ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 9, Issue 3, September 2019

A New Method to Solve the Principal Minors Assignment Problem and Related Computations

Gonzalez, H.E.[1], and Carmona, L., J.J.[2]
[1] Information Technology Department, ININ,ITTLA, [2] Information Technology Department, ININ

***Abstract*— The inverse problem of finding a matrix with prescribed principal minors is considered. A condition that implies a constructive algorithm for solving this problem will always succeed is presented. The algorithm is based on reconstructing matrices from their principal sub-matrices and new linear transform in a recursive manner. The algorithm guarantee to work if the matrix to be reconstructed is off-diagonal full. Among other conditions, this requires that no off-diagonal entry be equal to zero.**

***Index Terms*— Principal sub-matrix, Inverse eigenvalue problem, new linear transform.**

I. INTRODUCTION

The algorithm presented here is based on an explained method in [1], [2] that computes all the principal minors of a matrix in a recursive manner and it is order $O(2^n)$.

The assignment problem has attracted some attention in recent years. Griffin and Tsatsomeros [3] proposed an algorithm which is guaranteed to work if the matrix to be reconstructed is off-diagonal full. Among other conditions, this requires that no off-diagonal entry be equal to zero (the algorithm proposed here also makes those assumptions).

As an n×n matrix has 2^n-1 principal minors and n^2 entries, PMAP is an over-determined problem for $n \geq 5$. As a consequence, the existence of a solution to PMAP depends on relations among the (principal) minors of the matrix being satisfied. Generally, such relations (e.g., Newton identities for each principal sub-matrix) are theoretically and computationally hard to verify and fulfill [3, pp. 126]

Furthermore, their algorithm has a running time of $O(n^5)$ and involves multiple matrix inversions, which raises concerns about its numerical stability[4, pp. 1]. Again, the algorithm proposed here uses the adjugate matrix [1, pp. 420] and has greater numerical stability.

The main virtue of the method proposed in this work is that it is no longer necessary to calculate the pivots that are used in the algorithm proposed by [3, pp. 129, Fig. 1], [5] using Schur's contraction, but that the tree-shaped diagram of the Gonzalez's method, the intermediate results obtained at any stage are divided by the pivot of the previous immediate stage, modifying in the same way all intermediate results from initial stage to the final stage and thus the PMAP is resolved following the methodology explained in [3].

Manuscript received: 19 August 2019
Manuscript received in revised form: 15 September 2019
Manuscript accepted: 02 October 2019
Manuscript Available online: 10 October 2019

II. RELATED COMPUTATIONS

As the method to calculate the principal minors already explained in [2] next we will illustrate its methodology with the same numerical example [3] -[pp. 141-145].

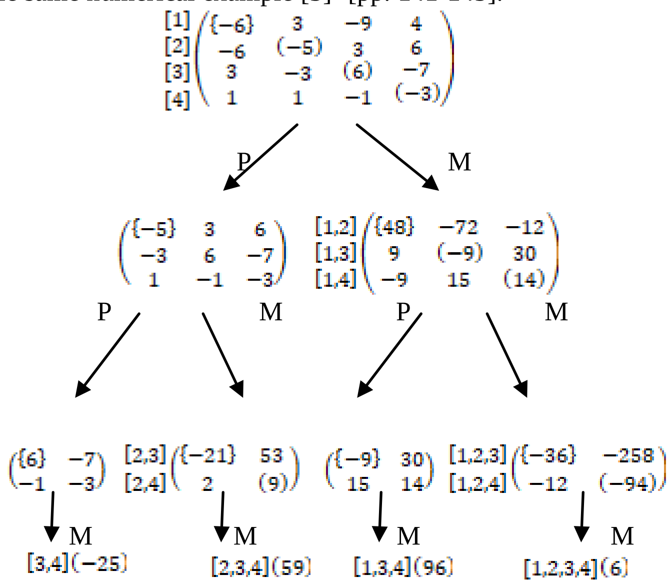

Fig. 1. Gonzalez's Method

The symbols P and M represent partition and modification respectively of the corresponding matrix or sub-matrix and the parentheses within the sub-matrices are the principal minor. Partition refers to eliminating the first row and the first column of the corresponding matrix or sub-matrix until it reaches a 2x2 sub-matrix for which there is only the modification. The modification refers to the application of the new linear transformation [1, pp. 416] to the corresponding matrix or sub-matrix and the results in brackets represent the principal minor of the corresponding matrix or sub-matrix. Each of the elements enclosed in a curly parentheses indicates the previous pivots that were used to calculate the principal minor enclosed in parentheses, namely, first level {-6}, second level, {-5},{48}, and finally the third level, {6},{21},{-9},{-36}.

The size of the principal minors can be seen by the square brackets of the same number of levels, namely, first level [1],[2],[3],[4], second level, [1,2],[1,3],[1,4],[2,3],[2,4],[3,4], third level, [1,2,3],[1,2,4],[1,3,4],[2,3,4] and finally the fourth level [1,2,3,4], which reveals the principal minors of the same size and whose sum (or trace) form the corresponding coefficients of the characteristic polynomial, namely:

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 9, Issue 3, September 2019

$$x^4 - \left[\left[\{-6\} + (-5) + (6)\right] + (-3)\right]x^3 +$$
$$+\left[\{48\} + (-9) + (14) + \{-21\} + (9) + (-25)\right]x^2 -$$
$$-\left[\{-36\} + (-94) + (96) + (59)\right]x + \left[(6)\right] = 0$$
$$x^4 + 8x^3 + 16x^2 - 25x + 6 = 0 \quad (1)$$

Until now the method is being used to calculate the principal minors of the matrix, however, the pivots are already identified. Now dividing the intermediate results between the previous immediate pivot, it has:

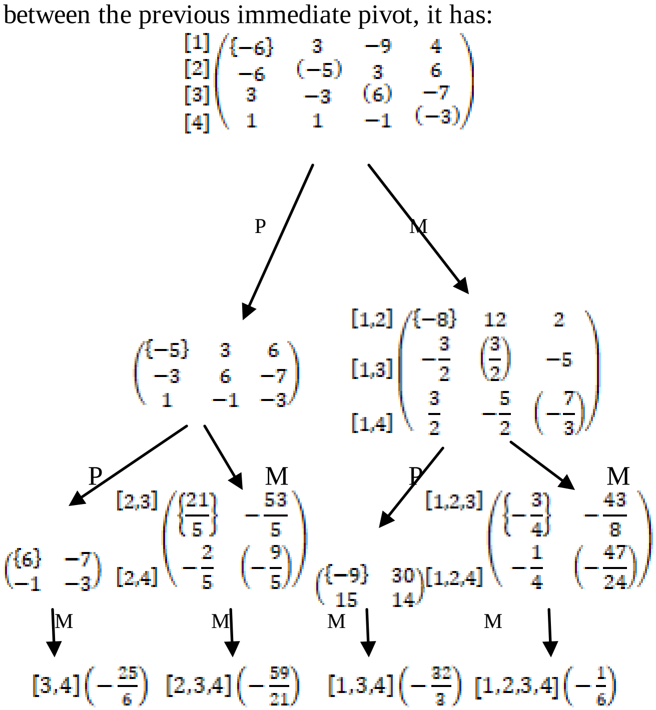

Fig. 2. Modified Gonzalez's Method

The Characteristic Polynomial Method of Gonzalez, allows at the same time, calculate the pivots and also the principal minors of the matrix, these being the following: (Taking the right side of the tree and from top to bottom, until the end, to continue from right to left, following the same directions for each branch and performing the indicated operations). Taking the same notation used by the author in the example illustrated in [3], we have:

$$PM = \left\{ \begin{array}{ccccccccccccccc} [1] & [2] & [3] & [4] & [1,2] & [1,3] & [1,4] & [1,2,3] & [1,2,4] & [1,2,3,4] & [1,3,4] & [2,3] & [2,4] & [2,3,4] & [3,4] \\ -6 & -5 & 6 & -3 & 48 & -9 & 14 & -36 & -94 & 6 & 96 & -21 & 9 & 59 & -25 \end{array} \right\}$$

Level 4 Pivots:

[4]=-3; [1,4] $=-\frac{7}{3}$; [2,4]=$-\frac{9}{5}$;[3,4]=$-\frac{25}{6}$;[1,2,4]= $-\frac{47}{24}$; [1,3,4]= $-\frac{32}{3}$; $[2,3,4] = -\frac{59}{21}$;[1,2,3,4]= $-\frac{1}{6}$;

Level 3 Pivots:

[3]=6; [1,3]=$\frac{3}{2}$; [2,3]= $\frac{21}{5}$; $[1,2,3]=-\frac{3}{4}$;

2x2 matrices formed by pivots levels 3 and 4:

$$B_1 = \begin{pmatrix} 6 & \\ & -3 \end{pmatrix};\ B_2 = \begin{pmatrix} \frac{3}{2} & \\ & -\frac{7}{3} \end{pmatrix};\ B_3 = \begin{pmatrix} \frac{21}{5} & \\ & -\frac{9}{5} \end{pmatrix};$$
$$B_4 = \begin{pmatrix} -\frac{3}{4} & \\ & -\frac{47}{24} \end{pmatrix};$$

To fill in the unknown values, the elements are simply matched: $b_{21} = [3]$ and the element $b_{12} = [4] - [3,4]$ for B_1:

$$B_1 = \begin{pmatrix} 6 & -3 - \left(-\frac{25}{6}\right) \\ 6 & -3 \end{pmatrix} = \begin{pmatrix} 6 & \frac{7}{6} \\ 6 & -3 \end{pmatrix};$$

And for the others: $b_{21} = [1,3]$ and the element $b_{12} = [1,4] - [1,3,4]$ for B_2 ; $b_{21} = [2,3]$ and the element $b_{12} = [2,4] - [2,3,4]$ for B_3 ; and finally $b_{21} = [1,2,3]$ and the element $b_{12} = [1,2,4] - [1,2,3,4]$ for the element B_4 :

$$B_2 = \begin{pmatrix} \frac{3}{2} & -\frac{7}{3} - \left(-\frac{32}{3}\right) \\ \frac{3}{2} & -\frac{7}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{25}{3} \\ \frac{3}{2} & -\frac{7}{3} \end{pmatrix};$$
$$B_3 = \begin{pmatrix} \frac{21}{5} & -\frac{9}{5} - \left(-\frac{59}{21}\right) \\ \frac{21}{5} & -\frac{9}{5} \end{pmatrix} = \begin{pmatrix} \frac{21}{5} & \frac{106}{105} \\ \frac{21}{5} & -\frac{9}{5} \end{pmatrix};$$
$$B_4 = \begin{pmatrix} -\frac{3}{4} & -\frac{47}{24} - \left(-\frac{1}{6}\right) \\ -\frac{3}{4} & -\frac{47}{24} \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} & -\frac{43}{24} \\ -\frac{3}{4} & -\frac{47}{24} \end{pmatrix};$$

Levels 2 Pivots: [2]=-5; [1,2]=-8

3x3 matrices formed by pivots levels 2, 3 and 4:

$$C_1 = \begin{pmatrix} -5 & & \\ & 6 & \frac{7}{6} \\ & 6 & -3 \end{pmatrix};\ C_2 = \begin{pmatrix} -8 & & \\ & \frac{3}{2} & \frac{25}{3} \\ & \frac{3}{2} & -\frac{7}{3} \end{pmatrix};$$

$$B_1 - B_3 = \begin{pmatrix} 6 & \frac{7}{6} \\ 6 & -3 \end{pmatrix} - \begin{pmatrix} \frac{21}{5} & \frac{106}{105} \\ \frac{21}{5} & -\frac{9}{5} \end{pmatrix} = \begin{pmatrix} 6 - \frac{21}{5} & \frac{7}{6} - \frac{106}{105} \\ 6 - \frac{21}{5} & -3 + \frac{9}{5} \end{pmatrix} =$$
$$= \begin{pmatrix} \frac{9}{5} & \frac{11}{70} \\ \frac{9}{5} & -\frac{6}{5} \end{pmatrix} = \begin{pmatrix} B1_{11} - B3_{11} & B1_{12} - B3_{12} \\ B1_{21} - B3_{21} & B1_{22} - B3_{22} \end{pmatrix}$$

(No range 1

From [5, p. 29], [6], you have:

Case m=2

If $B1, B3 \in M_2(\mathbb{C})$, and $B1_{12} \neq 0, B1_{21} \neq 0$ and B1, B3 have equal corresponding principal minor, then for some $t \in \mathbb{C}$:

$(B1 - B3)$ has the form: $(B1 - B3) = D(B1 - B3)D^{-1} =$

$$= \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} B1_{11} - B3_{11} & B1_{12} - B3_{12} \\ B1_{21} - B3_{21} & B1_{22} - B3_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{t} \end{bmatrix} =$$
$$= \begin{pmatrix} B1_{11} - B3_{11} & \frac{B1_{12} - B3_{12}}{t} \\ (B1_{21} - B3_{21})t & B1_{22} - B3_{22} \end{pmatrix}$$

For our case, this equivalent to:

$$\frac{B1_{11} - B3_{11}}{(B1_{21} - B3_{21})t} = \frac{\frac{B1_{12} - B3_{12}}{t}}{B1_{22} - B3_{22}}$$

Using the quadratic formula, , $At^2 - Bt + C = 0$, clearing and rearranging you have:

$$A = B1_{12}B3_{21};$$
$$B = [(B1_{11} - B3_{11})(B1_{22} - B3_{22}) - (B1_{12} - B1_{21}) - (B3_{12} - B3_{21})];$$
$$C = B1_{21}B3_{12}$$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 9, Issue 3, September 2019

$$D = (B1_{11} - B3_{11})^2(B1_{22} - B3_{22})^2 + (B1_{12} - B1_{21})^2(B3_{12} - B3_{21})^2 - \\ -2(B1_{11} - B3_{11})(B1_{22} - B3_{22})(B1_{12} - B1_{21}) - \\ -2(B1_{11} - B3_{11})(B1_{22} - B3_{22})(B3_{12} - B3_{21}) - \\ -2(B1_{12} - B1_{21})(B3_{12} - B3_{21})$$

Substituting values:

$$D^2 = \left(\frac{81}{25}\right)\left(\frac{36}{25}\right) + \left(\frac{49}{36}\right)(36) + \left[\frac{(11236)}{11025}\right]\left(\frac{441}{25}\right) - (2)\left(\frac{9}{5}\right)\left(-\frac{6}{5}\right)\left(\frac{7}{6}\right)(6) - \\ -(2)\left(\frac{9}{5}\right)\left(-\frac{6}{5}\right)\left(\frac{106}{105}\right)\left(\frac{21}{5}\right) - (2)\left(\frac{7}{6}\right)(6)\left(\frac{106}{105}\right)\left(\frac{21}{5}\right) = \\ = \frac{2916}{625} + 49 + \frac{11236}{625} + \frac{756}{25} + \frac{11448}{625} - \frac{1484}{25} = \frac{1521}{25}$$

$$D = \sqrt{\frac{1521}{25}} = \frac{39}{5}; \quad t_{1,2} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{B \pm \sqrt{D}}{2A} =$$

$$= \frac{-\left(\frac{9}{5}\right)\left(-\frac{6}{5}\right) + \left(\frac{7}{6}\right)(6) + \left(\frac{106}{105}\right)\left(\frac{21}{5}\right) \pm \frac{39}{5}}{(2)\left(\frac{7}{6}\right)\left(\frac{21}{5}\right)} = \frac{\frac{67}{5} \pm \frac{39}{5}}{\frac{49}{5}}; \quad t_2 = \frac{\frac{67}{5} + \frac{39}{5}}{\frac{49}{5}} = \frac{106}{49};$$

$$t_1 = \frac{\frac{67}{5} - \frac{39}{5}}{\frac{49}{5}} = \frac{28}{49} = \frac{4}{7};$$

Using $t_1 = \frac{4}{7}$;

You have: $B_1 - B_3 = \begin{pmatrix} 6 & \frac{7}{6} \\ 6 & -3 \end{pmatrix} - \begin{pmatrix} \frac{21}{5} & \frac{106}{5} \cdot \frac{7}{4} \\ \frac{21}{5} \cdot \frac{4}{7} & -\frac{9}{5} \end{pmatrix} =$

$$= \begin{pmatrix} 6 & \frac{7}{6} \\ 6 & -3 \end{pmatrix} - \begin{pmatrix} \frac{21}{5} & \frac{53}{30} \\ \frac{12}{5} & -\frac{9}{5} \end{pmatrix} = \begin{pmatrix} \frac{9}{5} & -\frac{3}{5} \\ \frac{18}{5} & -\frac{6}{5} \end{pmatrix}$$

From this last matrix we take the maximum of the principal minors (Diagonal) and obtain: $Máx\left\{\frac{9}{5}, -\frac{6}{5}\right\} = \frac{9}{5}$, Then we divide it between the pivot of level 2 and you have: $\frac{\frac{9}{5}}{-5} = -\frac{9}{25}$;

Following the procedure indicated in [5] to form the matrix

$$C_1 = \begin{pmatrix} C_1[1] & | & C_1[1,1) \\ - & | & - \\ C_1(1,1] & | & C_1(1) \end{pmatrix}$$; We complete the unknown

Column of $C_1(1,1] = \left\{ \begin{array}{l} \frac{\frac{9}{5}}{-\frac{9}{25}} = -5 \\ \frac{\frac{18}{5}}{-\frac{9}{25}} = -10 \end{array} \right\}$; It would also have: $C_1[1,1) = \left(\frac{9}{5} \quad -\frac{3}{5}\right)$ so the matrix would be completed $C_1 = \begin{pmatrix} -5 & \frac{9}{5} & -\frac{3}{5} \\ -5 & 6 & \frac{7}{6} \\ -10 & 6 & -3 \end{pmatrix}$; Now for $C_2 = \begin{pmatrix} C_2[1] & | & C_2[1,1) \\ - & | & - \\ C_2(1,1] & | & C_2(1) \end{pmatrix}$ you have:

$$B_2 - B_4 = \begin{pmatrix} \frac{3}{2} & \frac{25}{3} \\ \frac{3}{2} & -\frac{7}{3} \end{pmatrix} - \begin{pmatrix} -\frac{3}{4} & -\frac{43}{24} \\ -\frac{3}{4} & -\frac{47}{24} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} + \frac{3}{4} = & \frac{25}{3} + \frac{43}{24} = \\ \frac{3}{2} + \frac{3}{4} = & -\frac{7}{3} + \frac{47}{24} = \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{9}{4} & \frac{81}{8} \\ \frac{9}{4} & -\frac{3}{8} \end{pmatrix} = \begin{pmatrix} B2_{11} - B4_{11} & B2_{12} - B4_{12} \\ B2_{21} - B4_{21} & B2_{22} - B4_{22} \end{pmatrix}$$

(No range 1)

From [5, p. 29], [6], you have:

Case m=2

If $B2, B4 \in M_2(\mathbb{C})$, and $B2_{12} \neq 0, B2_{21} \neq 0$ and $B2, B4$ have equal corresponding principal minor, then for some $t \in \mathbb{C}$:

$(B2 - B4)$ has the form: $(B2 - B4) = D(B2 - B4)D^{-1} =$

$$= \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} B2_{11} - B4_{11} & B2_{12} - B4_{12} \\ B2_{21} - B4_{21} & B2_{22} - B4_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{t} \end{bmatrix} =$$

$$\begin{pmatrix} B2_{11} - B4_{11} & \frac{B2_{12} - B4_{12}}{t} \\ (B2_{21} - B4_{21})t & B2_{22} - B4_{22} \end{pmatrix} =$$

For our case, this equivalent to: $\frac{B2_{11} - B4_{11}}{(B2_{21} - B4_{21})t} = \frac{\frac{B2_{12} - B4_{12}}{t}}{B2_{22} - B4_{22}}$

Using the quadratic formula, , $At^2 - Bt + C = 0$, clearing and rearranging you have:

$$A = B2_{12}B4_{21};$$
$$B = [(B2_{11} - B4_{11})(B2_{22} - B4_{22}) - (B2_{12} - B4_{21}) - (B2_{12} - B4_{21})];$$
$$C = B2_{21}B4_{12}$$
$$D = (B2_{11} - B4_{11})^2(B2_{22} - B4_{22})^2 + (B2_{12} - B4_{21})^2(B2_{12} - B4_{21})^2 - \\ -2(B2_{11} - B4_{11})(B2_{22} - B4_{22})(B2_{12} - B4_{21}) - \\ -2(B2_{11} - B4_{11})(B2_{22} - B4_{22})(B2_{12} - B4_{21}) - \\ -2(B2_{12} - B4_{21})(B2_{12} - B4_{21})$$

Substituting values:

$$D^2 = \left(\frac{81}{16}\right)\left(\frac{9}{64}\right) + \left(\frac{625}{9}\right)\left(\frac{9}{4}\right) + \left(\frac{1849}{576}\right)\left(\frac{9}{16}\right) - \\ -(2)\left(\frac{9}{4}\right)\left(-\frac{3}{8}\right)\left(\frac{25}{3}\right)\left(\frac{3}{2}\right) - (2)\left(\frac{9}{4}\right)\left(-\frac{3}{8}\right)\left(-\frac{43}{24}\right)\left(-\frac{3}{4}\right) - \\ -(2)\left(\frac{25}{3}\right)\left(\frac{3}{2}\right)\left(-\frac{43}{24}\right)\left(-\frac{3}{4}\right) = \\ = \frac{729}{1024} + \frac{625}{4} + \frac{1849}{1024} + \frac{675}{32} + \frac{1161}{512} - \frac{1075}{32} = \frac{38025}{256}$$

$$D = \sqrt{\frac{38025}{256}} = \frac{195}{16}; \quad t_{1,2} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{B \pm \sqrt{D}}{2A} =$$

$$= \frac{-\left(\frac{9}{4}\right)\left(-\frac{3}{8}\right) + \left(\frac{25}{3}\right)\left(\frac{3}{2}\right) + \left(-\frac{43}{24}\right)\left(-\frac{3}{4}\right) \pm \frac{195}{16}}{(2)\left(\frac{25}{3}\right)\left(-\frac{3}{4}\right)} = \frac{\frac{27}{2} + \frac{25}{2} + \frac{43}{32} \pm \frac{195}{16}}{-\frac{25}{2}} =$$

$$= \frac{\frac{235}{16} \pm \frac{195}{16}}{-\frac{25}{2}}; \quad t_2 = \frac{\frac{235}{16} + \frac{195}{16}}{-\frac{25}{2}} = \frac{\frac{215}{8}}{-\frac{25}{2}} = -\frac{43}{20};$$

$$t_1 = \frac{\frac{235}{16} - \frac{195}{16}}{-\frac{25}{2}} = \frac{\frac{5}{2}}{-\frac{25}{2}} = -\frac{1}{5}$$

Using $t_1 = \frac{1}{5}$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 9, Issue 3, September 2019

You have:

$$B_2 - B_4 = \begin{pmatrix} \frac{3}{2} & \frac{25}{3} \\ \frac{3}{2} & -\frac{7}{3} \end{pmatrix} - \begin{pmatrix} -\frac{3}{4} & -\frac{43}{24}\cdot(-5) \\ -\frac{3}{4}\cdot\left(-\frac{1}{5}\right) & -\frac{47}{24} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{3}{2} & \frac{25}{3} \\ \frac{3}{2} & -\frac{7}{3} \end{pmatrix} - \begin{pmatrix} -\frac{3}{4} & \frac{215}{24} \\ \frac{3}{20} & -\frac{47}{24} \end{pmatrix} = \begin{pmatrix} \frac{9}{4} & -\frac{5}{8} \\ \frac{27}{20} & -\frac{3}{8} \end{pmatrix}$$

From this last matrix we take the maximum of the principal minors (Diagonal) and obtain: $Máx\ \left\{\frac{9}{4}, -\frac{3}{8}\right\} = \frac{9}{4}$, Then we divide it between the pivot of level 2 and you have: $\frac{\frac{9}{4}}{-8} = -\frac{9}{32}$;

Following the procedure indicated in [5] to form the matrix

$$C_2 = \begin{pmatrix} C_2[1] & | & C_2[1,1) \\ - & | & - \\ C_2(1,1] & | & C_2(1) \end{pmatrix}$$; we complete the unknown column of $C_2(1,1] = \left\{ \begin{array}{l} \frac{\frac{9}{4}}{-\frac{9}{32}} = -8 \\ \frac{\frac{27}{20}}{-\frac{9}{32}} = -\frac{24}{5} \end{array} \right\}$; It would also have: $C_2[1,1) = \begin{pmatrix} \frac{9}{4} & -\frac{5}{8} \end{pmatrix}$ so the matrix would be completed

$$C_2 = \begin{pmatrix} -8 & \frac{9}{4} & -\frac{5}{8} \\ -8 & \frac{3}{2} & \frac{25}{3} \\ -\frac{24}{5} & \frac{3}{2} & -\frac{7}{3} \end{pmatrix}$$

Level 1 Pivots: : {[1]} = -6;

4x4 matrix formed by pivots levels 1, 2, 3 and 4:

$$B = \begin{pmatrix} -6 & & & \\ & -5 & \frac{9}{5} & -\frac{3}{5} \\ & -5 & 6 & \frac{7}{6} \\ & -10 & 6 & -3 \end{pmatrix};$$

$$C_1 - C_2 = \begin{pmatrix} -5 & \frac{9}{5} & -\frac{3}{5} \\ -5 & 6 & \frac{7}{6} \\ -10 & 6 & -3 \end{pmatrix} - \begin{pmatrix} -8 & \frac{9}{4} & -\frac{5}{8} \\ -8 & \frac{3}{2} & \frac{25}{3} \\ -\frac{24}{5} & \frac{3}{2} & -\frac{7}{3} \end{pmatrix} =$$

$$= \begin{pmatrix} -5+8=3 & \frac{9}{5}-\frac{9}{4}=-\frac{9}{20} & -\frac{3}{5}+\frac{5}{8}=\frac{1}{40} \\ -5+8=3 & 6-\frac{3}{2}=\frac{9}{2} & \frac{7}{6}-\frac{25}{3}=-\frac{43}{6} \\ -10+\frac{24}{5}=-\frac{26}{5} & 6-\frac{3}{2}=\frac{9}{2} & -3+\frac{7}{3}=-\frac{2}{3} \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & -\frac{9}{20} & \frac{1}{40} \\ 3 & \frac{9}{2} & -\frac{43}{6} \\ -\frac{26}{5} & \frac{9}{2} & -\frac{2}{3} \end{pmatrix}$$

(No range 1)

From [5, p. 29], [6], you have:

Case m=3

You have to find **r**, **s** and **t** such that:

$$C_1 - C_2' = \begin{pmatrix} -5 & \frac{9}{5} & -\frac{3}{5} \\ -5 & 6 & \frac{7}{6} \\ -10 & 6 & -3 \end{pmatrix} - \begin{pmatrix} -8 & \frac{\frac{9}{4}}{s} & \frac{-\frac{5}{8}}{t} \\ -8\cdot s & \frac{3}{2} & \frac{\frac{25}{3}}{r} \\ -\frac{24}{5}\cdot t & \frac{3}{2}\cdot r & -\frac{7}{3} \end{pmatrix} = 1$$

So you have to solve the quadratic equations for each 2x2 sub-matrix of $C_1 - C_2'$ for **r**, **s** and **t** independently, that is, the following sub-matrices:

$$C_1^1 - C_2^1 = \left[\begin{pmatrix} -5 & \frac{9}{5} \\ -5 & 6 \end{pmatrix} - \begin{pmatrix} -8 & \frac{9}{4\cdot s} \\ -8\cdot s & \frac{3}{2} \end{pmatrix} \right] = 1\ ;$$

Proceeding in a similar way to the previous ones you have:

$$D^2 = (9)\left(\frac{81}{4}\right) + \left(\frac{81}{25}\right)(25) + \left(\frac{81}{16}\right)(64) - (2)(3)\left(\frac{9}{2}\right)\left(\frac{9}{5}\right)(-5) -$$
$$-(2)(3)\left(\frac{9}{2}\right)\left(\frac{9}{4}\right)(-8) - (2)\left(\frac{9}{5}\right)(-5)\left(\frac{9}{4}\right)(-8) =$$
$$= \frac{729}{4} + 81 + 324 + 243 + 486 - 324 = \frac{3969}{4}$$

$$D = \sqrt{\frac{3969}{4}} = \frac{63}{2}\ ;$$

$$s_{1,2} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{B \pm \sqrt{D}}{2A} = \frac{-(3)\left(\frac{9}{2}\right) + \left(\frac{9}{5}\right)(-5) + \left(\frac{9}{4}\right)(-8) \pm \frac{63}{2}}{(2)\left(\frac{9}{5}\right)(-8)} = \frac{-\frac{81}{2} \pm \frac{63}{2}}{-\frac{144}{5}}$$

$$s_2 = \frac{-\frac{81}{2} + \frac{63}{2}}{-\frac{144}{5}} = \frac{-9}{-\frac{144}{5}} = \frac{5}{16};\quad s_1 = \frac{-\frac{81}{2} - \frac{63}{2}}{-\frac{144}{5}} = \frac{-72}{-\frac{144}{5}} = \frac{5}{2};$$

$$C_1^2 - C_2^2 = \left[\begin{pmatrix} -5 & -\frac{3}{5} \\ -10 & -3 \end{pmatrix} - \begin{pmatrix} -8 & -\frac{5}{8\cdot t} \\ -\frac{24}{5}\cdot t & -\frac{7}{3} \end{pmatrix} \right] = 1\ ;$$

Proceeding in a similar way to the previous ones you have:

$$D^2 = (9)\left(\frac{4}{9}\right) + \left(\frac{9}{25}\right)(100) + \left(\frac{25}{64}\right)\left(\frac{576}{25}\right) - (2)(3)\left(-\frac{2}{3}\right)\left(-\frac{3}{5}\right)(-10) -$$
$$-(2)(3)\left(-\frac{2}{3}\right)\left(-\frac{5}{8}\right)\left(-\frac{24}{5}\right) - (2)\left(-\frac{3}{5}\right)(-10)\left(-\frac{5}{8}\right)\left(-\frac{24}{5}\right) =$$
$$= 4 + 36 + 9 + 24 + 12 - 36 = 49$$

$$D = \sqrt{49} = 7\ ;\quad t_{1,2} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{B \pm \sqrt{D}}{2A} =$$

$$= \frac{-(3)\left(-\frac{2}{3}\right) + \left(-\frac{3}{5}\right)(-10) + \left(-\frac{5}{8}\right)\left(-\frac{24}{5}\right) \pm 7}{(2)\left(-\frac{3}{5}\right)\left(-\frac{24}{5}\right)} = \frac{2+6+3\pm 7}{\frac{144}{5}} =$$

$$= \frac{11 \pm 7}{\frac{144}{5}};\ t_2 = \frac{11+7}{\frac{144}{5}} = \frac{(25)(18)}{144} = \frac{25}{8};$$

$$t_1 = \frac{11-7}{\frac{144}{5}} = \frac{(25)(4)}{144} = \frac{25}{36};$$

Finally:

$$C_1^3 - C_2^3 = \left[\begin{pmatrix} 6 & \frac{7}{6} \\ 6 & -3 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} & \frac{25}{3r} \\ \frac{3}{2}\cdot r & -\frac{7}{3} \end{pmatrix}\right] = 1\ ;$$

Similarly:

$$D^2 = \left(\frac{81}{4}\right)\left(\frac{4}{9}\right) + \left(\frac{49}{36}\right)(36) + \left(\frac{625}{9}\right)\left(\frac{9}{4}\right) - (2)\left(\frac{9}{2}\right)\left(-\frac{2}{3}\right)\left(\frac{7}{6}\right)(6) -$$

$$-(2)\left(\frac{9}{2}\right)\left(-\frac{2}{3}\right)\left(\frac{25}{3}\right)\left(\frac{3}{2}\right) - (2)\left(\frac{7}{6}\right)(6)\left(\frac{25}{3}\right)\left(\frac{3}{2}\right) =$$

$$= 9 + 49 + \frac{625}{4} + 42 + 75 - 175 = \frac{625}{4}$$

$$D = \sqrt{\frac{625}{4}} = \frac{25}{2};\quad r_{1,2} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{B \pm \sqrt{D}}{2A} =$$

$$= \frac{-\left(\frac{9}{2}\right)\left(-\frac{2}{3}\right) + \left(\frac{7}{6}\right)(6) + \left(\frac{25}{3}\right)\left(\frac{3}{2}\right) \pm \frac{25}{2}}{(2)\left(\frac{7}{6}\right)\left(\frac{3}{2}\right)} = \frac{3 + 7 + \frac{25}{2} \pm \frac{25}{2}}{\frac{7}{2}} = \frac{\frac{45}{2} \pm \frac{25}{2}}{\frac{7}{2}};$$

$$r_2 = \frac{\frac{45}{2} + \frac{25}{2}}{\frac{7}{2}} = \frac{70}{7} = 10;\quad r_1 = \frac{\frac{45}{2} - \frac{25}{2}}{\frac{7}{2}} = \frac{20}{7};$$

Now we will form a tree with all the possibilities of choosing the various values obtained:

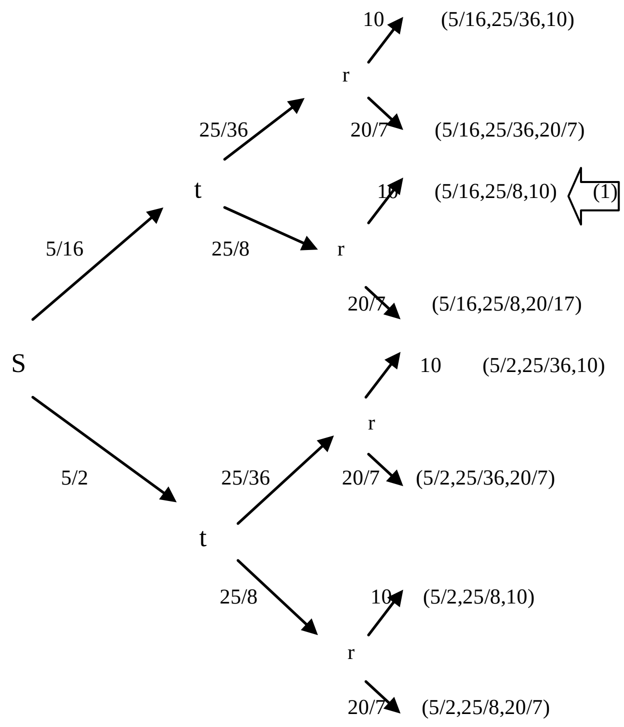

Fig. 3. Tree of possibilities

(1) Because it complies:

$$\left(\frac{7}{6} - \frac{25}{3r}\right)(-5 - (-8)) - (-5 - (-8)s)\left(-\frac{3}{5} - \frac{-\frac{5}{8}}{t}\right) = 0$$

Then you have:

$$C_2' = \begin{pmatrix} -8 & \frac{9}{4\cdot\frac{5}{16}} & -\frac{5}{8\cdot\left(\frac{25}{8}\right)} \\ -8\cdot\frac{5}{16} & \frac{3}{2} & \frac{\frac{25}{3}}{10} \\ -\frac{24\cdot\left(\frac{25}{8}\right)}{5} & \frac{3}{2}\cdot 10 & -\frac{7}{3} \end{pmatrix} = \begin{pmatrix} -8 & \frac{36}{5} & -\frac{1}{5} \\ -\frac{5}{2} & \frac{3}{2} & \frac{5}{6} \\ -15 & 15 & -\frac{7}{3} \end{pmatrix};$$

On the other hand, if:

$$C_2' = DC_2D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}\begin{pmatrix} -8 & \frac{9}{4} & -\frac{5}{8} \\ -8 & \frac{3}{2} & \frac{25}{3} \\ -\frac{24}{5} & \frac{3}{2} & -\frac{7}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{d_{22}} & 0 \\ 0 & 0 & \frac{1}{d_{33}} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}\begin{pmatrix} -8 & \frac{9}{4\cdot d_{22}} & -\frac{5}{8\cdot d_{33}} \\ -8 & \frac{3}{2\cdot d_{22}} & \frac{25}{3\cdot d_{33}} \\ -\frac{24}{5} & \frac{3}{2\cdot d_{22}} & -\frac{7}{3\cdot d_{33}} \end{pmatrix} =$$

$$= \begin{pmatrix} -8 & \frac{9}{4\cdot d_{22}} & -\frac{5}{8\cdot d_{33}} \\ -8d_{22} & \frac{3}{2} & \frac{25\cdot d_{22}}{3\cdot d_{33}} \\ -\frac{24}{5}d_{33} & \frac{3\cdot d_{33}}{2\cdot d_{22}} & -\frac{7}{3} \end{pmatrix} \equiv \begin{pmatrix} -8 & \frac{36}{5} & -\frac{1}{5} \\ -\frac{5}{2} & \frac{3}{2} & \frac{5}{6} \\ -15 & 15 & -\frac{7}{3} \end{pmatrix}$$

Since the last two matrices must be identical, they must be term by term, so it must be fulfilled that:

$\frac{9}{4\cdot d_{22}} = \frac{36}{5}$; from which it follows that:

$d_{22} = \frac{5(9)}{4(36)} = \frac{45}{144} = \frac{5}{16}$; so that: $d_{33} = \frac{25}{3}\left(\frac{6}{5}\right)\left(\frac{5}{16}\right) = \frac{25}{8}$;

Now if:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{16} & 0 \\ 0 & 0 & \frac{25}{8} \end{pmatrix}$$

Then:

$$C_2' = DC_2D^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{16} & 0 \\ 0 & 0 & \frac{25}{8} \end{pmatrix}\begin{pmatrix} -8 & \frac{9}{4} & -\frac{5}{8} \\ -8 & \frac{3}{2} & \frac{25}{3} \\ -\frac{24}{5} & \frac{3}{2} & -\frac{7}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{16}{5} & 0 \\ 0 & 0 & \frac{8}{25} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{16} & 0 \\ 0 & 0 & \frac{25}{8} \end{pmatrix}\begin{pmatrix} -8 & \frac{36}{5} & -\frac{1}{5} \\ -8 & \frac{24}{5} & \frac{8}{3} \\ -\frac{24}{5} & \frac{24}{5} & -\frac{56}{75} \end{pmatrix} = \begin{pmatrix} -8 & \frac{36}{5} & -\frac{1}{5} \\ -\frac{5}{2} & \frac{3}{2} & \frac{5}{6} \\ -15 & 15 & -\frac{7}{3} \end{pmatrix}$$

and

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 9, Issue 3, September 2019

$$C_1 - C_2' = \begin{pmatrix} -5 & \frac{9}{5} & -\frac{3}{5} \\ -5 & 6 & \frac{7}{6} \\ -10 & 6 & -3 \end{pmatrix} - \begin{pmatrix} -8 & \frac{36}{5} & -\frac{1}{5} \\ -\frac{5}{2} & \frac{3}{2} & \frac{5}{6} \\ -15 & 15 & -\frac{7}{3} \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & -\frac{27}{5} & -\frac{2}{5} \\ -\frac{5}{2} & \frac{9}{2} & \frac{1}{3} \\ 5 & -9 & -\frac{2}{3} \end{pmatrix};$$

The maximum pivot of row two is taken:

$$Máx\left\{3, \frac{9}{2}, -\frac{2}{3}\right\} = \frac{9}{2}$$

Then we divide the first element of row two by the pivot of level 1and you have:

$$\frac{-\frac{5}{2}}{-6} = \frac{5}{12};$$

Following the procedure indicated in [5] to form matrix B

$$B = \begin{pmatrix} B[1] & | & B[1,1) \\ - & | & - \\ B(1,1] & | & B(1) \end{pmatrix}$$

We complete the unknown column of

$$B(1,1] = \begin{Bmatrix} \frac{\frac{3}{5}}{\frac{5}{12}} = \frac{36}{5} \\ \frac{-\frac{5}{2}}{\frac{5}{12}} = -6 \\ \frac{5}{\frac{5}{12}} = 12 \end{Bmatrix};$$

It would also have: $B[1,1) = \left(-\frac{5}{2} \quad \frac{9}{2} \quad \frac{1}{3}\right)$, so the matrix would be completed:

$$B = \begin{pmatrix} -6 & -\frac{5}{2} & \frac{9}{2} & \frac{1}{3} \\ \frac{36}{5} & -5 & \frac{9}{5} & -\frac{3}{5} \\ -6 & -5 & 6 & \frac{7}{6} \\ 12 & -10 & 6 & -3 \end{pmatrix};$$

On the other hand, it must be fulfilled that [7] :

$$A = DBD^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix} \begin{pmatrix} -6 & -\frac{5}{2} & \frac{9}{2} & \frac{1}{3} \\ \frac{36}{5} & -5 & \frac{9}{5} & -\frac{3}{5} \\ -6 & -5 & 6 & \frac{7}{6} \\ 12 & -10 & 6 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{d_{22}} & 0 & 0 \\ 0 & 0 & \frac{1}{d_{33}} & 0 \\ 0 & 0 & 0 & \frac{1}{d_{44}} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d_{22} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ 0 & 0 & 0 & d_{44} \end{pmatrix} \begin{pmatrix} -6 & -\frac{5}{2 \cdot d_{22}} & \frac{9}{2 \cdot d_{33}} & \frac{1}{3 \cdot d_{44}} \\ \frac{36}{5} & -\frac{5}{d_{22}} & \frac{9}{5 \cdot d_{33}} & -\frac{3}{5 \cdot d_{44}} \\ -6 & -\frac{5}{d_{22}} & \frac{6}{d_{33}} & \frac{7}{6 \cdot d_{44}} \\ 12 & -\frac{10}{d_{22}} & \frac{6}{d_{33}} & -\frac{3}{d_{44}} \end{pmatrix} =$$

$$= \begin{pmatrix} -6 & -\frac{5}{2 \cdot d_{22}} & \frac{9}{2 \cdot d_{33}} & \frac{1}{3 \cdot d_{44}} \\ 36 d_{22} & -5 & \frac{9 \cdot d_{22}}{5 \cdot d_{33}} & -\frac{3 \cdot d_{22}}{5 \cdot d_{44}} \\ -6 d_{33} & -\frac{5 \cdot d_{33}}{d_{22}} & 6 & \frac{7 \cdot d_{33}}{6 \cdot d_{44}} \\ 12 d_{44} & -\frac{10 \cdot d_{44}}{d_{22}} & \frac{6 \cdot d_{44}}{d_{33}} & -3 \end{pmatrix} \equiv$$

$$\equiv \begin{pmatrix} (-6) & 3 & -9 & 4 \\ -6 & (-5) & 3 & 6 \\ 3 & -3 & (6) & -7 \\ 1 & 1 & -1 & (-3) \end{pmatrix}$$

Since the last two matrices must be identical, they must be term by term, so it must be fulfilled that:

$-\frac{5}{2 \cdot d_{22}}$; from which it follows that: $d_{22} = -\frac{5}{6}$;

$$\frac{9 \cdot d_{22}}{5 \cdot d_{33}} = 3$$

So that:

$$d_{33} = \frac{9}{5}\left(-\frac{5}{6}\right)\left(\frac{1}{3}\right) = -\frac{45}{90} = -\frac{1}{2}$$

Finally: $\frac{7 \cdot d_{33}}{6 \cdot d_{44}} = -7$; and then you have:

$$d_{44} = \frac{7 d_{33}}{6(-7)} = -\frac{\frac{7}{2}}{-42} = \frac{1}{12}$$

Now, if: $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{5}{6} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{12} \end{pmatrix}$; Then, it must be fulfilled that:

$$A = DBD^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{5}{6} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{12} \end{pmatrix} \begin{pmatrix} -6 & -\frac{5}{2} & \frac{9}{2} & \frac{1}{3} \\ \frac{36}{5} & -5 & \frac{9}{5} & -\frac{3}{5} \\ -6 & -5 & 6 & \frac{7}{6} \\ 12 & -10 & 6 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{6}{5} & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 12 \end{pmatrix} = \begin{pmatrix} (-6) & 3 & -9 & 4 \\ -6 & (-5) & 3 & 6 \\ 3 & -3 & (6) & -7 \\ 1 & 1 & -1 & (-3) \end{pmatrix}$$

III. NUMERICAL COMPUTATIONAL VERIFICATION

The first author has a program encoded in language C of the Gonzalez's Algorithm that calculates the principal minors, so by feeding the new matrix, it is verified that the result, rounding the numbers to integers, are the same as those obtained with the original matrix, so it is checked again

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 9, Issue 3, September 2019

that the algorithm is correct, namely:
Reading Matrix
Coefficients...
c[0] = 6.11
c[1] = -24.73
c[2] = 16.08
c[3] = 8.00
c[4] = 1.00
Principal Minors...
m[0] = -6.00
m[1] = -5.00
m[2] = 6.00
m[3] = -3.00
m[4] = 48.00
m[5] = -9.00
m[6] = 14.04
m[7] = -36.00
m[8] = -93.96
m[9] = 6.11
m[10] = 95.77
m[11] = -21.00
m[12] = 9.00
m[13] = 58.92
m[14] = -24.96

IV. CONCLUSION

Given the importance of the principal minor's assignment problem in various fields of science and technology, it is pending for future work; apply this algorithm to a tri-diagonal symmetric matrix.

REFERENCES

[1] Gonzalez, H.E. "A New LU Decomposition on Hybrid GPU-Accelerated Multicore Systems". Computation y Sistemas (CYS). Vol. 17. No. 3. (2013), pp. 413-422.

[2] González, H.E. "Teorema del Polinomio Característico. Nuevo método para obtener los coeficientes del Polinomio Característico". Núm. de Registro: 03-2009-101810321200-01 y 03-2009-091810340700-14.

[3] Kent Griffin, Michael J.Tsatsomeros. "Principal minors, Part II: The principal minor assignment problem". Linear Algebra and Its Applications. Volume 419, Issue 1, 1 November 2006, Pages 125-171.

[4] Rising,J., Kuleza,A.,Taskar,B. "An Efficient Algorithm for the Symmetric Principal Minor Assignment Problem ". www.alexkuleza.com, November, 2014.

[5] Griffin, Kent E. "Solving the Principal Minor Assignment Problem and related computations". Dissertation for the degree of Doctor of Philosophy. Pp. 62. August, 2006.

[6] Hartfiel, D.J., Loewy R. On Matrices Having Equal Corresponding Principal Minors. Linear Algebra and Its Applications 58:147-157 (1984).

[7] Loewy, R. Principal Minors and Diagonal Similarity of Matrices. . Linear Algebra and Its Applications 78:23-64 (1986).

AUTHOR BIOGRAPHY

H. E. González. He received the PhD in Operations Research at the National Autonomous University of Mexico (UNAM) in 2005. He has been working in research activities for more than twenty years. He has published a book and more than ten scientific papers. Presently, he is a full-time researcher at the National Institute of Nuclear Research (ININ) at the Department of Systems.

J.J. Carmona L., He received his B.S. in Electronic Engineering from UAM. Presently, he is a full-time researcher at the National Institute of Nuclear Research (ININ) at the Department of Systems.