ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 5, Issue 3, September 2015

Thermal Stresses of Semi Infinite Rectangular Beam with Internal Heat Source: Steady-State Problem

R. N. Pakade, Sachin Chauthale, and N. W. Khobragade
Department of Mathematics, MJP Educational Campus,
RTM Nagpur University, Nagpur 440 033, India.

Abstract- ***This paper is concerned with inverse transient thermoelastic problem in which we need to determine the temperature distribution, displacement function and thermal stresses of a semi-infinite rectangular beam when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem. The results are depicted graphically.***

Key Words: **Semi-infinite rectangular beam, inverse transient problem, Integral transform.**

I. INTRODUCTION

Khobragade et al. [2-7, 9] have investigated temperature distribution, displacement function, and stresses of a thin rectangular plate and **Khobragade et al**. [8] have established displacement function , temperature distribution and stresses of a semi-infinite rectangular beam.

In this Section, an attempt has been made to determine the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of a semi-infinite rectangular beam occupying the region D : $-a \le x \le a$;$0 \le y \le b$, $0 \le z \le \infty$ with known boundary conditions. Here Marchi-Fasulo transforms and Fourier cosine transform techniques have been used to find the solution.

II. STATEMENT OF THE PROBLEM

Consider a thin rectangular plate occupying the space D: $a \le x \le a$; $0 \le y \le b$, $0 \le z \le \infty$. The displacement components u_x and u_y u_z in the x and y and z directions respectively as **Tanigawa et al.** [1] are

$$u_x = \int_{-a}^{a}\left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \nu\frac{\partial^2 U}{\partial x^2}\right) + \lambda T\right]dx \tag{1}$$

$$u_y = \int_{0}^{b}\left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} - \nu\frac{\partial^2 U}{\partial y^2}\right) + \lambda T\right]dy \tag{2}$$

$$u_z = \int_{0}^{\infty}\left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \nu\frac{\partial^2 U}{\partial z^2}\right) + \lambda T\right]dz \tag{3}$$

where E, ν, and λ are the young's modulus, Poisson's ratio and the linear coefficient of the thermal expansion of the material of the beam respectively and U (x,y,z) is the Airy's stress functions which satisfy the differential equation as **Tanigawa et al.** [1] is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^2 U(x,y,z) = -\lambda E\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) T(x,y,z) \tag{4}$$

where T(x,y,z) denotes the temperature of a rectangular beam satisfy the following differential equation as Tanigawa et al. [1] is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g(x,y,z)}{k} = 0 \tag{5}$$

where k is the thermal conductivity and α is the thermal diffusivity of the material, subject to the boundary conditions

$$\left[T(x,y,z) + k_1\frac{\partial T(x,y,z)}{\partial x}\right]_{x=a} = f_1(y,z) \tag{6}$$

$$\left[T(x,y,z) + k_2\frac{\partial T(x,y,z)}{\partial x}\right]_{x=-a} = f_2(y,z) \tag{7}$$

$$\left[\frac{\partial T(x,y,z)}{\partial y}\right]_{y=0} = f_3(x,z) \tag{8}$$

$$\left[\frac{\partial T(x,y,z)}{\partial y}\right]_{y=\xi} = f_4(x,z) \tag{9}$$

$$\left[T(x,y,z)\right]_{y=b} = G(x,z) \quad \text{(Unknown)} \tag{10}$$

$$\left[T(x,y,z)\right]_{Z=0} = 0 \tag{11}$$

$$\left[T(x,y,z)\right]_{Z=\infty} = 0 \tag{12}$$

The stress components in terms of U(x, y, z) **Tanigawa et al**. [1] are given by

$$\sigma_{xx} = \left[\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right] \tag{13}$$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 5, Issue 3, September 2015

$$\sigma_{yy} = \left[\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2}\right] \quad (14)$$

$$\sigma_{zz} = \left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right] \quad (15)$$

The equations (1) to (15) constitute the mathematical formulation of the problem under consideration.

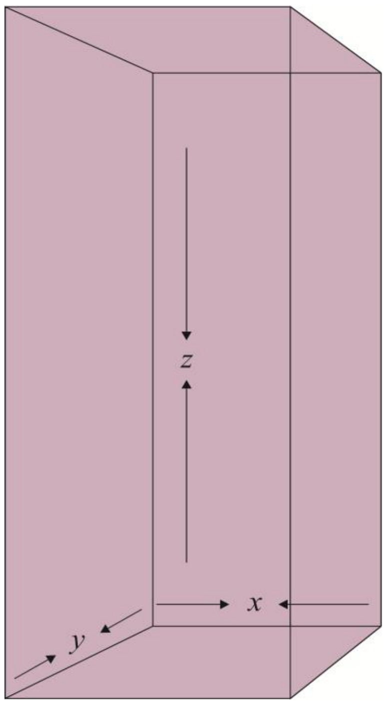

Fig 1: Geometry of the problem

III. SOLUTION OF THE PROBLEM

Applying finite Marchi-Fasulo transform and Fourier sine transform to the equation (5), we get

$$\frac{d\bar{T}^*}{dt} - q^2\bar{T}^* = \Psi \quad (16)$$

Where, $q^2 = \lambda_n^2 + p^2$

$$\psi = \frac{P_n(-a)}{k_2} f_2^* - \frac{P_n(a)}{k_1} f_1^* - \frac{\bar{g}^*}{k}$$

Equation (16) is a linear equation whose solution is given by

$$\bar{T}^* = Ae^{qy} + Be^{-qy} + F(y) \quad (17)$$

Where $F(y)$ is the P.I.

Using boundary conditions (8) and (9) we get

$$A = \frac{e^{-q\xi}\left(F'(0) - \bar{f}_3^*\right) + F'(\xi)}{2q\sinh(q\xi)}$$

$$B = \frac{e^{q\xi}\left(F'(0) - \bar{f}_3^*\right) + \bar{f}_4^* - F'(\xi)}{2q\sinh(q\xi)}$$

Substituting the values of A and B in equation (17) one obtains

$$\bar{T}^* = \frac{[\left(F'(0) - \bar{f}_3^*\right)\cosh\left(q(y-\xi)\right) + \left(\bar{f}_4^* - F'(\xi)\right)\cosh(qy)]}{q\sinh(q\xi)} + F(y) \quad (18)$$

Applying inverse Fourier sine transform and inverse Marchi-Fasulo transform on equation (18) we get,

$$T = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{P_n(x)}{\lambda_n} \times \int_0^{\infty}\left\{\frac{\left(F'(0) - \bar{f}_3^*\right)\cosh\sqrt{p^2+\lambda_n^2}\,(y-\xi) + \left(\bar{f}_4^* - F'(\xi)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,y\right)}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)} + F(y)\right\} \times \sin(pz)dp \quad (19)$$

$$G = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{P_n(x)}{\lambda_n} \times \int_0^{\infty}\left\{\frac{\left(F'(0) - \bar{f}_3^*\right)\cosh\sqrt{p^2+\lambda_n^2}\,(b-\xi) + \left(\bar{f}_4^* - F'(\xi)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,b\right)}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)} + F(b)\right\} \quad (20)$$

IV. AIRY'S STRESS FUNCTIONS

Substituting the value of temperature distribution (17) T(x,y,z) from (19) in equation (4) one obtains

$$U = \frac{-2\lambda E}{\pi}\sum_{n=1}^{\infty}\frac{P_n(x)}{\lambda_n} \int_0^{\infty}\left\{\frac{\left(F'(0) - \bar{f}_3^*\right)\cosh\sqrt{p^2+\lambda_n^2}\,(y-\xi) + \left(\bar{f}_4^* - F'(\xi)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,y\right)}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)} + F(y)\right\} \times \sin(pz)dp \quad (20)$$

(18)

(14)

V. DISPLACEMENT COMPONENTS

Substituting the values of Airy's stress function from equation (20) in the equation (1) to (3), one obtains (15)

$$u_x = \frac{-2\lambda}{\pi}\int_{-a}^{a}\int_{0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left\{\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{p^2+\lambda_n^2}\,(y-\xi)+\left(\bar{f}_4^{\,*}-F'(\xi)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,y\right)}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)}\right.$$

$$\left[\left(\lambda_n^2\right)P_n(x)-\nu\,P_n''(x)\right]$$

$$\left.+\left[F''(y)-\left(P^2+1\right)F(y)\right]P_n(n)-\nu\,F(y)P_n''(n)\right\}\sin(pz)dp\,dx \quad (21)$$

$$u_y = \frac{2\lambda}{\pi}\int_{-a}^{b}\int_{0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left\{\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,(y-\xi)\right)+\left(\bar{f}_4^{\,*}-F'(\xi)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,y\right)}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)}\right.$$

$$\left[\left((\nu+1)P^2+\nu\,\lambda_n^2+1\right)P_n(x)-\nu\,P_n''(x)\right]$$

$$\left.+\left[\left(P^2+1\right)F(y)+\nu\,F''(y)\right]P_n(n)-F(y)P_n''(n)\right\}\sin(pz)dp\,dy \quad (22)$$

$$u_z = \frac{-2\lambda}{\pi}\int_{0}^{\infty}\int_{0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left\{\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{p^2+\lambda_n^2}\,(y-\xi)+\left(\bar{f}_4^{\,*}-F'(\xi)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,y\right)}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)}\right.$$

$$\left[\left((\nu-1)P^2+\lambda_n^2-1\right)P_n(x)+P_n''(x)\right]$$

$$\left.+\left[\left(\nu\,P^2-1\right)F(y)+\nu\,F''(y)\right]P_n(n)+F(y)P_n''(n)\right\}$$
$$\times\sin(pz)dp\,dz \quad (23)$$

VI. DETERMINATION OF STRESS FUNCTION

Substituting the value of Airy's stress function U(x,y,z) from equation (20) in the equation (14) to (16) one obtain the stress functions as,

$$\sigma_{xn} = \frac{-2\lambda E}{\pi}\sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{P_n(x)}{\lambda_n}$$

$$\left\{\lambda_n^2\,\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{p^2+\lambda_n^2}\,(y-\xi)+\left(\bar{f}_4^{\,*}-F'(\xi)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,y\right)}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)}\right.$$

$$\left.+F''(y)-P^2F(y)\right\}\sin(pz)dp \quad (24)$$

$$\sigma_{yy} = \frac{2\lambda E}{\pi}\sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{P_n(x)-P_n''(x)}{\lambda_n}$$

$$\left[\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{p^2+\lambda_n^2}\,(y-\xi)+\left(\bar{f}_4^{\,*}-F'(\xi)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,y\right)}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)}+F(y)\right]$$

$$\sin(pz)dp \quad (25)$$

$$\sigma_{zz} = \frac{-2\lambda E}{\pi}\sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{1}{\lambda_n}$$

$$\left\{\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{p^2+\lambda_n^2}\,(y-\xi)+\left(\bar{f}_4^{\,*}-F'(\xi)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,y\right)}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)}\right.$$

$$\left[\left(P^2+\lambda_n^2\right)P_n(x)+P_n''(x)\right]$$

$$\left.+F''(y)P_n(x)+F(y)P_n''(x)\right\}\sin(pz)dp \quad (26)$$

Equations (20) and (21) are the required solutions.

VII. SPECIAL CASE

Set

$$f(x,y,z,t)=(x-a)^2(x+a)^2(z+e^{-z})(e^{y-t})$$

$$\overline{f}(n,y,z,t)=(z+e^{-z})(e^{y-t})$$

$$\times\left[\frac{a_n\cos^2(a_n a)-\cos(a_n a)\sin(a_n a)}{{a_n}^2}\right]$$

Substituting this value in equation (20) we get

$$G=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{P_n(x)}{\lambda_n}$$

$$\int_0^\infty \left\{ \frac{\begin{matrix}\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{p^2+\lambda_n^2}\,(b-\xi) \\ \left(\bar{f}_4^{\,*}-F'(\xi)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,b\right)\end{matrix}}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)} + F(b) \right\}$$

$$\sin(pz)dp \qquad (27)$$

VIII. NUMERICAL RESULTS

Set $a=2$, $k=0.86$, $b=3$, $\xi=2$, $t=1$ sec in the equations (20) to obtain

$$G=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{P_n(x)}{\lambda_n}$$

$$\int_0^\infty \left\{ \frac{\begin{matrix}\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{p^2+\lambda_n^2} \\ \left(\bar{f}_4^{\,*}-F'(2)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,y\right)\end{matrix}}{\sqrt{p^2\lambda_n^2}\,\sinh\left(\xi\sqrt{p^2+\lambda_n^2}\right)} + F(3) \right\}$$

$$\sin(pz)dp \qquad (28)$$

IX. MATERIAL PROPERTIES

The numerical calculations has been carried out for an Aluminum (pure) rectangular beam with the material properties as,

Density ρ =169 lb/ft ^{3}

Specific heat = 0.208 Btu/lbOF

Thermal conductivity K = 117 Btu/(hr. ftOF)

Thermal diffusivity α = 3.33 ft^2/hr.

Poisson ratio ν = 0.35

Coefficient of linear thermal expansion α_t = 12.84 x 10^{-6} 1/F

Lame constant μ = 26.67

Young's modulus of elasticity E = 70G Pa

X. DIMENSIONS

The constants associated with the numerical calculation are taken as

Length of rectangular beam x = 4ft

Breath of rectangular beam y = 3 ft

Height of rectangular beam z = 10^3ft

XI. CONCLUSION

In this article, the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of a semi-infinite rectangular beam have been obtained, when the boundary conditions are known with the aid of finite Marchi-Fasulo transform and semi-infinite Fourier cosine transform techniques. The results are obtain in the form of infinite series in terms of Bessel's function and depicted graphically.

REFERENCES

[1] Noda, N; Hetnarski, R.B; Tanigawa, Y: Thermal Stresses, second edition Taylor & Francis, New York (2003), 260.

[2] Dange, W. K; Khobragade, N.W, and Durge, M. H: "Three Dimensional Inverse Transient Thermoelastic Problem Of A Thin Rectangular Plate", Int. J. of Appl. Maths, Vol.23, No.2, 207-222, (2010).

[3] Ghume Ranjana S and Khobragade, N. W: "Deflection Of A Thick Rectangular Plate", Canadian Journal on Science and Engg. Mathematics Research, Vol.3 No.2, pp. 61-64, (2012).

[4] Roy, Himanshu and Khobragade, N.W: "Transient Thermoelastic Problem Of An Infinite Rectangular Slab", Int. Journal of Latest Trends in Maths, Vol. 2, No. 1, pp. 37-43, (2012).

[5] Lamba, N.K; and Khobragade, N.W: "Thermoelastic Problem of a Thin Rectangular Plate Due To Partially Distributed Heat Supply", IJAMM, Vol. 8, No. 5, pp.1-11, (2012).

[6] Sutar C. S. and Khobragade, N.W: "An inverse thermoelastic problem of heat conduction with internal heat generation for the rectangular plate", Canadian Journal of Science & Engineering Mathematics, Vol. 3, No.5, pp. 198-201, (2012).

[7] Khobragade N. W., Payal Hiranwar, H. S.Roy and Lalsingh Khalsa: Thermal Deflection of a Thick Clamped Rectangular Plate, Int. J. of Engg. And Information Technology, vol. 3, Issue 1, pp. 346-348, (2013).

[8] Roy H. S., Bagade S. H. and N. W. Khobragade: Thermal Stresses of a Semi infinite Rectangular Beam, Int. J. of Engg. And Information Technology, vol. 3, Issue 1, pp. 442-445, (2013).

[9] Jadhav, C.M; and Khobragade, N.W: "An Inverse Thermoelastic Problem of a thin finite Rectangular Plate due to Internal Heat Source", Int. J. of Engg. Research and Technology, vol.2, Issue 6, pp. 1009-1019, (2013).

AUTHOR BIOGRAPHY

Dr. N.W. Khobragade for being M.Sc in statistics and Maths, he attained Ph.D in both subjects. He has been teaching since 1986 for 29 years at PGTD of Maths, RTM Nagpur University, Nagpur and successfully handled different capacities.

At present he is working as Professor. Achieved excellent experiences in Research for 15 years in the area of Boundary value problems (Thermoelasticity in particular) and Operations Research. Published more than 180 research papers in reputed journals. Sixteen students awarded Ph.D Degree and six students submitted their thesis in University for award of Ph.D Degree under their guidance.

R.N.Pakade For being M.SC. in Maths, he is a working as a Lecture in PCE, Nagpur.

Sachin Chauthale For being M.SC. in Maths, he is a Research Scholar of Gondwana University, Gadchiroli.