Theoretical Investigation on Saltwater Movement in Lake Mouth

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Abstract—The present study investigates the basic equation for the saltwater intrusion in estuary, especially, about the material transfer between the upper and lower divided water layers. Taking the blowing phenomenon and the sinking phenomenon in the material transfer between the divided water layers, the reasonable expression of the material transfer are originally given in the present study. And, the boundary condition at the moment when the backflow begins from the sea to the lake is given reasonably. The theory for the saltwater movement in the estuary develops in the present study. It is shown that the theory and the field observations are a good agreement. The present study shows that also if there is even one value of the salinity observation, one can predict a time and spatial changing of salinity.

Index Terms—material diffusion, saltwater intrusion, estuary, blowing phenomenon, sinking phenomenon, material transfer.

I. INTRODUCTION

The theory concerning the movement of salt water in the mouth of a river has been shown by Sasaki, Tanaka and Umeda (2012) [1]. The theory has been progressed more by Sasaki, Tanaka and Umeda (2017) [2] to make clear the vertical distribution of salinity. However, there is a little unreasonable expression in their theory about the material transfer in the perpendicular direction. This study aims to show a certain physical expression about the material transfer, and investigates the characteristics of exact solution shown by Sasaki, Tanaka and Umeda [1], [2] for the saltwater movement in estuary. In the theory given by Sasaki, Tanaka and Umeda [1], [2], there are a little vague, uncertain expressions, particularly about the material transfer between the divided horizontal water layers. The present study gives physically the expression concerning the material transfer between the water layers with a clear meaning. The theory for the saltwater movement in the estuary develops in the present study. The saltwater movement is shown by the turbulent diffusion equation in three dimensions. The diffusion field can be divided into several water layers in the perpendicular direction along the water depth. The modified diffusion equation shows the seawater movement in the one of the layers. The theoretical solution given by Sasaki, Tanaka and Umeda [1], [2] is investigated by the certain boundary condition in the diffusion field. The present study also shows that the exact solution can generate well the movement of seawater intrusion in the lake mouth as well as the study shown by Sasaki, Tanaka and Umeda [2]. The solution shows that also if there is even one value of the salinity observation, one can calculate a time and spatial changing of salinity.

In the present study, the basic equation concerning the behavior of salt water is examined first. The material transfer between the divided water layers is considered in the basic equation. Next, the exact solution to the basic equation is checked. The boundary condition that expresses the phenomenon well is examined in the analytical solution, and a physical expression of the boundary condition is reasonably shown. Next, to examine the accuracy of the theory, the field observation is done. The field observation is executed in the Lake Ogawara. Finally, the theory and the field observation are compared, and the phenomenon reproducibility of the theory is confirmed.

The research method in the present study becomes as follows. The present study investigates about the material transfer between the upper and lower divided water layers and shows the reproducibility of the phenomenon of the exact solution given by Sasaki, Tanaka and Umeda [1], [2]. By using the results of the field observation shown by Sasaki, Tanaka and Umeda [3], the basic equation of the salt water movement is verified first. Investigating the blowing phenomenon and the sinking phenomenon in the material transfer between the divided water layers, the reasonable expressions of the material transfer are originally given. A boundary condition has not still explained well in the theory by Sasaki, Tanaka and Umeda [1], [2]. Then, the boundary condition at the moment the flow changes from the seaward flow to the lake ward flow is given reasonably. In the present study, Lake Ogawara is selected as a field observation, and the forecast of the salinity was executed. Finally, theory and the field observation on salinity are comparing.

II. INVESTIGATION OF BASIC EQUATION FOR SALTWATER INTRUSION

According to the results of the field observation shown by Sasaki, Tanaka and Umeda [3], the salt water movement can be given by the salinity change. The salinity change shows the material diffusion phenomena. Now, the movement of the mass per unit volume, c, of the diffusion material like salinity can be taken in estuary in three dimensions. The velocity components of the flow in x, y and z directions in the flow field are u, v, and w. And time is shown by t. The diffusion phenomenon is shown by the average variable and the minuteness change variable in the turbulent flow field. In addition, as turbulent diffusion coefficients D_x, D_y, and D_z in
the directions of x, y, and z are introduced based on the law of Fick given as follows.

\[
\frac{\partial c}{\partial t} + u \frac{\partial (c)}{\partial x} + v \frac{\partial (c)}{\partial y} + w \frac{\partial (c)}{\partial z} =
\]

\[
\frac{\partial}{\partial x} \left( D_0 \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial y} \left( D_0 \frac{\partial c}{\partial y} \right) + \frac{\partial}{\partial z} \left( D_0 \frac{\partial c}{\partial z} \right)
\]

Equation (1) is perfect for the basic equation of the salt water movement in the estuary. However, it is too difficult to get an analytical solution of the equation. Then, it is necessary to model the phenomenon further. On the seawater movement in the mouth of a river, Sasaki, Tanaka and Umeda [1] have taken the assumption that the phenomenon is the same in the direction of the crossing. This assumption is quite correct in the diffusion phenomenon in the river channel and the waterway between the sea and the lake. Fig. 1 shows the waterway, the coordinates x and z. As shown in Fig. 1, the coordinate x is taken in the direction of the water channel and the coordinate z is taken in a perpendicular direction. Then, the equation (1) can be rewritten as follows.

\[
\frac{\partial c}{\partial t} + u \frac{\partial (c)}{\partial x} + w \frac{\partial (c)}{\partial z} =
\]

\[
\frac{\partial}{\partial x} \left( D_0 \frac{\partial c}{\partial x} \right) + \frac{\partial}{\partial z} \left( D_0 \frac{\partial c}{\partial z} \right)
\]

In addition, the phenomenon is modeled. As shown in Fig. 1, the field of diffusion is divided vertically into some horizontal water layers along the total water depth. The change in a perpendicular direction in the new water layer is assumed to be omissible small compared with the change of horizontal direction in each layer. In general, it is reasonable that the change in a perpendicular direction is large all over the depth from the bottom to the water surface; however, the change in a perpendicular direction is small just in the new layers that are formed horizontally in water depth. In each layer, the salinity change is small vertically. Then, the perpendicular change of the diffusion material can be neglected in each layer. However, there must be the material transfer coming into a water layer from the upper and lower water layers, or going out from the water layer to the upper and lower water layers. The material transfer is shown by q now. Therefore, the basic equation can be rewritten in a new horizontal water layer as the next equation.

\[
\frac{\partial c_k}{\partial t} + u_k \frac{\partial (c_k)}{\partial x} = \frac{\partial}{\partial x} \left( D_{sk} \frac{\partial c_k}{\partial x} \right) + q_k
\]

Where the suffix k means the k-th horizontal water layer. The second term in the right of the above (3) is a material movement generated by having divided the water depth. Namely, the second term, \(q_k\), shows the material transfer coming into the k-th water layer from the upper and lower water layers, or going out from the k-th water layer to the upper and lower water layers. If the flow is seaward, the material transfer occurs as the blowing phenomenon. On the other hand, if the flow is landward from the sea, the material transfer occurs as the sinking phenomenon. Thus, the material transfer \(q_k\) can be given as follows.

During seaward flow

\[
q_k = -f_{ck} |u_k| \frac{\partial c_k}{\partial x}, \quad u > 0
\]

During backflow from sea to lake

\[
q_k = f_{ck} |u_k| \frac{\partial c_k}{\partial x}, \quad u < 0
\]

Where \(f_{ck}\) is a coefficient for the material transfer between the divided horizontal water layers. Equation (4) is the expressions given by the present study for the first time. According to Sasaki, Tanaka and Umeda [1], the turbulent diffusion coefficients \(D_{sk}\) can be given as follows.

\[
D_{sk} = l_{sk} |u_k|
\]

Where \(l_{sk}\) is the mixing length in the field of the diffusion In the waterway connecting the sea and the lake, because a spatial change in the speed is smaller than the temporal change, the flow velocity can be given as follows.

\[
u(x, t) = u(t) + u'(x, t) \approx u(t)
\]

Now, Using (4a and 4b), (5), and (6), equation (3) can be rewritten as follows.

\[
\frac{\partial c_k}{\partial t} + (1 + f_{ck})u_k \frac{\partial c_k}{\partial x} = l_{sk} |u_k| \frac{\partial^2 c_k}{\partial x^2}
\]

Equation (7) represents the basic equation for the saltwater movement in the estuary.
III. INVESTIGATION OF EXACT SOLUTION FOR BASIC EQUATION OF DIFFUSION

Now, as well as Sasaki, Tanaka and Umeda [1], a new variable is introduced as follows.

During backflow from the sea to the lake

\[ \xi_k = \beta_{2k} \int_0^t \left[ u_k \right] dt + \beta_{2k} x_k L_0 + X_{o bk} L_0, \]  
(8a)

During seaward flow from the lake to the sea

\[ \xi = \beta_{1k} \int_0^t \left[ u_k \right] dt + \beta_{2k} x_k L_0 + X_{o lk} L_0, \]  
(8b)

Where \( \beta_{1k} \) and \( \beta_{2k} \) are arbitrary constants, t is time that indicates the time is counted from the moment when the backflow from the sea to the lake and the seaward flow from the lake to the sea begin, \( X_{o bk} \) and \( X_{o lk} \) is the distance that relates to arrival of the seawater front when the flow changes, where the suffixes b and s mean the backflow and seaward flow, and \( L_o \) is a typical length of the diffusion field.

The new variable \( \xi_k \) shown in (8a and 8b) must become always positive value. That is as follows.

\[ \xi_k = 0 \quad \text{if} \quad \xi_k < 0 \]  
(9)

By introducing the new variable, equation (7) can be rewritten as follows. During backflow from sea to lake

\[ \frac{d^2 c}{d \xi^2} + \alpha_1 \frac{d c}{d \xi} = 0 \quad u<0 \]  
(10a)

During seaward flow from lake to sea

\[ \frac{d^2 c}{d \xi^2} + \alpha_2 \frac{d c}{d \xi} = 0 \quad u>0 \]  
(10b)

Where

\[ \alpha_1 = \begin{cases} \frac{-\beta_{1k}}{L_o} + \frac{\beta_{2k}}{L_o} (1 + C_{ek}) & u<0 \\ \frac{\left( \beta_{2k} \right)^2 L_s}{L_o} & \end{cases} \]  
(11a)

\[ \alpha_2 = \begin{cases} \frac{-\beta_{1k}}{L_o} + \frac{\beta_{2k}}{L_o} (1 + C_{ek}) & u>0 \\ \frac{\left( \beta_{2k} \right)^2 L_s}{L_o} & \end{cases} \]  
(11b)

Where \( \alpha_1 \) and \( \alpha_2 \) are the arbitrary constants that must become always positive value. As shown in (10a and 10b), the two parameters rule the diffusion phenomenon. The parameter \( \alpha_1 \) is the constant during the backflow from the sea to the lake, and the parameter \( \alpha_2 \) is the constant during the seaward flow from the lake to the sea. Sasaki, Tanaka and Umeda [1] have shown the exact solutions of the basic equations (10a) and (10b). However, their solutions have an uncertain expression on the boundary condition when the water flow changes from the seaward flow into the backflow from the sea to the lake. Their solution can be modified as follows. For the backflow, \( u<0 \)

\[ c_k = \left( C_{zk} - C_{ik} \right) \left[ 1 - \exp \left( -\alpha_1 \left( \xi_k + \xi_{ok} \right) \right) \right] + C_{ik} \]  
(12)

Where

\[ \xi_{ok} = -\frac{\ln \left( \frac{C_{3k} - C_{ok}}{C_{3k} - C_{ik}} \right)}{\alpha_{ik}} \]  
(13)

For the seaward flow, \( u>0 \)

\[ c_k = \left( C_{pk} - C_{ik} \right) \exp \left( -\alpha_2 \xi_k \right) + C_{ik} \]  
(14)

Where constants \( C_{ik} \) and \( C_{3k} \) in (12) are the minimum value of the salinity and the maximum value of the salinity at any point \( x \) in the divided horizontal water layer of \( k \)-th, the constant \( C_{ek} \) in (13) is the salinity at the moment when the backflow begins from the sea to the lake, and the constant \( C_{pk} \) is the salinity at the moment when the seaward flow begins from the lake to the sea. Equation (13) is given for the first time in the present study. In the present study, the arbitrary constants \( \beta_{1k} \) and \( \beta_{2k} \) are taken as follows.

\[ \beta_{1k} = 1, \quad \beta_{2k} = 2 \quad u<0 \]  
(15)

\[ \beta_{1k} = 1, \quad \beta_{2k} = -2 \quad u>0 \]  
(16)

In the present study, the typical length of the diffusion field can be given as follows.

\[ L_o = 3600 \text{ m} \]  
(17)

By using (15) and (16), the variable \( \xi_k \) defined by (8) can be calculate after the distance \( X_{o lk} \) is decided.

IV. PREDICTION OF FLOW VELOCITY NEAR THE LAKE MOUTH

We need to get the flow velocity for the calculation of salinity. However, we can forecast the velocity of the flow by using the water level data once we can decide the velocity coefficient. In general, the water level has been measured in the river which is managed by the country. We can easily take the observational data of the water level from the web. Hereafter, salinity is calculated as an example of Lake Ogawara that is located in the east of Aomori Prefecture. Fig. 2 shows Lake Ogawara. As shown in the Fig.2, Lake Ogawara faces to Pacific Ocean, and the lake is connected with the Pacific Ocean in the Takase River. Takase River is a river channel of the length of 6km. In the figure, an observation station is shown, which is taken near the mouth of the lake.
The field observation for the flow velocity and the salinity was made at the station for two months from October 6\textsuperscript{th} to September 10\textsuperscript{th}, 2016.

Flow velocity in the river channel is given by Bernoulli’s theory by the next equation.

\[ u = C_{vb} \sqrt{2g \Delta z} \quad \Delta z = z_{bc} - z_{rm} \]  

\[ u = C_{vs} \sqrt{2g | \Delta z |} \quad \Delta z = z_{sc} - z_{rm} \]  

Where the suffixes b, c, l, m, r, and s mean the backflow, the center of the lake, the lake, the mouth of river, river, and the seaward flow.

\[ C_{vb} \] is the velocity coefficient when the backflow. The constant \[ C_{vs} \] is the velocity coefficient when the seaward flow, the variable \[ z_{bc} \] is the water level at the lake center, and the variable \[ z_{rm} \] is the water level in the river mouth near sea. By comparing the velocities given by the field observation, and by (18), the velocity coefficients are decided as follows.

During the seaward flow
\[ C_{vb} = 0.122 \]  

During the backflow from the sea to lake
\[ C_{vs} = 0.22 \]  

Fig. 3 shows the velocity obtained from the field observation, and the velocity given by the equations (18) and (19). As mentioned above, because the coordinate \( x \) is taken from the lake to the sea, the minus value of the velocity shows the backflow from the sea to the lake, and the positive value of the velocity represents the seaward flow from the lake to the sea. The fig.3 shows that the flow velocity becomes the maximum of 40 cm/s when the flow is seaward from the lake to the sea, and reaches the maximum 40 cm/s when the flow is back from the sea to the lake, and that the velocity by the observation and the velocity given by the calculation indicate a good agreement. In the mouth of the lake, because the cross-sectional area of the flow is large, and the width of the flow is wide, the flow is not so fast, thus flow velocity is the maximum of 0.4m/s. The observation was made in the lake mouth outside the river channel, therefore, the equation (19) can give the velocity near the lake mouth. Now we can predict at any time the flow velocity given by the (19) without the field observation.

Fig. 4 shows the velocity given by (19) from October 6\textsuperscript{th} to 25\textsuperscript{th}, 2016. The figure is showing that the backflow occurs every day except two days, October 11\textsuperscript{th} and 12\textsuperscript{th}, the strong backflow occurs for 8 days from 17 October to 23 October, and the strong backflow also occurs on 8 October.

V. PREDICTION FOR SALINITY IN THE LAKE MOUTH

Now, the salinity can be predicted by using the velocity given by the (19). Fig. 5 shows the salinity obtained from the field observation and the salinity given by the theory. Because we have only one salinometer, the field observation was carried out for the first two weeks from 6 October, 2016, at the depth \( z=16 \) cm over the lake bed, and for the next two weeks from 15 October to 11 November, 2016 at the depth \( z=32 \) cm over the lake bed. The total depth is \( h=72 \) cm at the observation station. Then, the relative depth is \( z/h=0.22 \) at the depth \( z=16 \) cm, and the relative depth is \( z/h=0.44 \) at the depth \( z=32 \) cm. Then, The depths of \( z=16 \) cm and \( z=32 \) cm represent the positions of the middle and the lower layers of total depth. Fig. 5 shows the results of the field observation for the first two weeks in October, 206. The mixture coefficients between the freshwater and the saltwater \( \alpha_1 \) and \( \alpha_2 \), and the minimum and maximum values of the salinity in the diffusion field \( C_{1k} \) and \( C_{3k} \), and the distances related to
the saltwater intrusion \( X_{olbk} \) and \( X_{olak} \) can be determined by comparing with the field observation at the depth \( z=16 \) cm as follows.

\[
\begin{align*}
\alpha_1 &= 3.5 \\
\alpha_2 &= 1.5 \\
C_{1k} &= 1.3 \\
C_{3k} &= 33.5 \\
X_{0lbk} &= -1200 \\
X_{0lak} &= 0 \\
\end{align*}
\]  

(20)

As shown in (21), the mixture coefficients between the freshwater and the saltwater are \( \alpha_1 = 2.7 \) and \( \alpha_2 = 1.5 \), and the maximum and minimum value of the salinity are \( C_{1k} = 33.5 \) and \( C_{3k} = 1.3 \). The distances related to the saltwater intrusion are \( X_{olbk} = -1300 \) when the flow is the backflow, \( X_{olak} = 0 \) when the flow is the seaward flow.

![Fig. 5. Salinities obtained from the field observation, and the calculation by using the theory at the depth \( z=16 \) cm over the lake bottom from 6 October to 24 October, 2016.](image1)

![Fig. 6. Salinities obtained from the field observation and the calculation by using the theory at the depth \( z=32 \) cm over the lake bottom from 25 October to 10 November, 2016.](image2)

As shown in Fig. 5, the salinity can be predicted by using the constants given by (20) and (21). The constants by (20) give the salinity in the lower water layer, and the constants by (21) give the salinity in the middle layer. Fig. 7 shows the prediction of the salinities in the lower and middle water layers in October, 2016. In the figure, \( z^* \) is the relative depth which is defined as the ratio of the height \( z \) from the lake bottom to the total depth \( h=72 \) cm. Once the constants of the exact solutions are decided like (20) and (21), we can predict the salinity on any day in any month. As shown in Fig. 7, the saltwater intrusion occurs for eight days from 17 October to 24 October. But, the salinity concentration is lower in the upper water layer than in the lower water layer. The saltwater intrusion occurs on 8, 16-24, 28, and 29 October in the lower water layer, however, there is no saltwater intrusion on 8, 16, 24, and 28 October in the middle water layer. The saltwater is going up on 31 October even in the middle water layer. In the figure, the minimum value of salinity is 1.3 psu. Therefore, salinity always flows out from the lake.
We can’t decide the constants in the analytical solutions given by (12 and 14), because there is no data about the salinity observation. However, we can decide the constants in the exact solutions from the change between the constants at the relative depth \( z^* = 0.22 \), and at the relative depth \( z^* = 0.44 \) as shown in (20) and (21). The constants can be given as follows.

\[
\begin{align*}
   z^* &= 0.67 & \alpha_1 &= 2.2 & \alpha_1 &= 0.75 \\
   \alpha_2 &= 1.5 & \alpha_2 &= 0.5 & C_{1k} &= 1.3 & C_{1k} &= 1.3 \\
   C_{3k} &= 33.5 & C_{3k} &= 33.5 & X_{01bk} &= 1400 & X_{01bk} &= 1500 \\
   X_{01bk} &= 0 & X_{01bk} &= 0
\end{align*}
\]

\( \text{(22)} \)

In the present study, we selected Lake Ogawara, Japan as a field observation, and forecasted the salinity in the lake mouth. Finally, theory and the field observation on salinity are comparing. The advection diffusion by the flow is predominant in the estuary. To utilize the theory shown in the present study, we have to get the velocity in the waterway or estuary. If one can observe the flow velocity in the estuary in a short term, the flow velocity can be predicted for the long term by using the date of water level in the estuary.

In the present study, the velocity at the lake mouth can be predicted at any time by using the (19) without the field observation. By using the velocity, the vertical distribution of salinity in the lake mouth can be predicted by the theory (12) and (14). The exact solution shown in the present study can generate well the saltwater intrusion in the estuary. The theory shown in the present study can be able to apply in other estuary.

**REFERENCES**


**AUTHOR BIOGRAPHY**

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**Fig.8. Vertical distribution of salinity in October, 2016.**

Fig.8 shows the salt water intrusion given by (20), (21), and (22) from 15 October to 25 October, 2016. The figure is showing that the salinity is higher in the lower water layer than in the upper layers, and that the salt water intrusion occurs as the stratified flow of the salt water. Then, the salt water may be going up by the same vertical density distribution the river channel, however, the salt water is going up by the shape of the saltwater wedged in the mouth of the lake.

**VI. DISCUSSIONS**

The present study investigates the basic equation for the saltwater intrusion in estuary. Especially, we investigate about the material transfer between the upper and lower divided water layers. The expression of the material transfer between upper and lower layers was shown physically for the first time in the present study. It is given as the blowing phenomenon during the seaward current and as the sinking phenomenon during the backflows. The theory develops in the present study.