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A New Algorithm to Obtain the Adjugate Matrix using CUBLAS on GPU

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Abstract*— *In this paper a parallel code for obtain the Adjugate Matrix with real coefficients are used. We postulate a new linear transformation in matrix product form and we apply this linear transformation to an augmented matrix (A|I) by means of both a minimum and a complete pivoting strategies, then we obtain the Adjugate matrix. Furthermore, if we apply this new linear transformation and the above pivot strategy to a augmented matrix (A|b), we obtain a Cramer's solution of the linear system of equations. That new algorithm present an $O(n^3)$ computational complexity when $(\mathbf{A},\mathbf{b}) \subset \mathbf{R}^{\mathbf{n}}$. We use subroutines of CUBLAS 2nd and 3rd levels in double precision and we obtain correct numeric results.

***Index Terms*—Adjoint matrix, Adjugate matrix, Cramer's rule, CUBLAS, GPU.**

I. INTRODUCTION

A Linear System of Equations (**LSE**) can be defined as a set of *m* equations with *n* unknowns represented by a matrix **A**, a vector **b** and an unknown vector **x**, namely, $\mathbf{Ax} = \mathbf{b}$.Many methods have been proposed to solve such linear equations. A famous one is Cramer's rule, where each component of the solution is determined as the ratio of two determinants.

When trying to solve a system of *n* equations using Cramer's rule, one needs to compute *n+1* determinants, each of order *n*. If these are computed in a straightforward way, using the Laplace Expansion, the solution to the linear system takes $(n+1)(n!)(n-1)$ multiplications, plus a similar number of additions. Although Cramer´s rule possesses a fundamental theoretical importance, it may result impractical in computations. It is for that reason that this method is seldom recommended [1]-[6]. Cramer´s rule has at least one attractive property: it computes every element of the solutions independently. For this reason, it can be a practical method for some special linear systems on parallel computers [7].

Another approach, with a certain mathematical appeal but considerable computational pitfalls, finds the solution to a linear system of equations using the inverse matrix $\mathbf{A}^{-1}$. However, in virtually every application, it is unnecessary and inadvisable to compute the inverse matrix explicit. The inverse requires more arithmetic and produces a less accurate answer. Therefore, neither of the above methods are recommended [8]. Again, in this paper, we will propose a new efficient method to calculate the adjugate matrix.

The Gaussian Transformation (**GT**) for solving an **LSE** has proved to be the best option for most practical applications. The new transformation proposed here can be obtained from it. Next, we briefly review this topic.

II. LU-MATRICIAL DECOMPOSITION WITH GT. NOTATION AND DEFINITIONS

The problem of solving a linear system of equations $\mathbf{Ax} = \mathbf{b}$ is central to the field of matrix computation. There are several ways to perform the elimination process necessary for its matrix triangulation. We will focus on the Doolittle-Gauss elimination method: the algorithm of choice when **A** is square, dense, and un-structured.

Let us assume that $\mathbf{A} \in \mathbf{R}^{\mathbf{nxn}}$ is nonsingular and that we wish to solve the linear system $\mathbf{Ax} = \mathbf{b}$. Here we show how for exact arithmetic and partial pivoting and column interchanges some Gauss transformations $\mathbf{M_1,..., M_{n-1}}$ can almost always be found such that $\mathbf{M_{n-1},..., M_2M_1A = U}$ is upper triangular [9]. The original $\mathbf{Ax} = \mathbf{b}$ problem is then equivalent to the upper triangular system $\mathbf{Ux} = (\mathbf{M_{n-1},..., M_2M_1})\ \mathbf{b}$ wich can be solved through back-substitution.

Suppose, then, that $\mathbf{A} \in \mathbf{R}^{\mathbf{nxn}}$ and that, for some *k<n*, we have determined the Gauss transformations

$\mathbf{M_1,..., M_{k-1}} \in \mathbf{R}^{\mathbf{nxn}}$ such that

$$\mathbf{A}^{(\mathbf{k-1})} \equiv \mathbf{M_{k-1}} \cdots \mathbf{M_1 A} = \begin{pmatrix} \mathbf{A_{11}}^{(\mathbf{k-1})} & \mathbf{A_{12}}^{(\mathbf{k-1})} \\ \mathbf{0} & \mathbf{A_{22}}^{(\mathbf{k-1})} \end{pmatrix} \begin{matrix} (k-1) \\ (n-k+1) \end{matrix}$$
$$\qquad (k-1) \qquad (n-k+1)$$

Where: $\mathbf{A_{11}}^{(\mathbf{k-1})}$ is an upper triangular matrix?

$$\text{Now, if } \mathbf{A_{22}}^{(\mathbf{k-1})} = \begin{pmatrix} a_{kk}^{(k-1)} & . & . & . & a_{kn}^{(k-1)} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ a_{nk}^{(k-1)} & . & . & . & a_{nn}^{(k-1)} \end{pmatrix}$$

and $a_{kk}^{(k-1)} \neq 0$, then the *multiplicators* :

$m_i = \dfrac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}; i = k+1,\ldots,n$; with $a_{kk} \neq 0$ are well defined.

So, we have the following

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Definition. An elementary lower triangular matrix of order n and index k is a matrix of the form [10]

$$\mathbf{M_k} = \mathbf{I_n} - \mathbf{m}\mathbf{e_k^T}$$

Where:

$$\mathbf{e_k^T} = (0,\ldots,0,\underset{k}{1},0,\ldots,0)^T, \quad \mathbf{I_n} = \begin{pmatrix} 1 & . & . & . & 0 \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ 0 & . & . & . & 1 \end{pmatrix}$$

$$\mathbf{m^T} = (\underbrace{0,0,0,\ldots,0}_{k-times}, m_{k+1},\ldots,m_n)$$

In general an elementary lower triangular matrix has the above form.

The computational significance of elementary lower triangular matrices is that they can be used to introduce zero components into a vector. Thus,

$$\mathbf{M_k} \cdot \begin{pmatrix} a_{11} \\ . \\ . \\ . \\ a_{k1} \\ a_{k+1,1} \\ . \\ . \\ . \\ a_{n1} \end{pmatrix} = \begin{pmatrix} a_{11} \\ . \\ . \\ . \\ a_{k1} \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}.$$

The matrix $\mathbf{M_k}$ is said to be a **GT.** The vector **m** is referred to as the Gauss vector. The components of **m** are known as multipliers. Then it follows that

$$\mathbf{A^{(k)}} \equiv \mathbf{M_k A^{(k-1)}} = \begin{pmatrix} \mathbf{A_{11}}^{(k)} & \mathbf{A_{12}}^{(k)} \\ \mathbf{0} & \mathbf{A_{22}}^{(k)} \end{pmatrix} \begin{matrix} (k) \\ (n-k) \end{matrix}$$
$$(k) \quad (n-k)$$

Where $\mathbf{A_{11}}^{(k)}$ is an upper triangular matrix?

This process illustrates the *k-th* step of the decomposition process, in which we used

$$(\mathbf{M_k} \cdots \mathbf{M_1})^{-1} = \mathbf{M_1}^{-1} \cdots \mathbf{M_k}^{-1} = \prod_{i=1}^{k} (\mathbf{I_n} + \mathbf{m^{(i)}e_i^T}) = \mathbf{I_n} + \sum_{i=1}^{k} \mathbf{m^{(i)}e_i^T}$$

We find the final expression for the decomposition process as

$$\mathbf{A} = \begin{pmatrix} \mathbf{L_{11}}^{(k)} & \mathbf{0} \\ \mathbf{L_{21}}^{(k)} & \mathbf{I_{n-k}} \end{pmatrix} \begin{pmatrix} \mathbf{I_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{A_{22}}^{(k)} \end{pmatrix} \begin{pmatrix} \mathbf{A_{11}}^{(k)} & \mathbf{A_{12}}^{(k)} \\ \mathbf{0} & \mathbf{I_{n-k}} \end{pmatrix} \Rightarrow$$

$$(\mathbf{M_k \ldots M_1})^{-1} \equiv \begin{pmatrix} \mathbf{L_{11}}^{(k)} & \mathbf{0} \\ \mathbf{L_{21}}^{(k)} & \mathbf{I_{n-k}} \end{pmatrix} = \mathbf{I_n} + (\mathbf{m^{(1)}},\ldots,\mathbf{m^{(k)}},0,\ldots,0).$$

In general, the forward elimination consists of *n-1* steps. At the *k-th* step, multiples of the *k-th* equations are subtracted from the remaining equations to eliminate the *k-th* variable. If the pivot element $a_{k,k}^{(k)}$ is null or "small", it is advisable to interchange equations before this is done through **P**, a permutation matrix that records the row exchanges as detailed below.

A. *LU Decomposition Theorem*

Using the above expression, the following can be established [11]:

Theorem. Let $\mathbf{A_k}$ denote the leader or main sub-matrix (k x k) of $\mathbf{A} \in \mathbf{R^{nxn}}$. If $\mathbf{A_k}$ is non-singular for $k=1,\ldots,n$; then there exist a lower triangular matrix $\mathbf{L} \in \mathbf{R^{nxn}}$ and an upper triangular matrix $\mathbf{U} \in \mathbf{R^{nxn}}$ so that **A=PLU.** Furthermore, $|\mathbf{A_k}| = u_{11} \ldots u_{kk} \, \forall k = 1,\ldots, n.$

III. DERIVATION OF A NEW LINEAR TRANSFORMATION

In the previous section we have defined an elementary lower triangular matrix of order n and index k as $\mathbf{M_k} = \mathbf{I_n} - \mathbf{m}\mathbf{e_k^T}$. We shall find the above expression and the next theorem**,** whose proof can be found in [10] useful.

Theorem. The pivot elements $a_{k,k}^{(k)}$ $(k = 1,\ldots, n)$, are nonzero if and only if the leading principal sub matrices $\mathbf{A_k}$ $(k = 1,\ldots, n)$, are non-singular.

On this basis, the following can be stated

Corollary. Let $\mathbf{A} \in \mathbf{R^{nxn}}$ be a matrix with $\mathbf{A_k}$ $(k = 1,\ldots, n)$ non-singular leading principal sub-matrices. Then, there exists a unique diagonal matrix for $k=1$ $(a_{00}^{(1)} = 1)$ whose entries are:

$$\mathbf{D_k} = \begin{pmatrix} 1_1 & & & & & & & & \\ & . & & & & & & & \\ & & . & & & & & & \\ & & & . & & & & & \\ & & & & 1_k & & & & \\ & & & & & \left(\dfrac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_{k+1} & & & \\ & & & & & & . & & \\ & & & & & & & . & \\ & & & & & & & & . & \\ & & & & & & & & & \left(\dfrac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_{n} \end{pmatrix}$$

Where $a_{k,k}^{(k)}$ and $a_{k-1,k-1}^{(k)}$ are the pivots elements.

Now, if we scale the matrix $\mathbf{M_k}$ with that diagonal matrix, we will have

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$$\mathbf{M}_{\mathbf{k}}^{'} = \mathbf{D}_{\mathbf{k}}\mathbf{M}_{\mathbf{k}} =$$

$$= \begin{pmatrix} 1_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & 1_k & & & \\ & & & & -\dfrac{a_{k+1,k}^{(k)}}{a_{k-1,k-1}^{(k)}} & \left(\dfrac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_{k+1} & & \\ & & & & & & \cdot & \\ & & & & \cdot & & \cdot & \\ & & & & \cdot & & & \cdot \\ & & & & \cdot & & & \\ & & & & -\dfrac{a_{n,k}^{(k)}}{a_{k-1,k-1}^{(k)}} & \cdot & \cdot\ \cdot & \left(\dfrac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_n \end{pmatrix}$$

and, once simplified, it can be re-expressed as

$$\mathbf{M}_{\mathbf{k}}^{'} = \frac{1}{a_{k-1,k-1}^{(k)}} \cdot \begin{pmatrix} \left(a_{k-1,k-1}^{(k)}\right)_1 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \left(a_{k-1,k-1}^{(k)}\right)_k & & \\ & & & & -a_{k+1,k}^{(k)} & \left(a_{k,k}^{(k)}\right)_{k+1} & \\ & & & & & & \cdot \\ & & & & \cdot & & \cdot \\ & & & & \cdot & & \cdot \\ & & & & \cdot & & \\ & & & & -a_{n,k}^{(k)} & \cdot\ \cdot\ \cdot & \left(a_{k,k}^{(k)}\right)_n \end{pmatrix} \quad (1)$$

By using this new transformation and applying the elimination process to a matrix **A**, all of whose entries are integers, all intermediate results are integers too, forming a number ring [12], since they are obtained through additions and products. Thus, $\mathbf{A} \in \mathbf{I}^{\mathbf{nxn}} \Rightarrow (\mathbf{M}^{'},\mathbf{U}) \in \mathbf{I}^{\mathbf{nxn}}$.

The products of these results times the factor $\dfrac{1}{a_{k-1,k-1}^{(k)}}$ are integers too, because previously, in the $k-1$ step of the elimination process, such results had been multiplied by $a_{k-1,k-1}^{(k)}$. This multiplication process leads to a simplification of the final result.

We can re-express this linear transformation as: $(a_{00}^{(1)} = 1)$

$$\mathbf{M}_{\mathbf{k}}^{'} = \begin{pmatrix} \dfrac{1}{a_{k,k}^{(k)}}\mathbf{I}_{\mathbf{k}} & 0 \\ 0 & \dfrac{1}{a_{k-1,k-1}^{(k)}}\mathbf{I}_{\mathbf{n-k}} \end{pmatrix} \left[a_{k,k}^{(k)}\mathbf{I} - \begin{pmatrix} 0_1 \\ \cdot \\ \cdot \\ \cdot \\ 0_k \\ a_{k+1,k}^{(k)} \\ \cdot \\ \cdot \\ \cdot \\ a_{n,k}^{(k)} \end{pmatrix} \mathbf{e}_{\mathbf{k}}^{\mathbf{T}} \right]; \forall k = 1,\ldots,n-1 \quad (2)$$

A. *New Decomposition Theorem through Determinants*

Using any of the above expressions, one can state

Theorem. Let $\mathbf{A} \in \mathbf{R}^{\mathbf{nxn}}$ and $\mathbf{M} = \prod_{\mathbf{k=n-1}}^{\mathbf{1}} \mathbf{M}_{\mathbf{k}}^{'}$. Then $\mathbf{M}$ is a lower triangular matrix all whose components are determinants and $\mathbf{U} = \mathbf{MA}$ also has all components in determinant form. Furthermore, $|\mathbf{A}| = u_{n,n}$

Proof. It follows from an induction on *n*. For *n=1* the theorem is trivially true since $\mathbf{M} = \mathbf{M}_{\mathbf{1}}^{'} = 1$ and $\mathbf{U} = \mathbf{MA} = a_{11}$.
For the induction step $(k = 1,\ldots,n-1)$ we have:

n=2
k=1

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \ \mathbf{M}_{\mathbf{1}}^{'} = \begin{pmatrix} 1 & 0 \\ -a_{21} & a_{11} \end{pmatrix}; \ \mathbf{M} = \mathbf{M}_{\mathbf{1}}^{'};$$

$$\mathbf{U} = \mathbf{MA} = \begin{pmatrix} 1 & 0 \\ -a_{21} & a_{11} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

n=3
k=1,2

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; \ \mathbf{M}_{\mathbf{1}}^{'} = \begin{pmatrix} 1 & 0 & 0 \\ -a_{21} & a_{11} & 0 \\ -a_{31} & 0 & a_{11} \end{pmatrix};$$

$$\mathbf{U}_{\mathbf{1}} = \mathbf{M}_{\mathbf{1}}^{'}\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \end{pmatrix};$$

$$\mathbf{M}_{2}^{'} = \frac{1}{a_{11}} \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix};$$

$$\mathbf{M} = \mathbf{M}_{2}^{'}\mathbf{M}_{1}^{'} = \begin{pmatrix} 1 & 0 & 0 \\ -a_{21} & a_{11} & 0 \\ -\begin{vmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix};$$

The last row of the matrix **M** multiplied for the last column of the matrix **A** is equivalent to Laplace Expansion of **A** taking out the last column. Then we have $u_{33} = |\mathbf{A}|$.

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$$\mathbf{U}=\mathbf{MA}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ 0 & 0 & \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{pmatrix};$$

n
k=1,2,…,n-1

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{1,n-1} & a_{2,n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix};$$

$$\mathbf{M}=\mathbf{M}'_{\mathbf{n-1}}\ldots\mathbf{M}'_{\mathbf{1}}=$$

$$=\begin{pmatrix} m_{11} & 0 & 0 & \cdots & 0 & 0 \\ m_{21} & m_{22} & 0 & \cdots & 0 & 0 \\ m_{31} & m_{32} & m_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n-1,n-1} & 0 \\ m_{n,1} & m_{n,2} & m_{n,3} & \cdots & m_{n,n-1} & m_{n,n} \end{pmatrix}$$

where

$$m_{11}=1\ ;\ m_{21}=-a_{21}\ ;\ m_{22}=a_{11};$$

$$m_{31}=-\begin{vmatrix} a_{31} & a_{32} \\ a_{21} & a_{22} \end{vmatrix};\ m_{32}=-\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix};\ m_{33}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix};$$

$$m_{n-1,1}=-\begin{vmatrix} a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} \end{vmatrix};$$

$$m_{n-1,2}=-\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} \end{vmatrix};$$

$$m_{n-1,3}=-\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} \end{vmatrix};$$

$$m_{n-1,n-1}=-\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} \end{vmatrix}$$

$$m_{n,1}=-\begin{vmatrix} a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix};$$

$$m_{n,2}=-\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix};$$

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$$m_{n,3}=-\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2,n-1} \\ a_{n,1} & a_{n,2} & a_{n,3} & . & . & . & a_{n,n-1} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & . & . & . & a_{n-1,n-1} \end{vmatrix};$$

$$m_{n,n-1}=-\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3,n-1} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n,1} & a_{n,2} & a_{n,3} & . & . & . & a_{n,n-1} \end{vmatrix};$$

$$m_{n,n}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3,n-1} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & . & . & . & a_{n-1,n-1} \end{vmatrix}$$

and

$$\mathbf{U}=\mathbf{MA}=\begin{pmatrix} u_{11} & u_{12} & u_{13} & . & . & . & u_{1,n-1} & u_{1,n} \\ 0 & u_{22} & u_{23} & . & . & . & u_{2,n-1} & u_{2,n} \\ 0 & 0 & u_{33} & . & . & . & u_{3,n-1} & u_{3,n} \\ . & . & . & . & & & . & . \\ . & . & . & & . & & . & . \\ . & . & . & & & . & . & . \\ 0 & 0 & 0 & . & . & . & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & 0 & . & . & . & 0 & u_{n,n} \end{pmatrix}$$

where

$u_{11}=a_{11}$; $u_{12}=a_{12}$; $u_{13}=a_{13}$; $u_{1,n-1}=a_{1,n-1}$;

$u_{1,n}=a_{1,n}$

$$u_{22}=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix};\ u_{23}=\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix};$$

$$u_{2,n-1}=\begin{vmatrix} a_{11} & a_{1,n-1} \\ a_{21} & a_{2,n-1} \end{vmatrix};u_{2,n}=\begin{vmatrix} a_{11} & a_{1,n} \\ a_{21} & a_{2,n} \end{vmatrix}$$

$$u_{33}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix};\ u_{3,n-1}=\begin{vmatrix} a_{11} & a_{12} & a_{1,n-1} \\ a_{21} & a_{22} & a_{2,n-1} \\ a_{31} & a_{32} & a_{3,n-1} \end{vmatrix};$$

$$u_{3,n}=\begin{vmatrix} a_{11} & a_{12} & a_{1,n} \\ a_{21} & a_{22} & a_{2,n} \\ a_{31} & a_{32} & a_{3,n} \end{vmatrix}$$

The Laplace Expansion of sub-matrix $\mathbf{A_{n-1,n-1}}$ taking out the last column, is equivalent to multiply the *(n-1)- th* row of the matrix **M** by the *(n-1)- th* column of **A**. Then we have $u_{n-1,n-1}=\left|\mathbf{A_{n-1,n-1}}\right|$ and

$$u_{n-1,n-1}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3,n-1} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & . & . & . & a_{n-1,n-1} \end{vmatrix};$$

In a similar way, we have

$$u_{n-1,n}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2,n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3,n} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & . & . & . & a_{n-1,n} \end{vmatrix};$$

Finally, taking the last row of **M** multiplied by the last column of **A** we have

$$u_{n,n}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2,n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3,n} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n,1} & a_{n,2} & a_{n,3} & . & . & . & a_{n,n} \end{vmatrix}$$

Now, if **M** is the new transformation, then we have

$$\mathbf{M}=\prod_{\mathbf{k=n-1}}^{\mathbf{1}}\mathbf{M}_{\mathbf{k}}^{'}\ ;\ \mathbf{U}=\mathbf{MA}$$

In order to solve the linear system of equations $\mathbf{Ax}=\mathbf{b}$, we have

$$\mathbf{MAx}=\mathbf{Mb}$$

$$\mathbf{Ux}=\mathbf{Mb}$$

We can use the “backward process” and solve the linear system of equations using only determinants.

For a matrix **A** with floating point entries this process requires

$$\frac{n(n-1)(2n-1)}{6}+\frac{(n-2)(n-1)}{2}=\frac{n^3}{3}-\frac{4}{3}n+1$$

Floating point multiplications [13].

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B. New Decomposition Theorem with Determinants with Total Pivoting

Gauss elimination in real numbers is unstable due to the possibility of finding arbitrarily small pivots. This process can be alleviated, however, by exchanging rows during the elimination. In our case, only when a pivot is zero, we exchange the row or column in a similar way to the Gauss process. The following theorem is given without proof:

Theorem. Let $\mathbf{A}\in\mathbf{R}^{\mathbf{nxn}}$. Suppose that the New Transformation $\mathbf{M}_1^{'}\cdots\mathbf{M}_{k-1}^{'}$; row's permutation matrices $\mathbf{P}=\mathbf{P}_1\cdots\mathbf{P}_{k-1}$ and column's permutation matrices $\mathbf{\Pi}=\mathbf{\Pi}_1\cdots\mathbf{\Pi}_{k-1}$ have been determined so that $\mathbf{U}=\mathbf{M}_{k-1}^{'}\mathbf{P}_{k-1}\cdots\mathbf{M}_1^{'}\mathbf{P}_1\mathbf{A}\mathbf{\Pi}_1\ldots\mathbf{\Pi}_{k-1}$. Then, the Upper Matrix **U** is obtained from $\mathbf{PA\Pi}$ without exchanging any rows or columns and $\mathbf{U}=\mathbf{MPA\Pi}$. Furthermore, if *each ≡ number of row's exchanges plus numbers of column's exchanges*, we have $|\mathbf{A}|=(-1)^{exch}u_{n,n}$

C. Matrix L and U

Now, although it is not strictly necessary, should we wish to obtain the matrix **L**, then the elements of the matrices L = (l_{ij}) and **U** = (u_{ij}) can be computed starting from the following formulas [14]:

$u_{1k} = a_{1k}$; $k = 1,\ldots, n$

$l_{j1} = \frac{a_{j1}}{u_{11}}$; $j = 2,\ldots, n$

$u_{jk} = (a_{jk} - \sum_{s=1}^{j-1} l_{js}\, u_{sk})\, u_{j-1\, j-1}$; $k = j,\ldots, n$; $j \geq 2$

$l_{jk} = \frac{1}{u_{kk}} (a_{jk} - \sum_{s=1}^{k-1} l_{js}\, u_{sk})$; $j = k,\ldots, n$; $k \geq 2$

IV. Exact Calculation of the Adjugate Matrix ($\mathbf{A}^{\mathbf{Adj}}$) with Another New Linear Transformation $\mathbf{M}_{\mathbf{J_k}}$

The adjugate is also called the adjoint. We avoid this usage because, in functional analysis, it refers to the equivalent of the conjugate transpose of a matrix. The adjugate $\mathbf{A}^{\mathbf{Adj}}$ of a matrix **A** is the transpose of the co-factor´s matrix of the elements of **A** . The computation of the adjugate, from its definition, involves the calculation of n^2 determinants of order $(n-1)$. On the other hand, the calculation from the formula $\mathbf{A}^{\mathbf{Adj}}=|\mathbf{A}|\cdot\mathbf{A}^{-1}$ breaks down when **A** is singular and is potentially unstable when **A** is ill-conditioned with respect to inversion [15].

Expressing the new Linear Transformation in Jordan version we have:

For $(a_{00}^{(1)}=1)$ and $k=1,\ldots,n$

$$\mathbf{M}_{\mathbf{J_k}} = \frac{1}{a_{k-1k-1}^{(k)}}\cdot\begin{pmatrix} a_{kk}^{(k)} & & & & -a_{1k}^{(k)} & & & \\ & \cdot & & & \cdot & & & \\ & & \cdot & & \cdot & & & \\ & & & \cdot & \cdot & & & \\ & & & a_{kk}^{(k)} & & & & \\ & & & & a_{k-1k-1}^{(k)} & & & \\ & & & & & a_{kk}^{(k)} & & \\ & & & & \cdot & & \cdot & \\ & & & & \cdot & & & \cdot \\ & & & & \cdot & & & & \cdot \\ & & & & -a_{nk}^{(k)} & & & & a_{kk}^{(k)} \end{pmatrix} \quad (3)$$

It is sufficient to do the operations indicated by the new Linear Transformation $\mathbf{M_J}:\mathbf{I}^{\mathbf{nxn}}\to\mathbf{I}^{\mathbf{nxn}}$ to find the adjugate matrix of size n : $\mathbf{M_J}=\prod_{\mathbf{k=n}}^{\mathbf{1}}\mathbf{M}_{\mathbf{J_k}}=\mathbf{A}^{\mathbf{Adj}}$.

Calculating the matrix with $\mathbf{M_J}$ proves highly efficient when working with the augmented matrix **(A|I) :**

$$\mathbf{M}_{\mathbf{J_n}}\cdot\mathbf{M}_{\mathbf{J_{n-1}}}\cdot\ldots\cdot\mathbf{M}_{\mathbf{J_1}}(\mathbf{A_{n,n}}\,|\,\mathbf{I_n})=(|\,\mathbf{A}\,|\cdot\mathbf{I_n}\,|\,\mathbf{A}_{\mathbf{n,n}}^{\mathbf{Adj}}).$$

This process demands

$$\frac{(n^2-n)(2n-1)}{6}+\sum_{i=1}^{n-1}i(n-i-1)+\frac{(n-2)(n-1)}{2}=\frac{n^3}{2}-\frac{n^2}{2}-\frac{3}{2}n+1$$

Floating point multiplications.

Another form to express the new transformation is

$$\mathbf{M}_{\mathbf{J_k}}=\begin{pmatrix}\frac{1}{a_{k-1k-1}^{(k)}}\cdot\mathbf{e}_1^{\mathbf{T}}\\ \cdot\\ \cdot\\ \cdot\\ \frac{1}{a_{kk}^{(k)}}\cdot\mathbf{e}_k^{\mathbf{T}}\\ \cdot\\ \cdot\\ \cdot\\ \frac{1}{a_{k-1k-1}^{(k)}}\cdot\mathbf{e}_n^{\mathbf{T}}\end{pmatrix}\cdot\left[a_{kk}^{(k)}\mathbf{I}-\begin{bmatrix}a_{1k}^{(k)}\\ a_{2k}^{(k)}\\ \cdot\\ \cdot\\ \cdot\\ 0_{kk}^{(k)}\\ a_{k+1}^{(k)}\\ \cdot\\ \cdot\\ \cdot\\ a_{nk}^{(k)}\end{bmatrix}\cdot\mathbf{e_k}^{\mathbf{T}}\right]\quad(4)\ \ (k=1,\ldots,n);$$

$a_{00}^{(1)}=1$

$$\mathbf{A}^{\mathbf{Adj}}=\mathbf{M_J}=\prod_{\mathbf{k=n}}^{\mathbf{1}}\mathbf{M}_{\mathbf{J_k}}$$

By means of both a minimum and complete pivoting strategy, we have

$$\mathbf{A}^{\mathbf{Adj}}=\mathbf{\Pi M_J P}$$

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Proof. Let it suffice to explain how the algorithm works with the following case.

Let $\mathbf{A} \in \mathbf{I}^{\mathbf{nxn}}$ and

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; \mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \mathbf{\Pi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_{\mathbf{J}_3}\mathbf{M}_{\mathbf{J}_2}\mathbf{M}_{\mathbf{J}_1}\mathbf{PA\Pi} = |\mathbf{A}|\mathbf{I}$$

$$\mathbf{M}_{\mathbf{J}} = |\mathbf{A}|(\mathbf{PA\Pi})^{-1}$$

$$\mathbf{M}_{\mathbf{J}} = \mathbf{\Pi}^{-1}|\mathbf{A}|\mathbf{A}^{-1}\mathbf{P}^{-1}$$

$$\mathbf{\Pi M}_{\mathbf{J}}\mathbf{P} = \mathbf{A}^{\mathbf{Adj}}$$

For k=1,…,3 ; $(a_{00}^{(1)} = 1)$ we have:

$$\mathbf{A}^{\mathbf{Adj}} = \mathbf{\Pi M}_{\mathbf{J}_3}\mathbf{M}_{\mathbf{J}_2}\mathbf{M}_{\mathbf{J}_1}\mathbf{P} =$$

$$= \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

A. *Solution of Ax=b with* $\mathbf{M}_{\mathbf{J}}$

The simultaneous linear equations systems can also be solved with this new linear transformation.

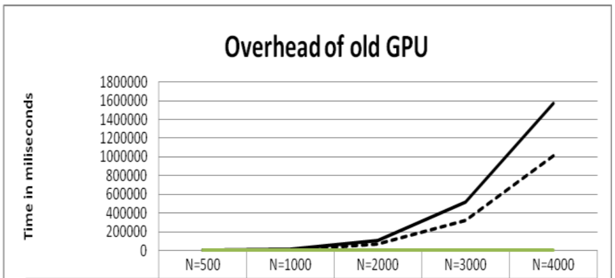
Since: $\mathbf{M}_{\mathbf{J}} = \prod_{\mathbf{k=n}}^{\mathbf{1}} \mathbf{M}_{\mathbf{J}_{\mathbf{k}}} = \mathbf{A}^{\mathbf{Adj}}$, then:

$$\mathbf{A}^{\mathbf{Adj}}\mathbf{Ax} = \mathbf{A}^{\mathbf{Adj}}\mathbf{b}; \; ; \; |\mathbf{A}|\mathbf{Ix} = \mathbf{A}^{\mathbf{Adj}}\mathbf{b} \; ; \; \mathbf{x} = \frac{\mathbf{A}^{\mathbf{Adj}}\mathbf{b}}{|\mathbf{A}|\mathbf{I}}$$

This result is a Cramer-type solution in $O(n^3)$. Consequently, this is a new result.

B. *Numeric Results*

We use two models of Tesla GPU: M2070-Q and K20c. We show the results in the following graph



	N=500	N=1000	N=2000	N=3000	N=4000
TESLA K20c	616.201	5217.65	67542.1	320821	1013020
TESLA_M2070-Q	1140	9659.87	110591	519390	1574420
% OVERHEAD	85.00%	85.14%	63.74%	61.89%	55.42%

V. CONCLUSION

In this paper we introduce a new theorem **LU** on the decomposition into determinants of matrix **A** and the new linear transformations, expressed as equations (1), (2), a modified Doolittle-Gauss elimination process. Additionally we expressed the equations (3) and (4), a modified Doolittle-Gauss-Jordan elimination process to calculate the adjugate matrix.

Most simultaneous linear equation systems can also be solved with these new linear transformations. The result is Cramer-type solutions in $O(n^3)$. This fact is new.

We have also proposed a modified Doolittle-Gauss-Jordan elimination process in two versions: the first one applied to the **A** matrix and the second to the augmented matrix **(A|b)** .

The first one is a new algorithm to compute determinants in exact form if and only if $\mathbf{A} \in \mathbf{I}^{\mathbf{n}}$, and the second is a new method to solve linear system of equations.

On the other hand, we have proposed a modified Doolittle-Gauss-Jordan elimination process in two version: the first one applied to the augmented matrix **(A|I)** and the second to the augmented matrix **(A|b)**. The first version is a new algorithm to calculate, in exact form, the Adjugate matrix, if $\mathbf{A} \in \mathbf{I}^{\mathbf{n}}$. The second version is a new direct method to solve linear system of equations in exact form if $(\mathbf{A},\mathbf{b}) \subset \mathbf{I}^{\mathbf{n}}$. Provided that $(\mathbf{A},\mathbf{b}) \subset \mathbf{R}^{\mathbf{n}}$ the above algorithms calculate, in approximate form, Cramer-type solutions of the linear systems of equations.

Gaussian elimination is usually the most economical way to solve $\mathbf{Ax} = \mathbf{b}$. Nevertheless there are three reasons why this new method might be relevant when the matrix´s coefficients are integers:

(1) The flop counts tend to exaggerate the Gaussian elimination advantage.

(2) The present method provides guaranteed stability; there is no “growth factor” to worry about as in Gaussian elimination.

(3) In cases of ill-conditioning, the reliability of the present method is unsurpassed when $\mathbf{A} \in \mathbf{I}^{\mathbf{n}}$.

Finally, when referring to the Cramer’s rule, it has been affirmed by G. Strang [16] that: “…Thus each component of *x* is a ratio of two determinants, a polynomial of degree *n* divided by another polynomial of degree *n*. This fact might have been recognized from Gauss elimination, but it never was”. This is made evident in the present paper.

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