

Thermal Stress Analysis of a Thick Rectangular Plate: An Inverse Problem

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ABSTRACT-This paper is concerned with inverse transient thermoelastic problem of a thin clamped rectangular plate in which we need to determine the temperature distribution, unknown temperature gradient, thermal stresses and deflection when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem.

KEY WORDS: Thin rectangular plate, transient problem, inverse thermo elastic problem, deflection.

I. INTRODUCTION

Khobragade et al. [1, 2] have derived thermal deflection of a thick clamped rectangular plate, Khobragade et al. [5, 6, 8-10] have investigated displacement function, temperature distribution and stresses of a thin rectangular plate and Khobragade et al. [11] have established displacement function, temperature distribution and stresses of a thick rectangular plate.

In the present paper, an attempt is made to determine the temperature distribution, unknown temperature gradient, thermal stresses and deflection of the plate occupying the space $D: \{(x, y, z) \in R^3 : 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq h\}$ with the known boundary conditions.

Finite Fourier sine transform and Laplace transform techniques are used to find the solution of the problem. Numerical estimate for the temperature distribution is obtained and depicted graphically.

II. STATEMENT OF THE PROBLEM

Consider a thin isotropic rectangular plate occupying the space D . The differential equation satisfied by the deflection $\omega(x, y, t)$ as Khobragade et al. [1] is

$$D\nabla^4 \omega(x, y, t) = \frac{-\nabla^2 M_T(x, y, t)}{1-\nu} \quad (1)$$

where,

ν is the Poisson's ratio of the plate material,

M_T denote the thermal momentum of the plate and

D denote the flexural rigidity,

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2}$$

The resultant thermal momentum M_T is defined as

$$M_T(x, y, t) = \alpha E \int_0^h z T(x, y, z, t) dz \quad (2)$$

where

α, E are the linear coefficient of thermal expansion of the material, and Young's modulus respectively.

Since the edge of the rectangular plate is fixed and clamped,

$$\omega = \frac{\partial^2 \omega}{\partial x^2} = \frac{\partial^2 \omega}{\partial y^2} = 0 \text{ at } x = a \text{ and } y = b \quad (3)$$

The temperature of the plate at time t satisfying the differential equation as Nowacki [15] is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g(x, y, z, t)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (4)$$

where k is the thermal diffusivity of the material of the plate,

subject to the initial and boundary conditions:

$$T(x, y, z, 0) = 0 \quad (5)$$

$$\left[T(x, y, z, t) + k_1 \frac{\partial T(x, y, z, t)}{\partial x} \right]_{x=-a} = f_1(y, z, t) \quad (6)$$

$$\left[T(x, y, z, t) + k_2 \frac{\partial T(x, y, z, t)}{\partial x} \right]_{x=a} = f_2(y, z, t) \quad (7)$$

$$\left[\frac{\partial T(x, y, z, t)}{\partial y} \right]_{y=0} = g_1(x, z, t) \quad (8)$$

$$\left[\frac{\partial T(x, y, z, t)}{\partial y} \right]_{y=b} = g_2(x, z, t) \quad (9)$$

$$[T(x, y, z, t)]_{z=0} = 0 \quad (10)$$

$$[T(x, y, z, t)]_{z=\xi} = g(x, y, t) \quad (11)$$

$$[T(x, y, z, t)]_{z=h} = f(x, y, t) \text{ (Unknown)} \quad (12)$$

The displacement components u_x and u_y, u_z in the x and y and z directions respectively as Tanigawa et al. [1] are

$$u_x = \int_{-a}^a \left[\frac{1}{E} \left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \lambda T \right] dx \quad (13)$$

$$u_y = \int_0^b \left[\frac{1}{E} \left(\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \lambda T \right] dy \quad (14)$$

$$u_z = \int_0^h \left[\frac{1}{E} \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial z^2} \right) + \lambda T \right] dz \quad (15)$$

where E, ν , and λ are the young's modulus, Poisson's ratio and the linear coefficient of the thermal expansion of the material of the beam respectively and $U(x, y, z, t)$ is the Airy's stress functions which satisfy the differential

equation as Tanigawa et al. [1] is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U(x, y, z, t) = -\lambda E \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \times T(x, y, z, t) \quad (16)$$

The stress components in terms of U(x, y, z, t) Tanigawa et al. [1] are given by

$$\sigma_{xx} = \left[\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right] \quad (17)$$

$$\sigma_{yy} = \left[\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2}\right] \quad (18)$$

$$\sigma_{zz} = \left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right] \quad (19)$$

Equations (1) to (19) constitute the mathematical formulation of the problem under consideration.

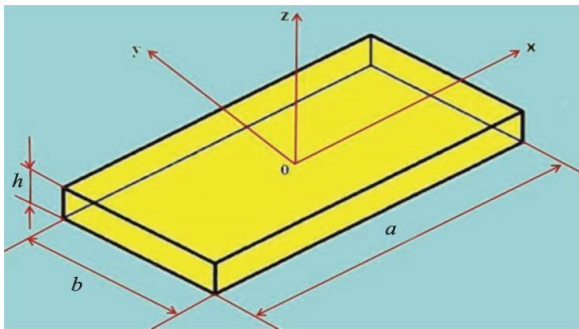


Fig 1: Geometry of the problem

III. SOLUTION OF THE PROBLEM

Applying Marchi-Fasulo, transform and finite Fourier cosine Transform and Fourier sine Transform to the equations one obtains.

$$\frac{d\bar{T}}{dt} + \alpha q^2 \bar{T} = \psi \quad (20)$$

Where

$$q^2 = \mu_n^2 + \frac{m^2 \pi^2}{b^2} + \frac{p^2 \pi^2}{\xi^2} \text{ and}$$

$$\psi = \alpha \left[\frac{P_n(a)}{k_2} \bar{f}_2^* - \frac{P_n(-a)}{k_1} \bar{f}_1^* + (-1)^m \bar{g}_2^* - \bar{g}_1^* + (-1)^{p+1} \left(\frac{p\pi}{\xi} \right) \bar{g}_3^* + \frac{g}{k} \right] \quad (21)$$

Solution of equation (20) is given by

$$\bar{T}^* = e^{-\alpha q^2 t} \left(\int_0^t \Psi e^{\alpha q^2 t'} dt' \right)$$

Applying inversion of Fourier sine, Fourier cosine and

Marchi-Fasulo transform we get the temperature distribution and unknown temperature gradient as

$$T(x, y, z, t) = \frac{4}{b\xi} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_n(x)}{\mu_n} \cos\left(\frac{m\pi y}{b}\right) \times \sin\left(\frac{p\pi z}{\xi}\right) \times \int_0^t \psi e^{-\alpha q^2 (t-t')} dt' \quad (22)$$

$$f(x, y, t) = \frac{4}{b\xi} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_n(x)}{\mu_n} \cos\left(\frac{m\pi y}{b}\right) \times \sin\left(\frac{p\pi h}{\xi}\right) \times \int_0^t \psi e^{-\alpha q^2 (t-t')} dt' \quad (23)$$

Where p, m, n are the positive integers

Using equation (22) in equation (16) we get

$$U(x, y, z, t) = \frac{-4(1+\nu)a_t}{q^2 b\xi} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_n(x)}{\mu_n} \cos\left(\frac{m\pi y}{b}\right) \times \sin\left(\frac{p\pi z}{\xi}\right) \times \int_0^t \psi e^{-\alpha q^2 (t-t')} dt' \quad (24)$$

Using equation (24) in equations (13) to (15) we get

$$u_x = \frac{4}{b\xi} \int_{-a}^a \sum_{m,n,p} \left\{ \frac{(1+\nu)a_t}{E\mu_n q^2} \left[\left(\frac{m^2 \pi^2}{b^2} - \frac{p^2 \pi^2}{\xi^2} \right) P_n(x) - \nu P_n''(x) \right] + \frac{P_n(x)}{\mu_n} \right\} \times \cos\left(\frac{m\pi y}{b}\right) \times \sin\left(\frac{p\pi z}{\xi}\right) \times \left(\int_0^t \psi e^{-\alpha q^2 (t-t')} dt' \right) dx \quad (25)$$

$$u_y = \frac{4}{b\xi} \int_0^b \sum_{m,n,p} \left\{ \frac{(1+\nu)a_t}{E\mu_n q^2} \left[\left(\frac{p^2 \pi^2}{\xi^2} - \frac{\nu m^2 \pi^2}{b^2} \right) P_n(x) - \nu P_n''(x) \right] + \frac{\lambda P_n(x)}{\mu_n} \right\} \times \cos\left(\frac{m\pi y}{b}\right) \times \sin\left(\frac{p\pi z}{\xi}\right) \times \left(\int_0^t \psi e^{-\alpha q^2 (t-t')} dt' \right) dy \quad (26)$$

$$u_z = \frac{4}{b\xi} \int_0^{\xi} \sum_{m,n,p} \left\{ \frac{(1+\nu)a_t}{E\mu_n q^2} \left[\left(\frac{m^2 \pi^2}{b^2} - \frac{\nu p^2 \pi^2}{\xi^2} \right) P_n(x) - P_n''(x) \right] + \frac{\lambda P_n(x)}{\mu_n} \right\} \times \cos\left(\frac{m\pi y}{b}\right) \times \sin\left(\frac{p\pi z}{\xi}\right) \times \left(\int_0^t \psi e^{-\alpha q^2 (t-t')} dt' \right) dz \quad (27)$$

Using equation (24) in equations (17) to (19) we get

$$\sigma_{xx} = \frac{4\pi^2(1+\nu)a_t}{q^2b^3\xi^3} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\xi^2m^2 + p^2b^2)}{\mu_n} P_n(x) \times \cos\left(\frac{m\pi y}{b}\right) \times \sin\left(\frac{p\pi z}{\xi}\right) \times \int_0^t \psi e^{-\alpha q^2(t-t')} dt' \quad (28)$$

$$\sigma_{yy} = \frac{-4(1+\nu)a_t}{q^2b\xi^3} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\xi^2P_n''(x) - p^2\pi^2P_n(x))}{\mu_n} \times \cos\left(\frac{m\pi y}{b}\right) \times \sin\left(\frac{p\pi z}{\xi}\right) \times \int_0^t \psi e^{-\alpha q^2(t-t')} dt' \quad (29)$$

$$\sigma_{zz} = \frac{-4(1+\nu)a_t}{q^2b^3\xi} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(b^2P_n''(x) - m^2\pi^2P_n(x))}{\mu_n} \times \cos\left(\frac{m\pi y}{b}\right) \times \sin\left(\frac{p\pi z}{\xi}\right) \times \int_0^t \psi e^{-\alpha q^2(t-t')} dt' \quad (30)$$

IV. DETERMINATION OF THERMAL DEFLECTION

Substituting the value of temperature distribution $T(x, y, z, t)$ from equation (22) in equation (2), one obtains

$$M_T(x, y, t) = \frac{4\alpha_t E \xi}{\pi} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{p+1}}{P} \frac{P_n(x)}{\mu_n} \cos\left(\frac{m\pi y}{b}\right) \times \int_0^t \psi e^{-\alpha q^2(t-t')} dt' \quad (31)$$

We assume that the solution of equation (1) satisfying equation (3) as

$$\omega(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}(t) \cos\left(\frac{m\pi y}{b}\right) \quad (32)$$

Using the equations (31) and (32) in (1), one obtains

$$c_{mn}(t) = \frac{4\alpha_t E \xi}{m^4 \pi^5 b^6} \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{P} (b^2 P_n''(x) - m^2 \pi^2) \times \int_0^t \psi e^{-\alpha q^2(t-t')} dt' \quad (33)$$

Substituting the value of $\omega_{mn}(t)$ in equation (32), one obtains the expression for thermal deflection as

$$\omega(x, y, t) = \frac{4\alpha_t E \xi}{b^6 \pi^5} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{p+1}}{m^4 P \mu_n} \times (b^2 P_n''(x) - m^2 \pi^2) \cos\left(\frac{m\pi y}{b}\right) \times \int_0^t \psi e^{-\alpha q^2(t-t')} dt' \quad (34)$$

V. SPECIAL CASE AND NUMERICAL RESULTS

Set $g(x, y, t) = (1 - e^{-t})(x^2 - ax)(y^2 - by)$, $a = 1$, $b = 2$, $h = 2$, $t = 1$ sec $\xi = 1.5$ and $k = 0.86$

in equations (22) we get

$$T(x, y, z, t) = (1.33) \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{P_n(x)}{\mu_n} \cos\left(\frac{m\pi y}{2}\right) \times \sin\left(\frac{p\pi z}{1.5}\right) \times \int_0^1 \psi e^{-\alpha q^2(1-t')} dt' \quad (35)$$

VI. CONCLUSION

The temperature distribution, unknown temperature gradient, displacement function and thermal deflection of a thin rectangular plate have been obtained, with the aid of finite Fourier sine transform and Laplace transform techniques when the stated boundary conditions are known. The results are obtained in the form of infinite series. The series solutions converge provided if we take sufficient number of terms in the series.

The expressions that are obtained can be applied to the design of useful structures or machines in engineering applications.

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