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Extended T-X Family of Distributions

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Abstract*—The motivation of the present article is introducing a more general family of distributions called extended T-X distribution. We examine the S-T-X family and T-X-Y family of distributions and then extend such families for any number of random variables X_n, n = 0, 1, … ,k, as " the transformers " or random variables T_n, n = 0, 1, …, k, as " the transformeds". We call such Families of distributions" extended T-X family of first kind (extended T-X (I))" and " extended T-X family of second kind (extended-X (II))", respectively. Some of their properties and special cases are discussed. Several known continuous distributions presented as members of this family.

***Index Terms*—Differ integral operator; Beta distribution; Exponential distribution; Kumaraswamy distribution.**

I. INTRODUCTION

In the recent studies by Alzaatreh et al. [1], a new method has been proposed for generating families of continuous distributions by using the composite function $(RoWoF)(x)$ with R and F being the c.d.f.s of the random variables T " the transformed " and X " the transformer ", respectively. W (.) function is defined to link the support of T to the range of X.

This class of distributions is defined as

$$G(x) = \int_a^{W(F(x))} r(t)dt, \tag{1}$$

Where $r(t)$ is p.d.f. of the random variable T and $F(x)$ is c.d.f. of any random variable. The corresponding p.d.f. to the c.d.f. in (1) is given by

$$g(x) = \{\frac{d}{dx} W(F(x))\} r(W(F(x))). \tag{2}$$

This study expands such family of distributions from two aspects: Firstly, operators theory allows one to generalize the T-X family to the extended T-X (I) by iterating differ integral operator, ψ , in which the upper limits of integrals is functions of c.d.f. of continuous random variables X_i and integrands are p.d.f. of a continuous random variable such as T. In each iteration of ψ, the upper limit of integral not only holds at the domain of random variable, but also it satisfy some additional conditions which must be met, so that, it can be a distribution function. Secondly, we present a direct extension of method presented for generating families of continuous distribution with Alzaatreh et al. [1] by iterated application of the composite function $(RoWoF)$ (x) and obtained the expression

$$(R_1 o W''' o R_2 o W'' o \ldots o R_{n-1} o W' o R_n o W o F)(x), \tag{3}$$

where R_i, i = 1,2, …,n, and F being the c.d.f.s of random Variables T_i and X; respectively. The functions in the form of $W(.),\ldots,W'(.),W''(.)$ and $W'''(.)$ are defined to link the supports of T_i to the range of X. We called this extended family of distributions, extended T-X (II) family of distributions. The extended T-X family of distributions of first and second kind can be further classified into several sub-families. The extended T-X (II) family includes three sub-families as: One sub-family has the same distributions for T_i and X. Second sub-family has the same distributions for T_i but different distribution for X. Third sub-family with different distributions for T_i and different distribution for X. New distributions discussed for the sub-families include Beta-Exponential-Y (B-E-Y) and extended Beta (I) (EB (I)) distributions. Also, extended T-X (I) family has three sub-families as: A sub-family has the same distributions for T and X_i; second sub-family with the same distributions for X_i but different distribution for T and third sub-family with different distributions for T as well as for X_i . New distributions discussed for these sub-families include Beta-Gamma-X (B-G-X) and extended Beta (II) (EB (II)) distributions. The rest of this article is organized as follows. In section 2, we define the extended T-X (I) family and provide some of its members. In section 3, we defined the extended T-X (I) family and provide some of its members.

II. PRELIMINARIES

In this section, we introduce notations, definitions and Preliminary facts which are used throughout this paper.

Definition 1. (a) With ${}^{u(x)}I_a$ we denote the operator that maps a function f , assumed to be (Riemann) integrable on the compact interval $[a,b]$; onto its primitive centered at a , i.e.

$${}^{u(x)}I_a = \int_a^{u(x)} f(t)dt, \tag{4}$$

for $a \le u(x) \le b$. When $u(x) = x$, we will simply write $I = \int_a^x f(t)dt$ instead of ${}^{x}I_a$.

(b) We denote the operator that maps a differentiable function onto its derivative with D, i.e.

$$Df(x) = f'(x) = \frac{d}{dx} f(x), \tag{5}$$

(c) For n $\in$ IN we use the symbols D^n and ${}^{u(x)}I_a^n$, which denote n-fold iterates of D and ${}^{u(x)}I_a$, respectively, i.e. we set

$$D^1 = D, \quad {}^{u(x)}I_a^1 = {}^{u(x)}I_a \tag{6}$$

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and

$$D^n = DD^{n-1}, \quad {}^{u(x)}I_a^n = {}^{u(x)}I_a \quad {}^{u(x)}I_a^{n-1} \qquad (7)$$

for n ≥ 2.

Definition 2. When u(x) is a differentiable function, we define the operator ${}^{u(x)}\Psi$ as

$$ {}^{u(x)}\Psi f = D \quad {}^{u(x)}I_a f, \qquad (8)$$

Whenever ${}^{u(x)}I_a f \in L^1[a,b]$ and we call it Differ integral operator. Here if u(x) = x then, from the calculus essential theorem, $\Psi f = \mathrm{I} f$, where I is Identical operator. For n ∈ IN we use the symbol Ψ^n for denoting n-fold iterates of ψ and we set

$$\Psi^1 = \Psi, \quad \Psi^n = \Psi\Psi^{n-1}. \qquad (9)$$

III. THE EXTENDED T-X (I) FAMILY OF DISTRIBUTIONS

Let r(t) be the p.d.f. of a random variable $T \in [a,b]$, for -∞≤ a < b ≤ ∞, and W(F(x)) be a function of the c.d.f. F(x) of random variable X, such that, $X \in [a,b]$, for -∞≤ c < d ≤ ∞. Suppose that W′(G(y)) be a function of the c.d.f. G(y) of any random variable Y, so that, W(F(x)) and W′(G(y)) satisfy the following conditions:

(i) W (F(x)) ∈ [a, b] and W′(G(y)) ∈ [c,d],

(ii) W (F(x)) and W′(G(y)) are differentiable and monotonically non-decreasing,

(iii) W (F(x)),W′(G(y)) $\rightarrow 0$ as x, y $\rightarrow -\infty$ and W(F(x)),W′(G(y)) $\rightarrow 1$ as x, y→ ∞.

We define the c.d.f. for the T-X-Y class of distributions for random variables X and Y, respectively, in the following definition.

Definition 3.Let X be a continuous random variable with p.d.f. f (x) and c.d.f. F(x) defined on [c,d] , T bea continuous random variable with p.d.f. r(t) defined on [a,b] and Y be a random variable with p.d.f. g(y) and c.d.f. G(y). The c.d.f. of a new family of distributions is defined as

$$H(y) = {}^{W'(G(y))}I_c \quad {}^{W(F(x))}\Psi r(t), \qquad (10)$$

and the corresponding p.d.f. associated with (10) is

$$h(y) = {}^{W'(G(y))}\Psi \quad {}^{W(F(x))}\Psi r(t), \qquad (11)$$

Where W(F(x)) and W′(G(y)) satisfy the above conditions (i) to (iii).

A first result, which will be most important for the later generalization to any number of random variable, can be obtained from definition 3. Utilizing the recursive formula of part three in definition 1, we can give an explicit formula of p.d.f. and c.d.f. for the extended T-X (I) family of distributions.

Definition 4.Let $X_1, X_2,..., X_n$ be continuous random variables with p.d.f. $f(x_i)$ and c.d.f. $F(x_i)$ defined $[c_i, d_i]$, i = 1,2, …, n , T be a continuous random variable with p.d.f. r(t) defined on [a,b] and X be a random variable with p.d.f. f (x) and c.d.f. F(x). The c.d.f. of a new family of distributions is defined as

$$H(y) = {}^{W'''(F(x))}I_{c_1} \quad {}^{W''(F(x_1))}\Psi \ldots {}^{W'(F(x_{n-1}))}\Psi \quad {}^{W(F(x_n))}\Psi r(t), \qquad (12)$$

and the corresponding p.d.f. associated with (12) is

$$h(y) = {}^{W'''(F(x))}\Psi \quad {}^{W''(F(x_1))}\Psi \ldots {}^{W'(F(x_{n-1}))}\Psi \quad {}^{W(F(x_n))}\Psi r(t), \qquad (13)$$

where W′′′(F(x)), W′′(F(x_1)), …,W′(F(x_{n-1})) and W(F(x_n)) satisfy the above conditions (i) and (ii) . In addition, we have W(F(x_n)) $\in [a,b]$, W′(F(x_{n-1})) $\in [c_n, d_n]$, …, W′′(F(x_1)) $\in [c_2, d_2]$, W′′′(F(x)) $\in [c_1, d_1]$.

Here, we are presented some families of extended T-X (I) distributions as:

Beta- X- Y family: If a random variable T follows the Beta distribution with parameters α and β, then

$$r(t) = \frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha,\beta)}, \quad t \in (0,1).$$

From (11), the p.d.f. of Beta- X- Y family is defined as

$$h(y) = \frac{g(y)DW'(G(y))f(W'(G(y)))(F(W'(G(y))))^{\alpha-1}(1-F(W'(G(y))))^{\beta-1}}{B(\alpha,\beta)}, \qquad (14)$$

Where F is c.d.f. of random variable X and f is its corresponding p.d.f. . Also G is c.d.f. of random variable Y and g is its corresponding p.d.f. . The c.d.f. of (14) can be expressed in terms of the incomplete beta function. The c.d.f. of Beta- X- Y family is

$$H(y) = \mathrm{I}_{W'(G(y))}(\alpha,\beta). \qquad (15)$$

For example, if a random variable X follows the Beta distribution with parameters α and β, also, if a random Variable T follows the Exponential distribution with parameter λ, then the p.d.f. of Beta- Exponential- Y is

$$h(y) = \lambda B^{-1}(\alpha,\beta)g(y)(1-G(y))^{\lambda\beta-1}(1-(1-G(y))^{\lambda})^{\alpha-1}, \qquad (16)$$

Since the p.d.f. of Beta- Exponential is

$$f(x) = \lambda B^{-1}(\alpha,\beta)e^{-\lambda\beta x}(1-e^{-\lambda x})^{\alpha-1}, \qquad (17)$$

By taking W′(G(y))=-log(1-G(y)), we get

$$h(y) = \frac{g(y)}{1-G(y)}\lambda B^{-1}(\alpha,\beta)e^{-\lambda\beta(-\log(1-G(y)))}(1-e^{-\lambda(-\log(1-G(y)))})^{\alpha-1}, \qquad (18)$$

Such that, it results (16).

The extended -Beta (I) family: In extended T-X (I) family, we considered a sub-family, which has the same distributions for

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T and X_i, i = 1, 2, …, n. If the support of T is bounded, without loss of generality, we assume that the support of T is [0,1]. Distributions for such T include uniform (0,1), Beta and Kumaraswamy. Also, W(F(x_i)) can be defined as F(x). In such case, we present the extended Beta (I) (E-B (I)) and extended Kumaraswam (I) (E-Kw (I)) families of distributions as following definitions.

Definition 5.Let $X_1, X_2,..., X_n$ be random variables from Beta distribution with α and β parameters i.e.

$$f(x_i) = \frac{x_i^{\alpha-1}(1-x_i)^{\beta-1}}{B(\alpha,\beta)}, \quad x \in (0,1), \; i = 1,2,\ldots,n.$$

Suppose that T be a random variable of Beta distribution with α and β parameters, r(t) p.d.f. and R(t) c.d.f.. Then the c.d.f. of the E-B (I) family of distributions is defined as

$$H(x) = {}^{F(x)}I_0 \; {}^{F(x)}\Psi^{n-1} r(t), \tag{20}$$

and corresponding p.d.f. associated with (20) is

$$h(x) = {}^{F(x)}\Psi^{n} r(t), \tag{21}$$

that is the p.d.f. of E-B (I) family is defined as

$$h(x) = (B(\alpha,\beta))^{-(n+1)} x^{\alpha-1}(1-x)^{\beta-1} F^{\alpha-1}(x)(1-F(x))^{\beta-1}$$

$$\times (F^2(x))^{\alpha-1}(1-F^2(x))^{\beta-1} ... (F^n(x))^{\alpha-1}(1-F^n(x))^{\beta-1},$$

Where, $F^0 = \mathrm{I}$, $F^2 = FoF, \ldots, F^n = FoF^{n-1}$. Special Cases of E-B family are as follow:

(1) When $n = 0$, the E-B family reduces to the Beta distribution.

(2) When $n = 1$, the E-B family reduces to the Beta-generated distribution (BG) which has been introduced by Eugene et al. [3].

IV. THE EXTENDED T-X (II) FAMILY OF DISTRIBUTIONS

We consider r(t) be the p.d.f. of a random variable T ∈ [a,b], for -∞≤ a < b ≤ ∞, h(s) be the p.d.f. of a random variable S ∈ [c,d], for -∞≤ c < d ≤ ∞ and W(F(x)) be a function of the F(x) which is c.d.f. of random variable X, so that W(F(x)) satisfies the following conditions:

(i)W(F(x)) ∈ [a;b] and W′(${}^{W(F(x))}I_a$) ∈ [c,d],

(ii) W(F(x)) and W′(${}^{W(F(x))}I_a$) is differentiable and monotonically non-decreasing,

(iii) W(F(x)), W′(${}^{W(F(x))}I_a$) $\to 0$ as x $\to -\infty$ and W(F(x)), W′(${}^{W(F(x))}I_a$) $\to 1$ as x $\to \infty$.

We define the c.d.f. for the S-T-X class of distributions for random variable X in the following definition.

Definition 6. Let T be a continuous random variable with p.d.f. r(t) defined on [a,b], S be the other continuous random variable with p.d.f. v(s) defined on [c,d] and X be a random variable with p.d.f. f (x) and c.d.f. F(x). The c.d.f. of a new family of distributions is defined as

$$H(x) = \int_c^{W'(R(W(F(x))))} v(s)ds, \tag{22}$$

Where, $R(W(F(x))) = \int_a^{W(F(x))} r(t)dt$, and corresponding p.d.f. associated with (22) is

$$= \mathrm{D}\{W'(R(W(F(x))))\}\; v(W'(R(W(F(x))))) \tag{23}$$

Such that, W(F(x)) and W′(${}^{W(F(x))}I_a$) satisfy the above Conditions (i) to (iii).

A first result, which will be most important for the later generalization to any number of random variable, can be obtained from definition 6. Utilizing the recursive formula of part three in definition 1, we can give an explicit formula of p.d.f. and c.d.f. for the Extended T-X (II) family of distributions.

Definition 7. Let $T_1, T_2,..., T_n$ be continuous random variables with p.d.f. $r(t_i)$ defined on $[a_i, b_i]$, i = 1, 2,.., n, and X be a random variable with p.d.f. f (x) and c.d.f. F(x). The c.d.f. of a new family of distributions is defined as

$$H(x) = R_1(W'''(R_2(W''(...(R_{n-1}(W'(R_n(W(F(x))))))....)))), \tag{24}$$

where $R_i(z) = \int_{a_i}^{z} r(t_i)dt_i$, i = 1, 2, …, n and corresponding p.d.f. associated with (24) is

$$h(x) = D\{W'''(R_2(W''(...(R_{n-1}(W'(R_n(W(F(x))))))...))))\}$$
$$\times t_1(W'''(R_2(W''(...(R_{n-1}(W'(R_n(W(F(x))))))...)))), \tag{25}$$

Where W‴(.), W″(.), …,W′(.) and W(.) satisfy the above conditions (i) to (iii).

In here, we are presented some families of extended T-X (II) distributions as:

Beta- Gamma- X family: If a random variable T follows the Gamma distribution with parameters α and β, then $r(t) = \Gamma^{-1}(\alpha)\beta^{-\alpha}t^{\alpha-1}e^{-t\beta^{-1}}$, $t \in [0,\infty)$, and other random variable S follows the Beta distribution with parameters a and b wirh $g(s) = B^{-1}(a,b)s^{a-1}(1-s)^{b-1}$. From (21), the p.d.f. of Beta-Gamma- Y family is defined as

$$H(x) = \mathrm{I}_{\Gamma^{-1}(\alpha)\gamma(\alpha,\frac{W(F(x))}{\beta})}(a,b), \tag{26}$$

where $\mathrm{I}_t(a,b)$ and $\gamma(\alpha,\beta)$ are incomplete beta and incomplete gamma functions, respectively. The corresponding p.d.f. associated with (26) is

$$h(x) = \frac{1}{B(a,b)} f(x)W'(f(x))(W(F(x)))^{\alpha-1}e^{-\beta^{-1}W(F(x))}\Gamma^{-1}(\alpha)\beta^{-\alpha}$$
$$\times(\Gamma^{-1}(\alpha)\gamma(\alpha,\frac{W(F(x))}{\beta}))^{a-1}(1-\Gamma^{-1}(\alpha)\gamma(\alpha,\frac{W(F(x))}{\beta}))^{b-1}. \tag{27}$$

where (′) means that derivative.

The extended -Beta (II) family: In extended T-X (II) family, if the supports of T_i, i = 1, 2, …,n, are bounded, without loss of generality, we assume that the support of T_i, i = 1, 2, …, n is [0,1]. Distributions for such T_i include uniform (0,1), Beta and Kumaraswamy. By taking W(F(x))=F(x), we present the extended Beta (II) (E-B (II)) and extended Kumaraswamy (II) (E-Kw (II)) families of distributions as following definitions.

Definition 8. Let $T_1, T_2, \ldots, T_n$ be random variables with Beta distribution and α and β parameters, i.e.

$$r(t_i) = \frac{t_i^{\alpha-1}(1-t_i)^{\beta-1}}{B(\alpha,\beta)}, \quad t_i \in (0,1),$$

i = 1, 2, …, n . Suppose that X be a random variable of Beta distribution with α and β parameters, f (x) p.d.f. and F(x) c.d.f. . The c.d.f. of the E-B (II) family of distributions is defined as

$$H(x) = F^k(x), \quad k = 1,2,\ldots, n, \tag{28}$$

and the corresponding p.d.f. associated with (28) is

$$h(x) = \prod_{k=0}^{n} f(F^k(x)), \quad n \in N_0, \tag{29}$$

where, $F^0 = \mathrm{I}$, $F^2 = FoF$, …, $F^n = FoF^{n-1}$. that is, p.d.f. of the E-B (II) family is defined as

$$h(x) = (B(\alpha,\beta))^{-(n+1)} x^{\alpha-1}(1-x)^{\beta-1} F^{\alpha-1}(x)(1-F(x))^{\beta-1}$$
$$\times (F^2(x))^{\alpha-1}(1-F^2(x))^{\beta-1} \ldots (F^n(x))^{\alpha-1}(1-F^n(x))^{\beta-1}.$$

Special cases of E-B (II) family are as follow:

(1) When n = 0, the E-B (II) family reduces to the Beta distribution.

(2) When n = 1, the E-B (II) family reduces to the Beta-generated distribution (BG) which is introduced by Eugene et al. [3].

Similarly, if in the above definition $T_1, T_2, \ldots, T_n$ be random variables with Kumaraswamy distribution with α and β parameters, then

$$r(t_i) = \alpha\beta t_i^{\alpha-1}(1-t_i^{\alpha})^{\beta-1}, t_i \in [0,1], i = 1,2,\ldots,n.$$

Suppose that X be a random variable of Kumaraswamy distribution with α and β parameters, p.d.f. f (x) and c.d.f. F(x). Then p.d.f. of the extended Kumaraswamy family (E-Kw (II)) becomes as

$$h(x) = (\alpha\beta)^{n+1} x^{\alpha-1}(1-x^{\alpha})^{\beta-1}(F(x))^{\alpha-1}(1-(F(x))^{\alpha})^{\beta-1}$$
$$\times (F^2(x))^{\alpha-1}(1-(F^2(x))^{\alpha})^{\beta-1} \ldots (F^n(x))^{\alpha-1}(1-(F^n(x))^{\alpha})^{\beta-1}.$$

An interesting case is that, when $T_1, T_2, \ldots, T_n$ and X be random variables following of Uniform (0,1) distribution, then the c.d.f. of the extended uniform family yields to,

$$H(x) = F(x), \tag{30}$$

and its corresponding p.d.f. is

$$h(x) = f(x). \tag{31}$$

That is, Extended Uniform (0,1) distribution acts as an Identical distribution among sub-family of Extended T-X (II) family with the same distributions T_i and X.

V. CONCLUSION

In this paper, we introduce extended T-X families of first kind and second kind, such that the extended T-X (II) family is the extension form of the T-X family proposed by Alzaatreh, Lee and Fomoye [1]. Some of their properties are derived and some members of the families are defined.

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