Abstract: We obtain non-trivial integral values for the sides of the Pythagorean triangle such that
its area \( a \) is
\[
a(Hypotenuse) - 4a \left( \frac{\text{Area}}{\text{Perimeter}} \right) = \alpha^2.
\]
A few interesting relations between the sides of the Pythagorean triangle are presented.

Key words: Integral solutions, Pythagorean triangles, MSC classification number 11D09.

I. INTRODUCTION

One well known set of solutions of the Pythagorean equation
\[
x^2 + y^2 = z^2
\]
are
\[
x = 2ab, \ y = a^2 - b^2 \text{ and } z = a^2 + b^2.
\]
Many mathematicians have used this set of solutions to obtain the non-zero integral values for \( x, y \) and \( z \) [1-3]. As a new approach, in this paper we introduce another set of solutions
\[
x = 2A + 1, \ y = 2A^2 + 2A \text{ and } z = 2A^2 + 2A + 1
\]
for the equation \( x^2 + y^2 = z^2 \). By using this solution we obtain three non-zero integers \( x, y \) and \( z \) under certain relations satisfying the equation \( x^2 + y^2 = z^2 \) [4-6]. In this communication, we present yet another interesting Pythagorean triangle where in each of which the ratio \( a(Hypotenuse) - 4a \left( \frac{\text{Area}}{\text{Perimeter}} \right) \) may be expressed as a perfect square.

II. METHOD OF ANALYSIS

Taking \( A > 0 \) to be the generators of the Pythagorean triangle \((x, y, z)\), the assumption that
\[
a(Hypotenuse) - 4a \left( \frac{\text{Area}}{\text{Perimeter}} \right) = \alpha^2
\]
leads to the Pellian equation
\[
Y^2 = DX^2 + a
\]
where \( D = 2a \), not a perfect square and \( A = X \).

For the clear understanding we consider the following two cases:

i) \( a = 3 \) (odd number) so that \( D = 6 \)

ii) \( a = 4 \) (even number) so that \( D = 8 \)

Case (i):
When \( a = 3 \) the equation
\[
Y^2 = DX^2 + a
\]
becomes
\[
Y^2 = 6X^2 + 3
\]
Let \((x_0, y_0) = (1, 3)\) be the initial solution of (2). Consider the Pellian
\[
Y^2 = 6X^2 + 1
\]
Let \((\tilde{x}_0, \tilde{y}_0) = (2, 5)\) be a solution of equation (3).

Using Brahmagupta lemma the general solution \((\tilde{x}_n, \tilde{y}_n)\) of equation (3) is given by
\[
\tilde{y}_n + \sqrt{6}\tilde{x}_n = \left[ 5 + 2\sqrt{6} \right]^n
\]
where \( n = 0, 1, 2, 3, \ldots \) (4)

Since irrational roots occur in pairs
\[
\tilde{y}_n - \sqrt{6}\tilde{x}_n = \left[ 5 - 2\sqrt{6} \right]^n
\]
where \( n = 0, 1, 2, 3, \ldots \) (5)
is also a solution.

From equations (4) and (5), we obtain
\[
\tilde{y}_n = \frac{1}{2} \left[ \left( 5 + 2\sqrt{6} \right)^{n+1} + \left( 5 - 2\sqrt{6} \right)^{n+1} \right]
\]
and
\[
\tilde{x}_n = \frac{1}{2\sqrt{6}} \left[ \left( 5 + 2\sqrt{6} \right)^{n+1} - \left( 5 - 2\sqrt{6} \right)^{n+1} \right]
\]

Using the equations (6) and (7), the solutions of equation (2) is given by

\[
A_{n+1} = X_{n+1} = \frac{1}{2\sqrt{6}} \left[ (3 + \sqrt{6}) (5 + 2\sqrt{6})^{n+1} - (3 - \sqrt{6}) (5 - 2\sqrt{6})^{n+1} \right]
\]

\[
y_{n+1} = \frac{1}{2} \left[ (3 + \sqrt{6}) (5 + 2\sqrt{6})^{n+1} - (3 - \sqrt{6}) (5 - 2\sqrt{6})^{n+1} \right]
\]

n = 0, 1, 2, 3, . . .

III. NUMERICAL EXAMPLES

Observations:
1. The recurrence relations for X and Y are
   \[X_{n+3} - 10X_{n+2} + X_{n+1} = 0\]
   \[Y_{n+3} - 10Y_{n+2} + Y_{n+1} = 0\]
   For all values of n, both X and Y are odd.
2. For all values of n, both X and Y are divisible by 3.
3. 24X_{n+1}Y_{n+1} is difference of two squares.
4. \[X_{n+3} + X_{n+1} \equiv 0 (\text{mod} 10)\]
   \[Y_{n+3} + Y_{n+1} \equiv 0 (\text{mod} 10)\]

Case (ii):

When a = 4, the equation (1) leads to

\[Y^2 = 8X^2 + 4\]  
(8)

Let \((x_0, y_0) = (2, 6)\) be initial solution of equation (8).

To obtain the general solution of (8) consider the Pellian

\[Y^2 = 8X^2 + 1\]  
(9)

Let \((x_0, y_0) = (1, 3)\) be the initial solution of equation (9).

Then the general solution of equation (9) is given by

\[
\tilde{y}_n = \frac{1}{2} \left[ (3 + \sqrt{8})^{n+1} + (3 - \sqrt{8})^{n+1} \right]
\]

and

\[
\tilde{x}_n = \frac{1}{2\sqrt{8}} \left[ (3 + \sqrt{8})^{n+1} - (3 - \sqrt{8})^{n+1} \right]
\]

n = 0, 1, 2, 3, . . .

Therefore, the general solution of equation (8) is

\[
A_{n+1} = X_{n+1} = \frac{1}{\sqrt{8}} \left[ (3 + \sqrt{8})^{n+1} - (3 - \sqrt{8})^{n+1} \right]
\]

\[
Y_{n+1} = \left[ (3 + \sqrt{8})^{n+1} + (3 - \sqrt{8})^{n+1} \right]
\]

n = 0, 1, 2, 3, . . .

Numerical Examples

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<th>Y_{n+1}</th>
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Observations:
1. The recurrence relations for X and Y are
   \[X_{n+3} - 6X_{n+2} + X_{n+1} = 0\]
   \[Y_{n+3} - 6Y_{n+2} + Y_{n+1} = 0\]
   For all values of n, both X_{n+1} and Y_{n+1} are even.
3. For all values of n, Y_{n+1} is divisible by 3.
4. \[X_{n+3} + X_{n+1} \equiv 0 (\text{mod} 6)\]
   \[Y_{n+3} + Y_{n+1} \equiv 0 (\text{mod} 6)\]

REFERENCES