

# Thermal Stresses of a Hollow Cylinder due to Heat Generation: An Inverse Problem

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**ABSTRACT**-In this paper, an attempt has been made to study thermoelastic response of an inverse thermoelastic problem of a hollow cylinder occupying the space  $D: a \leq r \leq b, 0 \leq z \leq h$ , with radiation type boundary conditions. We apply integral transform technique to find the thermoelastic solution.

**Keywords:** Thermo elastic Response, hollow cylinder, Thermal Stresses, inverse problem.

## I. INTRODUCTION

Khobragade et al. [2-18] have investigated temperature distribution, displacement function, and stresses of a thin as well as thick hollow cylinder and Khobragade et al. [13] have established displacement function, temperature distribution and stresses of a semi-infinite cylinder.

In the present paper, an attempt is made to study the theoretical solution for a thermoelastic problem to determine the temperature distribution, displacement and stress functions of a hollow cylinder with boundary conditions occupying the space

$$D = \{(x, y, z) \in R^3 : a \leq (x^2 + y^2)^{1/2} \leq b, 0 \leq z \leq h\},$$

where  $r = (x^2 + y^2)^{1/2}$ . A transform defined by Zgrablich et al. [2] is used for investigation which is a generalization of Hankel's double radiation finite transform and used to treat the problem with radiation type boundaries conditions.

## II. FORMULATION OF THE PROBLEM

Consider a hollow cylinder as shown in the figure 1. The material of the cylinder is isotropic, homogenous and all properties are assumed to be constant. We assume that the cylinder is of a small thickness and its boundary surfaces remain traction free. The initial temperature of the cylinder is the same as the temperature of the surrounding medium, which is kept constant.

The displacement function  $\phi(r, z, t)$  satisfying the differential equation as Khobragade [2] is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \left( \frac{1+\nu}{1-\nu} \right) a_t T \quad (1)$$

$$\text{with } \phi = 0 \text{ at } r = a \text{ and } r = b \quad (2)$$

where  $\nu$  and  $a_t$  are Poisson ratio and linear coefficient of thermal expansion of the material of the cylinder respectively and  $T(r, z, t)$  is the heating temperature of the cylinder at time  $t$  satisfying the differential equation

as Khobragade [2] is

$$\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] + \frac{g(r, z, t)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (3)$$

where  $\kappa = K / \rho c$  is the thermal diffusivity of the material of the cylinder,  $K$  is the conductivity of the medium,  $C$  is its specific heat and  $\rho$  is its calorific capacity (which is assumed to be constant) respectively, subject to the initial and boundary conditions

$$M_r(T, 1, 0, 0) = F(r, z) \text{ for all } a \leq r \leq b, 0 \leq z \leq h \quad (4)$$

$$M_r(T, 1, \bar{k}_1, a) = F_1(z, t), \text{ for all } 0 \leq z \leq h, t > 0 \quad (5)$$

$$M_r(T, 1, \bar{k}_2, b) = F_2(z, t) \text{ for all } 0 \leq z \leq h, t > 0 \quad (6)$$

$$M_z(T, 0, 1, 0) = F_3(r, t) \text{ for all } a \leq r \leq b, t > 0 \quad (7)$$

$$M_z(T, 0, 1, \xi) = F_4(r, t) \text{ for all } a \leq r \leq b, t > 0 \quad (8)$$

$$M_z(T, 1, 0, h) = G(r, t) \text{ (Unknown)}$$

$$\text{for all } a \leq r \leq b, t > 0 \quad (9)$$

being:

$$M_g(f, \bar{k}, \bar{k}, s) = (\bar{k} f + \bar{k} \hat{f})_{g=s}$$

where the prime ( $\wedge$ ) denotes differentiation with respect to  $g$ , radiation constants are  $\bar{k}$  and  $\bar{k}$  on the curved surfaces of the plate respectively.

The displacement function in the cylindrical coordinate system are represented by Love's function as Khobragade [4] are

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 L}{\partial r \partial z} \quad (10)$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu) \nabla^2 L - \frac{\partial^2 L}{\partial z^2} \quad (11)$$

The Love's function [14] must satisfy

$$\nabla^2 \nabla^2 L = 0 \quad (12)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The component of stresses are represented by the thermoelastic displacement potential  $\phi$  and Love's function  $L$  as Noda [3] are

$$\sigma_{rr} = 2G \left\{ \left( \frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi \right) + \frac{\partial}{\partial z} \left( \nu \nabla^2 L - \frac{\partial^2 L}{\partial r^2} \right) \right\} \quad (13)$$

$$\sigma_{\theta\theta} = 2G \left\{ \left( \frac{1}{r} \frac{\partial \phi}{\partial r} - \nabla^2 \phi \right) + \frac{\partial}{\partial z} \left( \nu \nabla^2 L - \frac{1}{r} \frac{\partial^2 L}{\partial r^2} \right) \right\} \quad (14)$$

$$\sigma_{zz} = 2G \left\{ \left( \frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi \right) + \frac{\partial}{\partial z} \left\{ (z - \nu) \nabla^2 L - \frac{\partial^2 L}{\partial z^2} \right\} \right\} \quad (15)$$

$$\sigma_{rz} = 2G \left\{ \frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left\{ (1 - \nu) \nabla^2 L - \frac{\partial^2 L}{\partial z^2} \right\} \right\} \quad (16)$$

For traction free surface stress function

$$\sigma_z = \sigma_{r\theta} = 0 \text{ at } z = \pm h \text{ for thick plate.}$$

Equations (1) to (16) constitute the mathematical formulation of the problem under consideration.

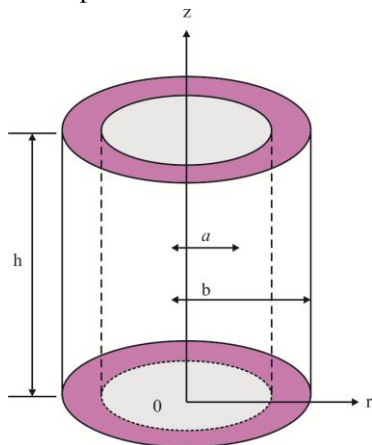


Fig 1: Geometry of the problem

### III. SOLUTION OF THE OF THE PROBLEM

Applying March Zgrablich transform and finite Fourier cosine transform on equation (3) one obtains

$$\frac{d\bar{T}^*}{dt} + \alpha p^2 \bar{T}^* = \Psi \quad (17)$$

$$\text{where } p^2 = \mu_m^2 + \frac{n^2 \pi^2}{\xi^2}$$

$$\begin{aligned} \Psi = \alpha \left[ \frac{b}{k_2} S_0(k_1, k_2, \mu_m b) F_2^* - \frac{a}{k_1} S_0(k_1, k_2, \mu_m a) F_1^* \right. \\ \left. + (-1)^n \bar{F}_4 - \bar{F}_3 + \frac{\bar{g}^*}{k} \right]. \end{aligned}$$

Equation (17) is a first order linear differential equation whose solution is given by

$$\bar{T}^* = e^{-\alpha p^2 t} \left( \bar{F}^* + \int_0^t \Psi e^{\alpha p^2 t'} dt' \right) \quad (18)$$

Applying inversion of Fourier cosine transform and Marchi-Zgrablich transform, one obtain the expression for temperature distribution and unknown temperature gradient respectively as

$$\begin{aligned} T = \frac{2}{\xi} \sum_{m,n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_m r)}{\mu_m} \cos\left(\frac{n\pi z}{\xi}\right) \\ \times e^{-\alpha p^2 t} \left( \bar{F}^* + \int_0^t \Psi e^{\alpha p^2 t'} dt' \right) \end{aligned} \quad (19)$$

$$\begin{aligned} G(r, t) = \frac{2}{\xi} \sum_{m,n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_m r)}{\mu_m} \cos\left(\frac{n\pi h}{\xi}\right) \\ \times e^{-\alpha p^2 t} \left( \bar{F}^* + \int_0^t \Psi e^{\alpha p^2 t'} dt' \right) \end{aligned} \quad (20)$$

Using equation (22) in (1) we get

$$\begin{aligned} \phi(r, z, t) = \frac{r^2 a t}{2\xi} \left( \frac{1+\nu}{1-\nu} \right) \sum_{m,n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_m r)}{\mu_m} \cos\left(\frac{n\pi z}{\xi}\right) \\ \times e^{-\alpha p^2 t} \left( \bar{F}^* + \int_0^t \Psi e^{\alpha p^2 t'} dt' \right) \end{aligned} \quad (21)$$

Using equation (24) in (12) and (13) one obtains

$$\begin{aligned} U = \frac{a_t}{2\xi} \left( \frac{1+\nu}{1-\nu} \right) \sum_{m,n=1}^{\infty} \frac{2r S_0(k_1, k_2, \mu_m r) + r^2 S_0'(k_1, k_2, \mu_m r)}{\mu_m} \\ \times \cos\left(\frac{n\pi z}{\xi}\right) e^{-\alpha p^2 t} \left( \bar{F}^* + \int_0^t \Psi e^{\alpha p^2 t'} dt' \right). \end{aligned} \quad (22)$$

$$\begin{aligned} W = \frac{-n\pi r^2 a_t}{2\xi^2} \left( \frac{1+\nu}{1-\nu} \right) \sum_{m,n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_m r)}{\mu_m} \sin\left(\frac{n\pi z}{\xi}\right) \\ \times e^{-\alpha p^2 t} \left( \bar{F}^* + \int_0^t \Psi e^{\alpha p^2 t'} dt' \right) \end{aligned} \quad (23)$$

Using expressions (22) and (23) in (16) to (19) one obtains

$$\begin{aligned} \sigma_r = \frac{a_t}{2\xi^3} \left( \frac{1+\nu}{1-\nu} \right) \sum_{m,n=1}^{\infty} \left\{ \xi^2 (\lambda + 2G) \left[ r^2 S_0''(k_1, k_2, \mu_m r) \right. \right. \\ \left. \left. + 4r S_0'(k_1, k_2, \mu_m r) + 2S_0(k_1, k_2, \mu_m r) \right] \right. \\ \left. + \lambda \left[ (2\xi^2 - n^2 \pi^2 r^2) S_0(k_1, k_2, \mu_m r) + r S_0'(k_1, k_2, \mu_m r) \right] \right\} \end{aligned}$$

$$\times \cos\left(\frac{n\pi z}{\xi}\right) \frac{e^{-\alpha p^2 t}}{\mu_m} \left( \bar{F}^* + \int_0^t \Psi e^{\alpha p^2 t'} dt' \right) \quad (24)$$

$$\begin{aligned} \sigma_z = & \frac{-a_t}{2\xi^3} \left( \frac{1+\nu}{1-\nu} \right) \sum_{m,n=1}^{\infty} \left\{ (\lambda + 2G)n^2 \pi^2 r^2 S_0(k_1, k_2, \mu_m r) \right. \\ & \left. - \lambda \xi^2 \left[ r^2 S''(k_1, k_2, \mu_m r) + 5r S'_0(k_1, k_2, \mu_m r) + 4r S_0(k_1, k_2, \mu_m r) \right] \right\} \\ & \times \cos\left(\frac{n\pi z}{\xi}\right) \frac{e^{-\alpha p^2 t}}{\mu_m} \left( \bar{F}^* + \int_0^t \Psi e^{\alpha p^2 t'} dt' \right). \quad (25) \end{aligned}$$

$$\begin{aligned} \sigma_\theta = & \frac{a_t}{2\xi^3} \left( \frac{1+\nu}{1-\nu} \right) \sum_{m,n=1}^{\infty} \left\{ (\lambda + 2G)\xi^2 \left[ r S'_0(k_1, k_2, \mu_m r) + 2S_0(k_1, k_2, \mu_m r) \right] \right. \\ & \left. + \lambda \left[ r^2 \xi^2 S''_0(k_1, k_2, \mu_m r) + 4r \xi^2 S'_0(k_1, k_2, \mu_m r) + (2\xi^2 - n^2 \pi^2 r^2) \right] \right. \\ & \left. \times S_0(k_1, k_2, \mu_m r) \right\} \cos\left(\frac{n\pi z}{\xi}\right) \frac{e^{-\alpha p^2 t}}{\mu_m} \left( \bar{F}^* + \int_0^t \Psi e^{\alpha p^2 t'} dt' \right) \quad (26) \end{aligned}$$

$$\begin{aligned} \tau_{rz} = & \frac{-G\pi a_t}{\xi^2} \left( \frac{1+\nu}{1-\nu} \right) \sum_{m,n=1}^{\infty} n \left[ r^2 S'_0(k_1, k_2, \mu_m r) + 2r S_0(k_1, k_2, \mu_m r) \right] \\ & \times \sin\left(\frac{n\pi z}{\xi}\right) \frac{e^{-\alpha p^2 t}}{\mu_m} \left( \bar{F}^* + \int_0^t \Psi e^{\alpha p^2 t'} dt' \right) \quad (27) \end{aligned}$$

#### IV. SPECIAL CASE

$$\text{Set } f(r, t) = (1 - e^{-t}) \delta(r - r_0) \quad (28)$$

Applying finite transform defined in Marchi Zgrablich [2] to the equation (28) one obtains

$$\bar{f}(n, t) = (1 - e^{-t}) r_0 S_0(k_1, k_2, \mu_n r_0) \quad (29)$$

Substituting the value of (29) in the equation (21) to obtain

$$\begin{aligned} T(r, z, t) = & \frac{2}{\xi} \sum_{n=1}^{\infty} \frac{1}{C_n} \sum_{m=0}^{\infty} \frac{\lambda_m \sin(\lambda_m z)}{\cos(\lambda_m \xi)} \\ & \times S_0(k_1, k_2, \mu_n r) \\ & \times \int_0^t (1 - e^{-t'}) r_0 S_0(k_1, k_2, \mu_n r_0) e^{-\alpha p^2 (t-t')} dt' \quad (30) \end{aligned}$$

#### V. CONCLUSION

In this paper, we modify the conceptual idea proposed by Khobragade et al [2] for hollow cylinder and the temperature distributions, displacement and stress functions at the edge  $z = h$  occupying the region of the cylinder  $a \leq r \leq b$ ,  $0 \leq z \leq h$  have been obtained with the known boundary conditions. We develop the analysis for the temperature field by introducing the transformation defined by Zgrablich et al, finite Fourier sine transform and Laplace transform techniques with boundaries conditions of radiations type. The series

solutions converge provided we take sufficient number of terms in the series. Since the thickness of cylinder is very small, the series solution given here will be definitely convergent. Assigning suitable values to the parameters and functions in the series expressions can derive any particular case. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications.

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