty} u_n = \alpha + L^{-1}(u_0) + \{L^{-1}(\sum_{j=0}^{n} u_j) - L^{-1}(\sum_{j=0}^{n-1} u_j)\}$$

which yield the recursive algorithm (2.10)

$$\begin{cases} u_0 = \alpha \\ u_1 = L^{-1}(u_0) \\ u_{n+1} = L^{-1}(u_0 + u_2 + \cdots + u_n) - L^{-1}(u_0 + u_2 + \cdots + u_{n-1}), \quad n \geq 1 \end{cases}$$

from where we obtain an n-th term approximate series solution to equation (2.1) as

$$u(t) = \sum_{i=0}^{n} u_i \qquad (2.11)$$

B.) Analysis of General Riccati Differential Equation

Consider the general Riccati differential equation [1.1] To solve equation by means of iterative decomposition algorithm, we construct the relation

$$\begin{cases} u_0(t) = g(t) + L^{-1}p(t), \\ u_1(t) = L^{-1}[q(t)u_0 + r(t)u_0^2] \\ u_{n+1}(t) = L^{-1}\{[r(t) \cdot L^{-1}(u_0 + u_1 + \cdots + u_n) - L^{-1}(u_0 + u_1 + \cdots + u_{n-1})r(t)\} \quad n \geq 1 \end{cases}$$

(2.12)

Which gives the approximate solution $\sum_{i=0}^{n} u_i(t)$ to the equation (1.1).

III. Numerical Examples

To show the accuracy, efficiency and convergence of the new algorithm. Three illustrative examples are presented. In all cases considered, where the exact solutions are known, we have defined our error as

$$E_j(t) = \max_{a \leq t \leq b} |u_j(t) - u_j^*(t)| \quad ; j = 1,2,\ldots,N$$

$u_j^*(t)$ is the approximate solution

$u_j(t)$ is the exact solution

Example 1.

Consider the following example

$$\frac{du}{dt} = -u^2(t) + 1 \qquad 0 \leq t \leq 1$$
$$u(0) = 0$$

The analytic solution is $u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}$. called from [3]

Solving by means of iterative decomposition method and using the recursive algorithm (2.12), we get the Result of This is shown in Appendix. The rest of the components of the iteration scheme (2.12) can be obtained using the Maple package.

Example 2

Consider the quadratic Riccati differential equation with variable coefficient.

$$\begin{cases} \frac{du}{dt} = t^3u^2 - 2t^4u + t^5 + 1 \\ u(0) = 0 \end{cases} \qquad (2.9)$$

In similar way, solving example 2 by means of iterative decomposition method and using the recursive algorithm (2.12), we get

$$u_0(t) = t + \frac{1}{6}t^6,$$

$$u_1(t) = \frac{-1}{6!}t^6 + \frac{1}{576}t^{16}$$

Using wazwaz [14] and in view of the presence of an effective noise term in u_0 and u_1 .Therefore, the exact solution is $u(t) = t$

Example 3

Consider the following a matrix Riccati differential equation

$$\begin{cases} \frac{du(t)}{dt} = -u^2(t) + Q \\ u(0) = 0 \end{cases}$$

Where $Q = \frac{1}{2}\begin{pmatrix} 101 & -99 \\ -99 & 101 \end{pmatrix}$

To obtain the solution of this equation via the recursive relation (2.12). we shall treat the matrix equation as a system of differentiate equations. So we express it as a system of integral equations.

$$\frac{du_{11}}{dt} = \frac{101}{2} - (u^2{}_{11} + u_{12}u_{21}) \Rightarrow u_{11}(t) = \frac{101}{2}t - L^{-1}(u^2{}_{11} + u_{12}u_{21})$$

$$\frac{du_{12}}{dt} = \frac{-99}{2} - (u_1u_{12} + u_{12}u_{22}) \Rightarrow u_{12}(t) = \frac{99}{2}t - L^{-1}(u_1u_{12} + u_{12}u_{22})$$

$$\frac{du_{21}}{dt} = \frac{-99}{2} - (u_1u_{12} + u_{12}u_{22}) \Rightarrow u_{12}(t) = \frac{99}{2}t - L^{-1}(u_1u_{12} + u_{12}u_{22})$$

$$\frac{du_{22}}{dt} = \frac{101}{2} - (u^2{}_{11} + u_{12}u_{21}) \Rightarrow u_{22}(t) = \frac{101}{2}t - L^{-1}(u^2{}_{11} + u_{12}u_{21})$$

On applying the recursive algorithm (2.12) to the above systems of integral equations, we get

$$u_{11,0}(t) = u_{22,0}(t) = \frac{101t}{2}$$

$$u_{11,1}(t) = u_{22,1}(t) = \frac{-10001t}{6}$$

$$u_{11,2}(t) = u_{22,2}(t) = \frac{1000001}{15}t^5 - \frac{100000001}{126}t^7$$

Also,

$$u_{12,0}(t) = u_{21,0}(t) = \frac{-99t}{2}$$

$$u_{12,1}(t) = u_{21,1}(t) = \frac{3333}{2}t^3$$

$$u_{12,2}(t) = u_{21,2}(t) = \frac{-333333}{5}t^5 + \frac{11111111}{14}t^7$$

In a similar approach, the rest of the components of the iterative scheme (2.12) can be obtained using the maple package. Hence, the series solutions are

$$u_{11}(t) = u_{22}(t) = \frac{101t}{2} - \frac{10001}{6}t^3 + \frac{1000001}{15}t^5 - \frac{100000001}{126}t^7 + \cdots$$

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$$u_{12}(t) = u_{21}(t) = \frac{-99t}{2} + \frac{3333}{2}t^3 - \frac{333333}{5}t^5 - \frac{11111111}{14}t^7 + \cdots$$

A.)TABLE 0F RESULTS

Table1 (Table of error estimate for example 1) is shown in APPENDIX

Table 2: Numerical comparison for example 3

	$u_{11} = u_{22}$			$u_{12} = u_{21}$		
	RK4	VIM	IDM	RK4	VIM	IDM
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	3.85780	3.85780	3.85780	-3.7581	-3.7581	-3.7581
0.2	4.91883	4.91883	4.91883	-4.7215	-4.7215	-4.7215
0.3	5.12093	5.12093	5.12093	-4.8296	-4.8296	-4.8296
0.4	5.18662	5.18662	5.18662	-4.8067	-4.8067	-4.8067
0.5	5.23060	5.23056	5.23056	-4.7685	-4.7685	-4.7685
0.6	5.26846	5.26821	5.26820	-4.7314	-4.7312	-4.7310
0.7	5.30218	5.30126	5.30124	-4.6978	-4,6971	-4,6969
0.8	5.33202	5.32945	5.32944	-4.6680	-4.6660	-4.6660
0.9	5.35815	5.35223	5.35221	-4.6419	-4.6372	-4.6371
1.0	5.38080	5.36903	5.36900	-4.6192	-4.6100	-4.6103

IV. CONCLUSION

An iterative decomposition method for direct solution of Riccati differential equation is developed. The efficiency of the method is encouraging judging from the small error values recorded in the tables. Furthermore, the method is economical to implement since it does not involve cumbersome computer coding. The method also performs favourably well when compared to some existing methods. It is worth noting to point out that this method does not make use of the so called special polynomials yet experiences fast convergence to the solution are all added advantage.

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APPENDIX

$$u_o(t) = t,\ \ u_1(t) = \frac{-1}{3}t^3,\ \ u_2(t) = \frac{2}{15}t^5 - \frac{t^7}{63},$$

$$u_3(t) = \frac{-t^{15}}{59535} + \frac{4t^{13}}{122285} - \frac{134t^{11}}{51975} + \frac{38t^9}{2835} - \frac{4t^7}{105}$$

$$u_4(t) = \frac{-t^{31}}{109876902975} + \frac{8t^{29}}{21210236775} - \frac{100732t^{27}}{14119435204875} + \frac{256948t^{25}}{3016973334375} - \frac{12676238t^{23}}{16962094524375} + \frac{29756t^{21}}{5746615875}$$
$$- \frac{24022t^{19}}{820945125} + \frac{1522814t^{17}}{10854718875} - \frac{366292t^{15}}{638512875} + \frac{11344t^{13}}{6081075} - \frac{148t^{11}}{31185} - + \frac{8t^9}{945}.$$

$$u(t) = t - \frac{t^{31}}{1098769029\ \ 75} + \frac{8t^{29}}{2121023677\ \ 5} - \frac{100732\ t^{27}}{1411943520\ \ 4875} + \frac{256948\ t^{25}}{3016973334\ \ 375} - \frac{12676238\ t^{23}}{1696209452\ \ 4375} + \frac{29756\ t^{21}}{5746615875}$$
$$- \frac{24022\ t^{19}}{820945125} + \frac{1522814\ t^{17}}{1085471887\ \ 5} - \frac{366292\ t^{15}}{638512875} + \frac{11344\ t^{13}}{6081075} - \frac{1142\ t^{11}}{155925} + \frac{62t^9}{2835} - \frac{11t^7}{315} + \frac{2t^5}{15} - \frac{t^3}{3}.$$

Table 1:

t	Exact	Variation Iteration Method[5]		New Modified Adomian Decomposition Method[15]		Iterative decomposition method	
		Approximate	Error	Approximate	error	Approximate	error
0.1	0.09966799456	0.09966799461	5.000E-11	0.09966799461	5.000E-11	0.09966799457	1.000E-11
0.2	0.1973533203	0.1973753160	4.300E-9	0.1973753204	1.000E-10	0.1973753203	0.000E-10
0.3	0.2913126124	0.2913124564	1.560E-7	0.2913126276	1.520E-8	0.2913126149	2.500E-9
0.4	0.3799489622	0.3799469862	1.976E-6	0.3799493114	3.492E-7	0.3799490183	5.610E-8
0.5	0.4621171572	0.4621033328	1.382E-5	0.4621210868	3.930E-6	0.4621177611	6.039E-7
0.6	0.5370495670	0.5369833784	6.619E-5	0.5370776284	2.806E-6	0.5370536631	4.096E-6
0.7	0.6043677771	0.6041244734	2.433E-4	0.6045139949	1.462E-4	0.6043878962	2.012E-5
0.8	0.6640367702	0.6633009217	7.358e-4	0.6646413099	6.045E-4	0.6641146411	7.787E-5
0.9	0.7162978702	0.7143823394	1.916E-3	0.7183918393	2.094E-3	0.7165487969	2.509E-4
1.0	0.7615941560	0.7571662670	4.428E-3	0.7679012345	6.310E-3	0.7622933377	6.992E-4