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Transient thermo elastic problem of a semi-infinite circular beam

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Abstract- This paper is concerned with transient thermo elastic problem in which we need to determine the temperature distribution, displacement function and thermal stresses of a semi-infinite circular beam when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem.

Key Words: **Semi-infinite circular beam, transient problem, Integral transform, internal heat source.**

I. INTRODUCTION

Khobragade et al. [2-16] studied the various problems on circular plate, rectangular plate and hollow cylinder.

In this paper, we analyzed inverse thermo elastic problem of temperature and thermal stresses of thick, semi-infinite circular beam due to heat generation. The governing heat conduction equation has been solved by using Marchi-Zgrablich and Fourier Cosine transform techniques. The result presented here will be more useful in engineering applications.

II. STATEMENT OF THE PROBLEM

Consider a thick circular beam occupying the space D: a ≤ r≤ b, 0≤ z< ∞. The material is homogeneous and isotropic. The differential equation governing the displacement potential function ϕ (r ,z ,t) as Noda et al. [1] is

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=\left[\frac{1+\upsilon}{1-\upsilon}\right]\alpha_t T \qquad (1)$$

Where, ν and α_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate and T is temperature of the plate satisfying the differential equation as **Noda et al. [1]** is

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}+\chi(r,z,t)=\frac{1}{k}\frac{\partial T}{\partial t} \qquad (2)$$

Subject to initial condition:

$$T(r,z,0)=f(r,z) \qquad (3)$$

and boundary conditions are

$$\left[T(r,z,t)+k_1\frac{\partial T(r,z,t)}{\partial r}\right]_{r=a}=g_1(z,t) \qquad (4)$$

$$\left[T(r,z,t)+k_2\frac{\partial T(r,z,t)}{\partial r}\right]_{r=\xi}=g_2(z,t) \text{ (Known)} \qquad (5)$$

$$\left[T(r,z,t)\right]_{r=b}=G(z,t) \quad \text{(Unknown)} \qquad (6)$$

$$\left[\frac{\partial T(r,z,t)}{\partial z}\right]_{z=0}=f_1(r,t) \qquad (7)$$

$$\left[\frac{\partial T(r,z,t)}{\partial z}\right]_{z=\infty}=f_2(r,t) \quad ,0\le r\le a,\ \ t>0 \qquad (8)$$

Where k is the thermal diffusivity of the material of the plate. The displacement function in the cylindrical co-ordinate system are represented by the Goodier thermo elastic function ϕ and Love's function L as **Noda et al. [1]** are

$$u_r=\frac{\partial\phi}{\partial r}-\frac{\partial^2 L}{\partial r\partial z} \qquad (9)$$

$$u_z=\frac{\partial\phi}{\partial z}+2(1-\upsilon)\nabla^2 L-\frac{\partial^2 L}{\partial z^2} \qquad (10)$$

in which Goodier thermo elastic potential must satisfy the equation as **Noda et al. [1]** is

$$\nabla^2\phi=\left(\frac{1+\nu}{1-\nu}\right)a_t T \qquad (11)$$

The Love's function must satisfy

$$\nabla^2(\nabla^2 L)=0 \qquad (12)$$

where,

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$$

The component of stresses are represented by the use of the potential ϕ and Love's function L as **Noda et al. [1]** are

$$\sigma_{rr}=2G\left\{\left[\frac{\partial^2\phi}{\partial r^2}-\nabla^2\phi\right]+\frac{\partial}{\partial z}\left[\upsilon\nabla^2 L-\frac{\partial^2 L}{\partial r^2}\right]\right\} \qquad (13)$$

$$\sigma_{\theta\theta}=2G\left\{\left[\frac{1}{r}\frac{\partial\phi}{\partial r}-\nabla^2\phi\right]+\frac{\partial}{\partial z}\left[\upsilon\nabla^2 L-\frac{1}{r}\frac{\partial^2 L}{\partial r^2}\right]\right\} \qquad (14)$$

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$$\sigma_{zz} = 2G\left\{\left[\frac{\partial^2\phi}{\partial z^2} - \nabla^2\phi\right] + \frac{\partial}{\partial z}\left[(z-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right]\right\} \tag{15}$$

$$\sigma_{rz} = 2G\left\{\frac{\partial^2\phi}{\partial r\partial z} + \frac{\partial}{\partial r}\left[(1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right]\right\} \tag{16}$$

Equations (1) to (16) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

Applying finite **Marchi-Zgrablich transform** defined in [3] to the equations (2) and using equations (4), (5) one obtains

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \chi(r,z,t) = \frac{1}{k}\frac{\partial T}{\partial t} \tag{17}$$

By using the operational property of finite Marchi-Zgrablich transform, we get

$$\frac{\partial^2 \overline{T}}{\partial z^2} - \mu_n^2\overline{T} + \overline{\chi} = \frac{1}{k}\frac{\partial \overline{T}}{\partial t} + g(z,t) \tag{18}$$

Again, applying Fourier cosine transform to the equation (18), we get

$$\frac{d\overline{T}_c^*}{dt} + kp^2\overline{T}_c^* = \phi_1^* + \overline{\chi}_1^* \tag{19}$$

where

$$\overline{\chi}_1^* = k\overline{\chi}_c^* \quad and \quad \phi_1^* = k\mu - k\mu_n^2\overline{T}_c^* - kg_c^*$$

Equation (19) is a linear equation whose solution is given by

$$\overline{T}^*(n,z,t) = e^{-kp^2 t}\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2 t'}dt' + Ce^{-kp^2 t} \tag{20}$$

Using (3), we get

$$C = F^*(m,n)$$

Thus, we have,

$$\overline{T}^*(n,z,t) = e^{-kp^2 t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2 t'}dt' + \overline{F}^*(m,n)\right] \tag{21}$$

Applying inversion of Fourier cosine transform and Marchi-Zgrablich transform to the equation (21), one obtains

$$T(r,z,t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2 t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2 t'}dt' + \overline{F}^*(m,n)\right]\right\}\times S_0(k_1,k_2,\mu_n r) \tag{22}$$

$$G(z,t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2 t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2 t'}dt' + \overline{F}^*(m,n)\right]\right\}\times S_0(k_1,k_2,\mu_n b) \tag{23}$$

These are the desired solutions of the given problem. Let us assume Love's function L, which satisfy condition (11) as

$$L(r,z) = \sum_{n=1}^{\infty}\frac{1}{C_n}\psi S_0(k_1,k_2,\mu_n r) \tag{24}$$

Where,

$$\psi = e^{-kp^2 t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2 t'}dt' + \overline{F}^*(m,n)\right]$$

The displacement potential is given by

$$\phi = A\sum_{n=1}^{\infty}\frac{1}{C_n}\psi S_0(k_1,k_2,\mu_n r)[\psi + B(t)] \tag{25}$$

where, $A = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t$

$$B(t) = e^{-kp^2 t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2 t'}dt' + \overline{F}^*(m,n)\right]dt$$

IV. DETERMINATION OF DISPLACEMENT FUNCTION

Substituting the equations (24) and (25) in the equation (8) one obtains

$$u_r = A\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0'(k_1,k_2,\mu_n r)[\psi + B(t)] - \sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0'(k_1,k_2,\mu_n r) \tag{26}$$

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$$u_z = 2(1-\upsilon)\left[\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}\psi S_0'(k_1,k_2,\mu_n r)+\frac{1}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0'(k_1,k_2,\mu_n r)\right] \quad (27)$$

V. DETERMINATION OF STRESS FUNCTIONS

Substituting the values from the equation (24) and (25) in the equation (10) to (13) we get

$$\sigma_{rr} = 2G\left\{\begin{bmatrix} A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r)[\psi+B(t)] \\ -A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r)[\psi+B(t)]- \\ \frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0'(k_1,k_2,\mu_n r)[\psi+B(t)] \end{bmatrix} + \frac{\partial}{\partial z}\begin{bmatrix} \upsilon\begin{bmatrix} A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r) \\ +\frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0'(k_1,k_2,\mu_n r) \end{bmatrix} \\ -\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r) \end{bmatrix}\right\} \quad (28)$$

$$\sigma_{\theta\theta} = 2G\left\{\begin{matrix} \frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0'(k_1,k_2,\mu_n r)[\psi+B(t)] \\ \begin{bmatrix} A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r)[\psi+B(t)]+ \\ \frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0'(k_1,k_2,\mu_n r)[\psi+B(t)] \end{bmatrix} \\ +\frac{\partial}{\partial z}\begin{bmatrix} \upsilon\begin{bmatrix} A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}\psi S_0''(k_1,k_2,\mu_n r) \\ +\frac{1}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0'(k_1,k_2,\mu_n r) \end{bmatrix} \\ -\frac{1}{r}\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}\psi S_0''(k_1,k_2,\mu_n r) \end{bmatrix} \end{matrix}\right\} \quad (29)$$

$$\sigma_{zz} = 2G\left\{\begin{matrix} \begin{bmatrix} A\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}S_0''(k_1,k_2,\mu_n r)[\psi+B(t)] \\ +\frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0'(k_1,k_2,\mu_n r)[\psi+B(t)] \end{bmatrix} \\ +(1-\upsilon)\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}\psi S_0''(k_1,k_2,\mu_n r) \\ +\frac{(1-\upsilon)}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0'(k_1,k_2,\mu_n r) \end{matrix}\right\} \quad (30)$$

$$\sigma_{rz} = 2G\left[(1-\upsilon)\sum_{n=1}^{\infty}\frac{\mu_n^3}{C_n}\psi S_0'''(k_1,k_2,\mu_n r) - \frac{(1-\upsilon)}{r^2}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu_n^2}{C_n}\psi S_0''(k_1,k_2,\mu_n r)\right] \quad (31)$$

Where,

$$A = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \ \text{and}\ \psi = e^{-kp^2 t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{kp^2 t'}dt' + \overline{F}^*(m,n)\right]$$

$$B(t) = \int \psi\, dt$$

VI. SPECIAL CASE

Set $F(r,z) = \delta(r-r_0)(z-e^{-z})$ (32)

Applying finite transform defined in Marchi Zgrablich [35] to the equation (32) one obtains

$$\overline{F}(n,z) = r_0\,(z-e^{-z})S_0(k_1,k_2,\mu_n r_0) \quad (33)$$

Substituting the value of (33) in the equations (22) to (23) one obtains

$$T(r,z,t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2 t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2 t'}dt' + \overline{F}^*(m,n)\right]\right\} \times S_0(k_1,k_2,\mu_n r) \quad (34)$$

Where

$$\overline{F}^*(n,m) = r_0\, S_0(k_1,k_2,\mu_n r_0)\int_0^{\infty}(z-e^{-z})\cos\alpha z\, dz$$

$$G(z,t)=\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2t}\left[\int_0^t\left(\phi_1^*+\overline{\chi}_1^*\right)e^{-kp^2t'}dt'+\overline{F}^*(m,n)\right]\right\}$$
$$\times S_0(k_1,k_2,\mu_n b)$$

(35)

VII. NUMERICAL RESULTS

Put $a=2,\xi=2.3,\ b=2.5,\ t=1\sec$ in equations (34) to (35) one obtains

$$T(r,z,t)=\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2}\left[\int_0^1\left(\phi_1^*+\overline{\chi}_1^*\right)e^{-kp^2t'}dt'+\overline{F}^*(m,n)\right]\right\}$$
$$\times S_0(k_1,k_2,\mu_n r)$$

(36)

$$G(z,t)=\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2}\left[\int_0^1\left(\phi_1^*+\overline{\chi}_1^*\right)e^{-kp^2t'}dt'+\overline{F}^*(m,n)\right]\right\}$$
$$\times S_0(k_1,k_2,\mu_n(2.5))$$

(37)

VIII. MATERIAL PROPERTIES

The numerical calculation has been carried out for an Aluminum (pure) circular plate with the material properties as

Density ρ =169 lb/ft ^{3}

Specific heat = 0.208 Btu/lbOF

Thermal conductivity K = 15.9 x 10^6Btu/(hr. ftOF)

Thermal diffusivity α = 3.33 ft2/hr.

Poisson ratio ν = 0.35

Coefficient of linear thermal expansion

α_t = 12.84 x 10^{-6}1/F

Lame constantµ = 26.67

Young's modulus of elasticity E = 70G Pa

IX. DIMENSIONS

The constants associated with the numerical calculation are taken as

Radius of the disk *a* = 2ft

Radius of the disk b = 2.5 ft

X. CONCLUSION

In this paper, we develop the analysis for the temperature field by introducing the methods of the Marchi- Zgrablich and Fourier cosine transform techniques and determined the expression for temperature distribution, displacement and thermal stresses of a semi-infinite, thick circular beam with known boundary conditions which is useful to design of structure or machines in engineering applications.

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