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Confidence intervals for the signal-to-noise ratio of gamma distributions

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Abstract: ***The purpose of this paper is to provide three approaches for constructing confidence intervals for the signal-to-noise ratio (SNR) of gamma distribution based on large sample approach, Wald approach, and Score approach. We compare these approaches in terms of coverage probability and average length by using Monte Carlo simulation. Simulations show that the Wald approach performs well for constructing the confidence intervals for the SNR of gamma distribution. At the end, the proposed approaches are illustrated using real data example.***

Keywords: signal-to-noise ratio, gamma distribution, Wald approach, Score approach.

I. BACKGROUND

Coefficient of variation (CV) expresses the ratio of the standard deviation to the mean. It is a relative measure of dispersion. The CV is free from the unit of measurement. The CV can be used to compare the variation of two or more different measurement methods. For example, in public health, the CV is used to compare the variability in blood pressure measurement (mmHg) and cholesterol measurement (mg/dL) for assessing the overall health of an individual. Furthermore, some applications of the CV are insurance, stock market, actuarial risk theory on mergers of companies, and rainfall data.

The signal-to-noise-ratio (SNR) is the inverse of the CV, which is the ratio of one to the CV. The inverse of the CV is used to measure the slope of the indifference curve in the mean-standard deviation space. If this ratio is high value, then it implies higher mean-variance expected utility. The SNR measures how much signal has been corrupted by noise (McGibney and Smith, 1993). The inverse of the CV is applied in the analysis of portfolio selection models and in a market risk.

Gamma distribution is used to model life-time data due to its flexibility in shape. This distribution has found applications in the areas including engineering, quality control, economics, hydrology, and seismology. Chang et al. (2011) considered testing the equality of several gamma means based on a parametric bootstrap method with applications. Sangnawakij et al. (2016) constructed the confidence intervals for the ratio of coefficients of variation of gamma distributions.

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The literature on the confidence intervals for estimating the population SNR or the inverse of CV is very limited. The asymptotic distribution of the SNR without making any assumption about the distribution is developed by Sharma and Krishna (1994). George and Kibria (2012) compared several confidence intervals estimate for estimating SNR. Confidence interval estimation of the SNR using ranked set sampling is found in the literature by Albatineh et al. (2014). The objectives of this paper are to construct the confidence intervals for the SNR of gamma distribution. The confidence intervals for the SNR of gamma distribution are constructed based on three approaches; large sample approach, Wald approach, and Score approach. A simulation study is conducted to compare the performance of these confidence intervals.

The organization of the paper is as follows. In Section II, the confidence intervals for the SNR based on the large sample approach, the Wald approach, and the Score approach are constructed. Section III describes the simulation and the results concerning coverage probabilities and average lengths. Section IV presents numerical examples. Discussion and conclusions are given in Section V.

II. CONFIDENCE INTERVALS FOR THE SINGLE SIGNAL-TO-NOISE RATIO

Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from a gamma distribution with shape parameter α and scale parameter β, $X \sim Gamma(\alpha, \beta)$. The probability density function of X is

$$f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \exp(-\frac{x}{\beta}); 0 < x < \infty, \alpha > 0, \beta > 0 \quad (1)$$

The mean and variance of X are $E(X) = \alpha\beta$ and $Var(X) = \alpha\beta^2$. The CV is

$$\tau = \frac{\sqrt{Var(X)}}{E(X)} = \frac{\beta\sqrt{\alpha}}{\alpha\beta} = \frac{1}{\sqrt{\alpha}}. \quad (2)$$

The following SNR

$$\theta = \frac{1}{\tau} = \sqrt{\alpha}. \quad (3)$$

From Sangnawakij et al. (2018), the estimators of α and β are

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$$\hat{\alpha} = \frac{1}{2[\ln \bar{X} - \frac{1}{n}\sum_{i=1}^{n} \ln X_i]} \text{ and } \hat{\beta} = \frac{\bar{X}}{\hat{\alpha}}. \quad (4)$$

Furthermore, the variances of $\hat{\alpha}$ and $\hat{\beta}$ are

$$Var(\hat{\alpha}) = \frac{n}{\alpha} + \frac{n}{2\alpha^2} \text{ and } Var(\hat{\beta}) = \frac{n\alpha}{\beta^2}. \quad (5)$$

The estimator of θ is

$$\hat{\theta} = \sqrt{\hat{\alpha}} = \left[\frac{1}{2[\ln \bar{X} - \frac{1}{n}\sum_{i=1}^{n} \ln X_i]}\right]^{1/2}. \quad (6)$$

The mean and variance of $\hat{\theta}$ are

$$E(\hat{\theta}) = \sqrt{\alpha} \text{ and } Var(\hat{\theta}) = \frac{1}{4\alpha}\left[\frac{n}{\alpha} + \frac{n}{2\alpha^2}\right]; \text{ see Theorem.} \quad (7)$$

Hence, the $\hat{\theta}$ is an unbiased estimator of θ.

Theorem: Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from a gamma distribution. Let $\theta = \sqrt{\alpha}$ be the SNR of X. Also, let $\hat{\theta}$ be an estimator of θ. The mean and variance of $\hat{\theta}$ are $E(\hat{\theta}) = \sqrt{\alpha}$ and $Var(\hat{\theta}) = \frac{1}{4\alpha}\left[\frac{n}{\alpha} + \frac{n}{2\alpha^2}\right]$.

Proof: Suppose that the SNR is $\theta = \sqrt{\alpha}$. Using the delta method (Casella and Berger, 2002), let

$$\theta = g(\alpha) = \sqrt{\alpha}.$$

Differentiating $g(\alpha)$ with respect to α

$$g'(\alpha) = \frac{1}{2\sqrt{\alpha}}.$$

The mean of $\hat{\theta}$ is

$$E(\hat{\theta}) = E(g(\hat{\alpha})) \approx g(\alpha) = \sqrt{\alpha}.$$

The variance of $\hat{\theta}$ is

$$Var(\hat{\theta}) = Var(g(\hat{\alpha}))$$
$$\approx [g'(\alpha)]^2 Var(\hat{\alpha})$$
$$= \left[\frac{1}{2\sqrt{\alpha}}\right]^2 \left[\frac{n}{\alpha} + \frac{n}{2\alpha^2}\right]$$
$$= \frac{1}{4\alpha}\left[\frac{n}{\alpha} + \frac{n}{2\alpha^2}\right].$$

Hence, Theorem is proved.

a. The large sample approach for the signal-to-noise ratio

Using the normal approximation, the pivotal statistic is

$$Z = \frac{\hat{\theta} - E(\hat{\theta})}{\sqrt{Var(\hat{\theta})}} = \frac{\hat{\theta} - \theta}{\sqrt{Var(\hat{\theta})}}. \quad (8)$$

Therefore, the $100(1-\gamma)\%$ two-sided confidence interval for the SNR θ based on the large sample approach is

$$CI_{LS} = (L_{LS}, U_{LS}) = (\hat{\theta} - z_{1-\gamma/2}\sqrt{Var(\hat{\theta})}, \hat{\theta} + z_{1-\gamma/2}\sqrt{Var(\hat{\theta})}), \quad (9)$$

Where $\hat{\theta}$ and $Var(\hat{\theta})$ are defined in equations (6)-(7), respectively, and $z_{1-\gamma/2}$ denotes the $(1-\gamma/2)$-th quantile of the standard normal distribution.

b. The Wald approach for the signal-to-noise ratio

From Sangnawakij et al. (2018), the confidence interval for the coefficient of variation based on Wald approach is

$$\frac{1}{\sqrt{\hat{\alpha} + z_{1-\gamma/2}\sqrt{\frac{2\hat{\alpha}^2}{n}}}} \le \frac{1}{\sqrt{\alpha}} \le \frac{1}{\sqrt{\hat{\alpha} - z_{1-\gamma/2}\sqrt{\frac{2\hat{\alpha}^2}{n}}}}. \quad (10)$$

Hence, the confidence interval for the SNR based on the Wald approach is

$$\sqrt{\hat{\alpha} - z_{1-\gamma/2}\sqrt{\frac{2\hat{\alpha}^2}{n}}} \le \sqrt{\alpha} \le \sqrt{\hat{\alpha} + z_{1-\gamma/2}\sqrt{\frac{2\hat{\alpha}^2}{n}}}, \quad (11)$$

Where $z_{1-\gamma/2}$ denotes the $(1-\gamma/2)$-th quantile of the standard normal distribution. Therefore, the $100(1-\gamma)\%$ two-sided confidence interval for the SNR θ based on the Wald approach is

$$CI_W = (L_W, U_W) = \left(\sqrt{\hat{\alpha} - z_{1-\gamma/2}\sqrt{\frac{2\hat{\alpha}^2}{n}}}, \sqrt{\hat{\alpha} + z_{1-\gamma/2}\sqrt{\frac{2\hat{\alpha}^2}{n}}} \right), \quad (12)$$

Where $\hat{\alpha}$ is defined in equation (4) and $z_{1-\gamma/2}$ denotes the $(1-\gamma/2)$-th quantile of the standard normal distribution.

c. The Score approach for the signal-to-noise ratio

From Sangnawakij et al. (2018), the confidence interval for the CV based on Score approach is

$$\sqrt{\frac{2}{n}\left(z_1 - z_{1-\gamma/2}\sqrt{\frac{n}{2\hat{\alpha}^2}}\right)} \le \frac{1}{\sqrt{\alpha}} \le \sqrt{\frac{2}{n}\left(z_1 + z_{1-\gamma/2}\sqrt{\frac{n}{2\hat{\alpha}^2}}\right)}, \quad (13)$$

where $z_1 = n \ln \bar{X} - \sum_{i=1}^{n} \ln X_i$ and $z_{1-\gamma/2}$ denotes the $(1-\gamma/2)$-th quantile of the standard normal distribution. Hence, the confidence interval for the SNR based on the Score approach is

$$\frac{1}{\sqrt{\frac{2}{n}\left(z_1 + z_{1-\gamma/2}\sqrt{\frac{n}{2\hat{\alpha}^2}}\right)}} \le \sqrt{\alpha} \le \frac{1}{\sqrt{\frac{2}{n}\left(z_1 - z_{1-\gamma/2}\sqrt{\frac{n}{2\hat{\alpha}^2}}\right)}}. \quad (14)$$

Therefore, the $100(1-\gamma)\%$ two-sided confidence interval for the SNR θ based on the Score approach is

$$CI_S = (L_S, U_S) = \left(\frac{1}{\sqrt{\frac{2}{n}\left(z_1 + z_{1-\gamma/2}\sqrt{\frac{n}{2\hat{\alpha}^2}}\right)}}, \frac{1}{\sqrt{\frac{2}{n}\left(z_1 - z_{1-\gamma/2}\sqrt{\frac{n}{2\hat{\alpha}^2}}\right)}} \right), \quad (15)$$

Where $\hat{\alpha}$ is defined in equation (4), $z_1 = n \ln \bar{X} - \sum_{i=1}^{n} \ln X_i$, and $z_{1-\gamma/2}$ denotes the $(1-\gamma/2)$-th quantile of the standard normal distribution.

III. RESULTS

In this section for constructing confidence intervals of single SNR and difference of SNRs under heterogeneity, the approaches are compared according to coverage probabilities and average lengths for different combinations of parameters and sample sizes. For single SNR, we consider some cases from smaller to larger sample sizes with same scale parameter as $\beta =$ 2 and different SNRs as $\theta =$ 2, 3, 5, 10, 20. The shape parameter is computed by $\alpha = \theta^2$. For difference of SNRs, we carried out the simulation with sample sizes $(n, m) =$ (10,10), (30,30), (50,50), (100,100), and (200,200). We selected the scale parameters as $\beta_1 = \beta_2 =$ 2, the SNRs as $(\theta_1, \theta_2) =$ (5,2), (5,3), (5,5), (10,2.5), (10,5), (10,10), (20,3), (20,10), and (20,20). And the shape parameters are computed by $\alpha_1 = \theta_1^2$ and $\alpha_2 = \theta_2^2$. For specified nominal level of $\alpha =$ 0.05, we used $M = 10000$ times to calculate the coverage probabilities and average lengths of these confidence intervals. The performance of confidence interval, the coverage probability are greater than or close to the nominal confidence level $1-\alpha =$ 0.95 and the average length is shortest, is the better confidence interval.

The simulation results of the single SNR are displayed in Table 1. The large sample confidence interval, the Wald confidence interval, and the Score confidence interval are compared. The coverage probabilities of the large sample confidence interval are less than the nominal confidence level 0.95 when the sample size is small and the SNR is large, whereas the coverage probabilities are close to 1.00 when the sample size is large and the SNR is small. The coverage probabilities of the Wald confidence interval are greater than the nominal confidence level of 0.95 for all cases, except the coverage probabilities are less than the nominal confidence level of 0.95 when the sample size is large and the SNR is small. Moreover, the Score confidence interval provides the coverage probabilities under the nominal confidence level of 0.95 and close to nominal confidence level of 0.95 when the sample size is large.

Table 1 The coverage probabilities (CP) and average lengths (AL) of 95% two-sided confidence intervals for the signal-to-noise ratio of gamma distribution.

n	θ	CI_{LS}		CI_W		CI_S	
		CP	AL	CP	AL	CP	AL
10	2	0.7363	1.5707	0.9611	2.2920	0.8276	4.7615
	3	0.4143	0.6537	0.9679	3.4820	0.8178	7.2337
	5	0.0833	0.2266	0.9729	5.8552	0.8059	12.1638
	10	0.0093	0.0556	0.9729	11.7893	0.8015	24.4915
	20	0.0014	0.0139	0.9701	23.5126	0.8030	48.8461
30	2	0.9849	2.8791	0.9491	1.0724	0.9220	1.2434
	3	0.8154	1.2161	0.9513	1.6244	0.9027	1.8833
	5	0.2589	0.4206	0.9590	2.7339	0.8976	3.1698
	10	0.0338	0.1038	0.9596	5.4823	0.8957	6.3563
	20	0.0043	0.0260	0.9553	10.954	0.8930	12.7006
50	2	0.9994	3.7745	0.9477	0.8030	0.9441	0.8728
	3	0.9358	1.5850	0.9489	1.2188	0.9278	1.3248

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	5	0.4308	0.5531	0.9520	2.0440	0.9223	2.2218
	10	0.0514	0.1361	0.9571	4.1066	0.9158	4.4638
	20	0.0068	0.0340	0.9551	8.2106	0.9105	8.9249
100	2	1.0000	5.3681	0.9369	0.5559	0.9491	0.5785
	3	0.9972	2.2620	0.9435	0.8421	0.9422	0.8764
	5	0.7437	0.7905	0.9529	1.4116	0.9390	1.4692
	10	0.1080	0.1951	0.9508	2.8304	0.9373	2.9458
	20	0.0150	0.0485	0.9535	5.6719	0.9323	5.9032
200	2	1.0000	7.6432	0.9270	0.3882	0.9488	0.3959
	3	1.0000	3.2124	0.9426	0.5888	0.9471	0.6005
	5	0.9574	1.1226	0.9479	0.9873	0.9455	1.0068
	10	0.2182	0.2772	0.9504	1.9795	0.9466	2.0187
	20	0.0277	0.0691	0.9502	3.9619	0.9425	4.0403

IV. AN EMPIRICAL APPLICATION

Two examples are given to illustrate our proposed approaches for confidence interval estimation for the SNR of gamma distribution and difference of two SNRs of gamma distributions. The first example is used to estimate the SNR using large sample approach, Wald approach, and Score approach. The second example is used to evaluate the difference of SNRs using three approaches; large sample approach, MOVER approach based on Wald confidence interval, and MOVER approach based on Score confidence interval.

Example 1: The data for the first example are taken from Gross and Clark (1975), Grice and Bain (1980), and Fraser et al. (1997) who reported the data on survival times on 20 mice exposed to 240 rads of gamma radiation. The data are 152, 152, 115, 109, 137, 88, 94, 77, 160, 165, 125, 40, 128, 123, 136, 101, 62, 153, 83, and 69. The mean and SNR are 113.4500 and 2.938682, respectively.

Using the three approaches, the 95% two-sided confidence intervals for the SNR of gamma distribution were constructed. The large sample confidence interval is (2.4167,3.4607) and the interval length is 1.0440. The Wald confidence interval is (1.8120,3.7401) and the interval length is 1.9281. Furthermore, the Score confidence interval is (2.3090,4.7659) and the interval length is 2.4569. The interval length of the large sample confidence interval is shorter than the other confidence intervals. Hence, the results of this real data example are consistent with the results of the simulation studies in term of the average length.

V. DISCUSSION AND CONCLUSIONS

In this paper, we propose new confidence intervals for the single SNR of gamma distribution Three confidence intervals for the SNR are derived by three approaches; the large sample approach, the Wald approach, and the Score approach.

We use Monte Carlo simulations to compute the performance of the confidence intervals. For the single SNR, it can be concluded that the large sample approach is not recommended to use for constructing the confidence interval for SNR of gamma distribution. The Wald approach can be used as interval estimator for the SNR when the sample size is small. Moreover, the Wald approach can be an alternative approach when the sample size is large and the SNR is large. In addition, the Score approach should be used only when the sample size is large and the SNR is small.

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