Thermal Stresses of Semi Infinite Rectangular Beam with Internal Heat Source: Steady-State Problem

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Abstract- This paper is concerned with inverse transient thermoelastic problem in which we need to determine the temperature distribution, displacement function and thermal stresses of a semi-infinite rectangular beam when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem. The results are depicted graphically.

Key Words: Semi-infinite rectangular beam, inverse transient problem, Integral transform.

I. INTRODUCTION

Khobragade et al. [2-7, 9] have investigated temperature distribution, displacement function, and stresses of a thin rectangular plate and Khobragade et al. [8] have established displacement function, temperature distribution and stresses of a semi-infinite rectangular beam.

In this Section, an attempt has been made to determine the temperature distribution, unknown temperature gradient, displacement function and thermal stresses of a semi-infinite rectangular beam occupying the region D : \(-a \leq x \leq a ; 0 \leq y \leq b, 0 \leq z \leq \infty\) with known boundary conditions. Here Marchi-Fasulo transforms and Fourier cosine transform techniques have been used to find the solution.

II. STATEMENT OF THE PROBLEM

Consider a thin rectangular plate occupying the space D:

\[ a \leq x \leq a ; 0 \leq y \leq b, 0 \leq z \leq \infty \]

The displacement components \(u_x, u_y, u_z\) in the x and y and z directions respectively as Tanigawa et al. [1] are

\[
u_x = \int_a^{-a} \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - v \frac{\partial^2 U}{\partial x^2} \right) + \lambda T \right] dx \tag{1}
\]

\[
u_y = \int_0^b \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2} - v \frac{\partial^2 U}{\partial y^2} \right) + \lambda T \right] dy \tag{2}
\]

\[
u_z = \int_0^\infty \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - v \frac{\partial^2 U}{\partial z^2} \right) + \lambda T \right] dz \tag{3}
\]

where \(E, v, \lambda\) are the Young’s modulus, Poisson’s ratio and the linear coefficient of the thermal expansion of the material of the beam respectively and \(U(x, y, z)\) is the Airy’s stress functions which satisfy the differential equation as Tanigawa et al. [1] is

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) U(x, y, z) = -\lambda E \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2} \right)
\]

\[T(x, y, z)\]

where \(T(x, y, z)\) denotes the temperature of a rectangular beam satisfy the following differential equation as Tanigawa et al. [1] is

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\partial T}{\partial x} = 0
\]

(5)

In the boundary conditions

\[
\begin{align*}
T(x, y, z)|_{x=a} &= f_1(y, z) \quad \text{(Unknown)} \tag{6} \\
T(x, y, z)|_{x=-a} &= f_2(y, z) \quad \text{(Unknown)} \tag{7} \\
\frac{\partial T}{\partial y}|_{y=0} &= f_3(x, z) \quad \text{(Unknown)} \tag{8} \\
\frac{\partial T}{\partial y}|_{y=b} &= G(x, z) \quad \text{(Unknown)} \tag{9} \\
T(x, y, z)|_{z=0} &= 0 \quad \text{(Unknown)} \tag{10} \\
T(x, y, z)|_{z=\infty} &= 0 \quad \text{(Unknown)} \tag{11}
\end{align*}
\]

The stress components in terms of \(U(x, y, z)\) Tanigawa et al. [1] are given by

\[
\sigma_{xx} = \left[ \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right]
\]

(13)
\[ \sigma_{yy} = \left[ \frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} \right] \]

\[ \sigma_{zz} = \left[ \frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial y^2} \right] \]  

(14)

(15)

The equations (1) to (15) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

Applying finite Marchi-Fasulo transform and Fourier sine transform to the equation (5), we get

\[ \frac{dT^*}{dt} - q^2 T^* = \Psi \]  

(16)

Where, \( q^2 = \lambda_n^2 + p^2 \)

\[ \Psi = \frac{P_n(-a)}{k_2} f_2^* - \frac{P_n(a)}{k_1} f_1^* - \frac{G}{k} \]

Equation (16) is a linear equation whose solution is given by

\[ T^* = Ae^{\Psi t} + Be^{-\Psi t} + F(y) \]  

(17)

Where \( F(y) \) is the P.I.

Using boundary conditions (8) and (9) we get

\[ A = e^{\Psi} \left[ F'(0) - \tilde{f}_3^* \right] + F'(\xi) \]

\[ B = e^{\Psi} \left[ F'(0) - \tilde{f}_3^* \right] + \tilde{f}_4^* - F'(\xi) \]

Substituting the values of A and B in equation (17) one obtains

\[ \left[ \left( F'(0) - \tilde{f}_3^* \right) \cosh \left( q(y - \xi) \right) \right] \]

\[ \tilde{T}^* = \frac{+ \left( \tilde{f}_4^* - F'(\xi) \right) \cosh (qy)}{q \sinh (q \xi)} + F(y) \]  

(18)

Applying inverse Fourier sine transform and inverse Marchi-Fasulo transform on equation (18) we get,

\[ T = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{P_n(x)}{\lambda_n} \left[ \left( F'(0) - \tilde{f}_3^* \right) \cosh \left( p^2 + \lambda_n^2 \right) (y - \xi) \right] \]

\[ \times \int_0^{\infty} \frac{\left( \tilde{f}_4^* - F'(\xi) \right) \cosh \left( p^2 + \lambda_n^2 y \right)}{\sqrt{p^2 \lambda_n^2 \sinh (\xi \sqrt{p^2 + \lambda_n^2})}} + F(y) \]

\[ \times \sin (pz) dp \]  

(19)

\[ G = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{P_n(x)}{\lambda_n} \left[ \left( F'(0) - \tilde{f}_3^* \right) \cosh \left( p^2 + \lambda_n^2 (b - \xi) \right) \right] \]

\[ \times \int_0^{\infty} \frac{\left( \tilde{f}_4^* - F'(\xi) \right) \cosh \left( p^2 + \lambda_n^2 b \right)}{\sqrt{p^2 \lambda_n^2 \sinh (\xi \sqrt{p^2 + \lambda_n^2})}} + F(b) \]  

(20)

IV. AIRY'S STRESS FUNCTIONS

Substituting the value of temperature distribution \( T(x,y,z) \) from (19) in equation (4) one obtains

\[ U = -2 \pi \sum_{n=1}^{\infty} \frac{P_n(x)}{\lambda_n} \left[ \left( F'(0) - \tilde{f}_3^* \right) \cosh \left( p^2 + \lambda_n^2 (y - \xi) \right) \right] \]

\[ \times \int_0^{\infty} \frac{\left( \tilde{f}_4^* - F'(\xi) \right) \cosh \left( p^2 + \lambda_n^2 y \right)}{\sqrt{p^2 \lambda_n^2 \sinh (\xi \sqrt{p^2 + \lambda_n^2})}} + F(y) \]

\[ \times \sin (pz) dp \]  

(20)

V. DISPLACEMENT COMPONENTS

Substituting the values of Airy’s stress function from equation (20) in the equation (1) to (3), one obtains
\[ u_x = -\frac{2\lambda}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left( F'(0) - \tilde{f}_3^* \right) \cosh \sqrt{p^2 + \lambda_n^2} (y - \xi) \]
\[ + \left( \tilde{f}_4^* - F'(\xi) \right) \cosh \sqrt{p^2 + \lambda_n^2} y \left/ \sqrt{p^2 \lambda_n^2 \sinh (\xi \sqrt{p^2 + \lambda_n^2})} \right. \]
\[ + \left[ \frac{\tilde{f}_3^*}{\lambda_n} P_n(x) - \nu \frac{P_n^*(x)}{\lambda_n} \right] \]
\[ + \left[ (F'(y) - \tilde{f}_3^*) \cosh \sqrt{p^2 + \lambda_n^2} (y - \xi) \right. \]
\[ + \left( \tilde{f}_4^* - F'(\xi) \right) \cosh \sqrt{p^2 + \lambda_n^2} y \left/ \sqrt{p^2 \lambda_n^2 \sinh (\xi \sqrt{p^2 + \lambda_n^2})} \right. \]
\[ + \left[ (\nu + 1)P^2 + \nu \lambda_n^2 + 1 \right] P_n(x) - \nu P_n^*(x) \]
\[ + \left[ F'(y) \right. \left. + \nu F'(y) \right] P_n(n) - \nu F(y) P_n^*(n) \sin(pz) \] \[ \times \sin(pz) dp \] \[ d\xi \] \[ \quad (20) \]

\[ u_y = -\frac{2\lambda}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left( F'(0) - \tilde{f}_3^* \right) \cosh \sqrt{p^2 + \lambda_n^2} (y - \xi) \]
\[ + \left( \tilde{f}_4^* - F'(\xi) \right) \cosh \sqrt{p^2 + \lambda_n^2} y \left/ \sqrt{p^2 \lambda_n^2 \sinh (\xi \sqrt{p^2 + \lambda_n^2})} \right. \]
\[ + \left[ \frac{\tilde{f}_3^*}{\lambda_n} P_n(x) - \nu \frac{P_n^*(x)}{\lambda_n} \right] \]
\[ + \left[ (F'(y) + \nu F'(y)) P_n(n) + F(y) P_n^*(n) \right] \sin(pz) \] \[ \times \sin(pz) dp \] \[ d\xi \] \[ \quad (21) \]

\[ u_z = -\frac{2\lambda}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left( F'(0) - \tilde{f}_3^* \right) \cosh \sqrt{p^2 + \lambda_n^2} (y - \xi) \]
\[ + \left( \tilde{f}_4^* - F'(\xi) \right) \cosh \sqrt{p^2 + \lambda_n^2} y \left/ \sqrt{p^2 \lambda_n^2 \sinh (\xi \sqrt{p^2 + \lambda_n^2})} \right. \]
\[ + \left[ \frac{\tilde{f}_3^*}{\lambda_n} P_n(x) + \nu \frac{P_n^*(x)}{\lambda_n} \right] \]
\[ + \left[ (\nu - 1)P^2 + \lambda_n^2 - 1 \right] P_n(x) + P_n^*(x) \] \[ + \left[ (\nu - 1)F'(y) + \nu F'(y) \right] P_n(n) + F(y) P_n^*(n) \] \[ \times \sin(pz) dp \] \[ dz \] \[ \quad (22) \]

**VI. DETERMINATION OF STRESS FUNCTION**

Substituting the value of Airy’s stress function \( U(x,y,z) \) from equation (20) in the equation (14) to (16) one obtain the stress functions as,

\[ \sigma_{xx} = -\frac{2\lambda E}{\pi} \sum_{n=1}^{\infty} \frac{P_n(x)}{\lambda_n} \]

\[ \sigma_{yy} = \frac{2\lambda E}{\pi} \sum_{n=1}^{\infty} \frac{P_n(x)}{\lambda_n} \]

\[ \sigma_{zz} = \frac{2\lambda E}{\pi} \sum_{n=1}^{\infty} \frac{P_n(x)}{\lambda_n} \]

\[ \left( F'(0) - \tilde{f}_3^* \right) \cosh \sqrt{p^2 + \lambda_n^2} (y - \xi) \]
\[ \left( \tilde{f}_4^* - F'(\xi) \right) \cosh \sqrt{p^2 + \lambda_n^2} y \left/ \sqrt{p^2 \lambda_n^2 \sinh (\xi \sqrt{p^2 + \lambda_n^2})} \right. \]
\[ + \left[ \frac{\tilde{f}_3^*}{\lambda_n} P_n(x) + \nu \frac{P_n^*(x)}{\lambda_n} \right] \]
\[ + \left[ (\nu - 1)P^2 + \lambda_n^2 - 1 \right] P_n(x) + P_n^*(x) \] \[ + \left[ (\nu - 1)F'(y) + \nu F'(y) \right] P_n(n) + F(y) P_n^*(n) \] \[ \times \sin(pz) dp \] \[ dz \]

\[ \quad (23) \]

**VII. SPECIAL CASE**

Set

\[ f(x, y, z, t) = (x-a)^2 (x+a)^2 (z+e^{-z}) (e^{y-t}) \]
\[ \bar{f}(n, y, z, t) = (z+e^{-z}) (e^{y-t}) \]
\[ \times \left[ \frac{a_n \cos^2(a_n a) - \cos(a_n a) \sin(a_n a)}{a_n^2} \right] \]

Substituting this value in equation (20) we get

\[ G = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{P_n(x)}{\lambda_n} \]
in the form of infinite series in terms of Bessel’s function and depicted graphically.

REFERENCES


AUTHOR BIOGRAPHY

Dr. N.W. Khobragade for being M.Sc in statistics and Maths, he attained Ph.D in both subjects. He has been teaching since 1986 for 29 years at PGTD of Maths, RTM Nagpur University, Nagpur and successfully handled different capacities. At present he is working as Professor. Achieved excellent experiences in Research for 15 years in the area of Boundary value problems (Thermoelasticity in particular) and Operations Research. Published more than 180 research papers in reputed journals. Sixteen students awarded Ph.D Degree and six students submitted their thesis in University for award of Ph.D Degree under their guidance.
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