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Thermo elastic problem of semi-infinite Rectangular beam: steady-state problem

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Abstract- This paper is concerned with steady state thermoelastic problem in which we need to determine the temperature distribution, displacement function and thermal stresses of a semi-infinite rectangular beam when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem.

***Key Words:* Semi-infinite rectangular beam, steady-state problem, Integral transform, heat source.**

I. INTRODUCTION

Khobragade et al. [**2-7, 9**] have investigated temperature distribution, displacement function, and stresses of a thin rectangular plate and **Khobragade et al**. [8] have established displacement function, temperature distribution and stresses of a semi-infinite rectangular beam.

In this paper, an attempt has been made to determine the temperature distribution, displacement function and thermal stresses of a semi-infinite square beam occupying the region **D: *-a* ≤ x ≤ *a* ; 0 ≤ y ≤ *b*, 0 ≤ z ≤ ∞.** with known boundary conditions. Here Marchi-Fasulo transforms and Fourier cosine transform techniques have been used to find the solution of the problem.

II. STATEMENT OF THE PROBLEM

Consider a thin rectangular plate occupying the space D: -a ≤ x ≤ a ; 0 ≤ y ≤ b, 0 ≤ z ≤ ∞. The displacement components ux , uy and uz in the x and y and z directions respectively as Noda et al. [1] are

$$u_x = \int_{-a}^{a}\left[\frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\nu\frac{\partial^2\phi}{\partial x^2}\right)+\lambda T\right]dx \qquad (1)$$

$$u_y = \int_{0}^{b}\left[\frac{1}{E}\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial x^2}-\nu\frac{\partial^2\phi}{\partial y^2}\right)+\lambda T\right]dy \qquad (2)$$

$$u_z = \int_{0}^{\infty}\left[\frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}-\nu\frac{\partial^2\phi}{\partial z^2}\right)+\lambda T\right]dz \qquad (3)$$

where E, ν, and λ are the young's modulus, Poisson's ratio and the linear coefficient of the thermal expansion of the material of the beam respectively and ϕ (x,y,z) is the Airy's stress functions which satisfy the differential equation as Noda et al. [1] is

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)^2\phi(x,y,z) = -\lambda E\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)T(x,y,z) \qquad (4)$$

where T(x,y,z) denotes the temperature of a rectangular beam satisfy the following differential equation as Noda et al. [1] is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{g(x,y,z)}{k}=0 \qquad (5)$$

where k is the thermal conductivity of the material, subject to the boundary conditions

$$\left[T(x,y,z)+k_1\frac{\partial T(x,y,z)}{\partial x}\right]_{x=a} = f_1(y,z) \qquad (6)$$

$$\left[T(x,y,z)+k_2\frac{\partial T(x,y,z)}{\partial x}\right]_{x=-a} = f_2(y,z) \qquad (7)$$

$$\left[\frac{\partial T(x,y,z)}{\partial y}\right]_{y=0} = f_3(x,z) \qquad (8)$$

$$\left[\frac{\partial T(x,y,z)}{\partial y}\right]_{y=b} = f_4(x,z) \qquad (9)$$

$$\left[\frac{\partial T(x,y,z)}{\partial z}\right]_{Z=0} = 0 \qquad (10)$$

$$\left[\frac{\partial T(x,y,z)}{\partial z}\right]_{Z=\infty} = 0 \qquad (11)$$

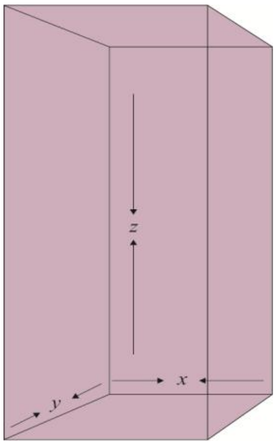

Fig 1: Geometry of the problem

The stress components in terms of ϕ (x, y, z) **Noda et al.** [**1**] are given by

$$\sigma_{xx} = \left[\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right] \qquad (12)$$

$$\sigma_{yy} = \left[\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial x^2}\right] \qquad (13)$$

$$\sigma_{zz} = \left[\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right] \qquad (14)$$

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Equations (1) to (14) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

Applying finite Marchi-Fasulo transform, finite Fourier cosine transform and Fourier sine transform to the equations, we get

$$\frac{d\bar{T}^*}{dy} - q^2\bar{T}^* = \Omega \qquad (15)$$

where, $q^2 = \lambda_n{}^2 + \mu_m{}^2$

$$\Omega = \frac{P_n(-a)}{k_2} f_2{}^* - \frac{P_n(a)}{k_1} f_1{}^* - \frac{\bar{g}^*}{k}$$

This is a linear differential equation whose solution is given by

$$\bar{T}^* = Ae^{qy} + Be^{-qy} + F(y) \qquad (16)$$

where $F(y)$ is the P.I.

$$A = \frac{e^{-q\xi}\left(F'(0) - \bar{f}_3{}^*\right) + F'(\xi)}{2q\sinh(q\xi)} \qquad (17)$$

$$B = \frac{e^{q\xi}\left(F'(0) - \bar{f}_3{}^*\right) + \bar{f}_4{}^* - F'(\xi)}{2q\sinh(q\xi)} \qquad (18)$$

Substituting the values of A and B in equation (16) one obtains

$$\bar{T}^* = \frac{\left(F'(0) - \bar{f}_3{}^*\right)\cosh(q(y-\xi)) + \left(\bar{f}_4{}^* - F'(\xi)\right)\cosh(qy)}{q\sinh(q\xi)} + F(y) \qquad (19)$$

Applying inverse Fourier sine transform and inverse Marchi-Fasulo transform to the equation (19) we get,

$$T = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{P_n(x)}{\lambda_n}\int_0^{\infty}\left\{\frac{\left(F'(0) - \bar{f}_3{}^*\right)\cosh\sqrt{\mu_m{}^2 + \lambda_n^2}\,(y-b) \quad \left(\bar{f}_4{}^* - F'(b)\right)\cosh\left(\sqrt{\mu_m{}^2 + \lambda_n^2}\,y\right)}{\sqrt{\mu_m{}^2\lambda_n^2}\,\sinh\left(b\sqrt{p^2 + \lambda_n^2}\right)} + F(y)\right\}\cos(\mu_m z)\,d\mu_m \qquad (20)$$

Equation (20) is the required solution.

IV. AIRY'S STRESS FUNCTIONS

Substituting the value of temperature distribution T(x,y,z) from (19) in equation (18) one obtains

$$\phi = \frac{-2\lambda E}{\pi}\sum_{n=1}^{\infty}\frac{P_n(x)}{\lambda_n}$$

$$\int_0^{\infty}\left\{\frac{\left(F'(0) - \bar{f}_3{}^*\right)\cosh\sqrt{\mu_m{}^2 + \lambda_n^2}\,(y-b) \quad \left(\bar{f}_4{}^* - F'(b)\right)\cosh\left(\sqrt{\mu_m{}^2 + \lambda_n^2}\,y\right)}{\sqrt{\mu_m{}^2\lambda_n^2}\,\sinh\left(b\sqrt{\mu_m{}^2 + \lambda_n^2}\right)} + F(y)\right\}\times\cos(\mu_m z)\,d\mu_m \qquad (21)$$

V. DISPLACEMENT COMPONENTS

Substituting the values of Airy's stress function ϕ from equation (21) in the equation (1) to (3), one obtains

$$u_x = \frac{-2\lambda}{\pi}\int_{-a}^{a}\int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left\{\frac{\left(F'(0) - \bar{f}_3{}^*\right)\cosh\sqrt{\mu_m{}^2 + \lambda_n^2}\,(y-b) + \left(\bar{f}_4{}^* - F'(b)\right)\cosh\left(\sqrt{\mu_m{}^2 + \lambda_n^2}\,y\right)}{\sqrt{\mu_m{}^2\lambda_n^2}\,\sinh\left(b\sqrt{\mu_m{}^2 + \lambda_n^2}\right)}\right.$$

$$\left[\left(\lambda_n^2\right)P_n(x) - \nu\,P_n''(x)\right]$$

$$\left. + \left[F''(y) - \left(\mu_m{}^2 + 1\right)F(y)\right]P_n(n) - \nu\,F(y)P_n''(n)\right\}$$

$$\cos(\mu_m z)\,d\mu_m\,dx \qquad (22)$$

$$u_y = \frac{2\lambda}{\pi}\int_0^{b}\int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left\{\frac{\left(F'(0) - \bar{f}_3{}^*\right)\cosh\left(\sqrt{\mu_m{}^2 + \lambda_n^2}\,(y-b)\right) + \left(\bar{f}_4{}^* - F'(b)\right)\cosh\left(\sqrt{\mu_m{}^2 + \lambda_n^2}\,y\right)}{\sqrt{\mu_m{}^2\lambda_n^2}\,\sinh\left(b\sqrt{\mu_m{}^2 + \lambda_n^2}\right)}\right.$$

$$\left[\left((\nu+1)p^2 + \nu\,\lambda_n^2 + 1\right)P_n(x) - \nu\,P_n''(x)\right]$$

$$\left. + \left[\left(p^2 + 1\right)F(y) + \nu\,F''(y)\right]P_n(n) - F(y)P_n''(n)\right\}$$

$$\cos(\mu_m z)\,d\mu_m\,dy \qquad (23)$$

$$u_z = \frac{-2\lambda}{\pi}\int_0^{\infty}\int_0^{\infty}\sum_{n=1}^{\infty}\frac{1}{\lambda_n}$$

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$$\left\{ \frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{{\mu_m}^2+\lambda_n^2}\,(y-b) + \left(\bar{f}_4^{\,*}-F'(b)\right)\cosh\left(\sqrt{{\mu_m}^2+\lambda_n^2}\,y\right)}{\sqrt{{\mu_m}^2\lambda_n^2}\,\sinh\left(b\sqrt{{\mu_m}^2+\lambda_n^2}\right)} \right.$$

$$\left[\left((\nu-1)p^2+\lambda_n^2-1\right)P_n(x)+P_n''(x)\right]$$

$$+\left[\left(\nu\,{\mu_m}^2-1\right)F(y)+\nu\,F''(y)\right]P_n(n)+F(y)P_n''(n)\Big\} \cos(\mu_m z)\,d\mu_m\,dz \qquad (24)$$

VI. DETERMINATION OF STRESS FUNCTION

Substituting the value of Airy's stress function ϕ (x,y,z) from equation (22) in the equation (23) to (24) one obtain the stress functions as,

$$\sigma_{xn}=\frac{-2\lambda E}{\pi}\sum_{n=1}^{\infty}\int_0^{\infty}\frac{P_n(x)}{\lambda_n}$$

$$\left\{\lambda_n^2\,\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{{\mu_m}^2+\lambda_n^2}\,(y-b)+\left(\bar{f}_4^{\,*}-F'(b)\right)\cosh\left(\sqrt{{\mu_m}^2+\lambda_n^2}\,y\right)}{\sqrt{{\mu_m}^2\lambda_n^2}\,\sinh\left(b\sqrt{{\mu_m}^2+\lambda_n^2}\right)}\right.$$

$$+F''(y)-{\mu_m}^2F(y)\Big\}\cos(\mu_m z)\,d\mu_m \qquad (25)$$

$$\sigma_{yy}=\frac{2\lambda E}{\pi}\sum_{n=1}^{\infty}\int_0^{\infty}\frac{P_n(x)-P_n''(x)}{\lambda_n}$$

$$\left[\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{{\mu_m}^2+\lambda_n^2}\,(y-b)+\left(\bar{f}_4^{\,*}-F'(b)\right)\cosh\left(\sqrt{{\mu_m}^2+\lambda_n^2}\,y\right)}{\sqrt{{\mu_m}^2\lambda_n^2}\,\sinh\left(b\sqrt{{\mu_m}^2+\lambda_n^2}\right)}+F(y)\right] \qquad (26)$$

$$\cos(\mu_m z)\,d\mu_m$$

$$\sigma_{zz}=\frac{-2\lambda E}{\pi}\sum_{n=1}^{\infty}\int_0^{\infty}\frac{1}{\lambda_n}$$

$$\left\{\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{{\mu_m}^2+\lambda_n^2}\,(y-b)+\left(\bar{f}_4^{\,*}-F'(b)\right)\cosh\left(\sqrt{{\mu_m}^2+\lambda_n^2}\,y\right)}{\sqrt{{\mu_m}^2\lambda_n^2}\,\sinh\left(b\sqrt{{\mu_m}^2+\lambda_n^2}\right)}\right.$$

$$\left[\left({\mu_m}^2+\lambda_n^2\right)P_n(x)+P_n''(x)\right]$$

$$+F''(y)P_n(x)+F(y)P_n''(x)\Big\}\cos(\mu_m z)\,d\mu_m \qquad (27)$$

VII. SPECIAL CASE

Set $f(x,y,z)=(x-a)^2(x+a)^2(z+e^{-z})(e^y)$ (28)

Applying Marchi-Fasulo transform to the equation (4.7.1) we get

$$\bar{f}(n,y,z)=(z+e^{-z})(e^y)$$

$$\times\left[\frac{a_n\cos^2(a_n a)-\cos(a_n a)\sin(a_n a)}{{a_n}^2}\right] \qquad (29)$$

Substituting equation (4.7.2) in equations (4.3.6), (4.4.1), (4.5.1)- (4.5.3), (4.6.1)-(4.6.3) we obtain

$$T=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{P_n(x)}{\lambda_n}$$

$$\int_0^{\infty}\left\{\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{{\mu_m}^2+\lambda_n^2}\,(y-b)\ \left(\bar{f}_4^{\,*}-F'(b)\right)\cosh\left(\sqrt{p^2+\lambda_n^2}\,y\right)}{\sqrt{{\mu_m}^2\lambda_n^2}\,\sinh\left(b\sqrt{{\mu_m}^2+\lambda_n^2}\right)}+F(y)\right\}\cos(\mu_m z)\,d\mu_m \qquad (30)$$

VIII. NUMERICAL RESULTS

Set $a=2$, $k=0.86$, $b=3$, in the equation (30) to obtain

$$T=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{P_n(x)}{\lambda_n}$$

$$\int_0^{\infty}\left\{\frac{\left(F'(0)-\bar{f}_3^{\,*}\right)\cosh\sqrt{{\mu_m}^2+\lambda_n^2}\,(y-3)\ \left(\bar{f}_4^{\,*}-F'(3)\right)\cosh\left(\sqrt{{\mu_m}^2+\lambda_n^2}\,y\right)}{\sqrt{{\mu_m}^2\lambda_n^2}\,\sinh\left(3\sqrt{{\mu_m}^2+\lambda_n^2}\right)}+F(y)\right\}\cos(\mu_m z)\,d\mu_m \qquad (31)$$

IX. MATERIAL PROPERTIES

The numerical calculations has been carried out for an Aluminum (pure) rectangular beam with the material

properties as,
Density ρ =169 lb/ft ^{3}
Specific heat = 0.208 Btu/lbOF
Thermal conductivity K = 117 Btu/(hr. ftOF)
Thermal diffusivity α = 3.33 ft^2/hr.
Poisson ratio ν = 0.35
Coefficient of linear thermal expansion α_t = 12.84 x 10^{-6} 1/F
Lame constant μ = 26.67
Young's modulus of elasticity E = 70G Pa

X. DIMENSIONS

The constants associated with the numerical calculation are taken as
Length of rectangular beam x = 4ft
Breath of rectangular beam y = 3 ft
Height of rectangular beam z = 10^3ft

XI. CONCLUSION

In this paper, the temperature distribution, displacement function and thermal stresses at any point of a semi-infinite rectangular beam have been obtained, when the boundary conditions are known with the aid of finite Marchi-Fasulo transform and finite Fourier cosine transform and Fourier sine transform techniques. The results are obtain in the form of infinite series in terms of Bessel's function.

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