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Bayesian Estimation for the Reliability Function of Pareto Type I Distribution under Generalized Square Error Loss Function

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Abstract— The main objective of this study is to obtain and compare the performance of the standard Bayesian estimators of the reliability function R(t) of the Pareto type I distribution under Generalized square error loss function in addition of Quadratic loss function, with informative and non-informative prior, with assuming that, the scale parameter, α, is known. Estimators are compared empirically using Monte Carlo simulation by employing the Integral mean squares error (IMSE).

Index Terms—**Pareto distribution; Maximum likelihood estimator, Reliability function, Generalized square error loss function, Quadratic loss function.**

I. INTRODUCTION

The Pareto distribution was introduced (Pareto, 1897) as a model for the distribution of income. In addition to economics, its models in several different forms are now being used in a wide range of fields such as insurance, business, engineering, survival analysis, reliability and life testing.[6] The probability density function of Pareto distribution type I distribution is defined as following [8]:

$$f(t,\theta)=\frac{\theta\alpha^{\theta}}{t^{\theta+1}} \qquad t\geq\alpha,\quad \alpha>0,\quad \theta>0 \qquad (1)$$

Where t is a random variable, θ and α are the shape and scale parameters respectively. The cumulative distribution function of Pareto distribution type I, is given by:

$$F(x)=1-\left(\frac{\alpha}{t}\right)^{\theta} \qquad t\geq\alpha,\alpha>0,\quad \theta>0 \qquad (2)$$

The reliability function is given by:

$$R(t)=P_r(T>t)$$

$$=\int_t^{\infty} f(t,\alpha,\theta)\,dt$$

$$=\int_t^{\infty}\frac{\theta\alpha^{\theta}}{u^{\theta+1}}du=\theta\alpha^{\theta}\left[\frac{1}{\theta t^{\theta}}\right]$$

$$R(t)=\left(\frac{\alpha}{t}\right)^{\theta} \quad , t\geq\alpha,\ \ \theta>0,\quad \alpha>0 \qquad (3)$$

The Pareto distribution belongs to the exponential family of distribution as the density function (1) can be written as:

$$f(x,\theta)=\theta e^{\theta\ln\alpha-(\theta+1)\ln t}=\theta e^{-\ln t}e^{-\theta\ln\left(\frac{t}{\alpha}\right)}$$

Hence,

$$a(\theta)=\theta\ ,\ \ b(t)=e^{-\ln t}\ ,\ \ c(\theta)=-\theta\ ,\ \ d(t)=\ln\left(\frac{t}{\alpha}\right)$$

Therefore, statistic $P=\sum_{i=1}^{n}\ln\left(\frac{t_i}{\alpha}\right)$ is a complete sufficient statistic for θ, and it is easy to show that, P is distributed as Gamma distribution with parameters n and θ.

II. MAXIMUM LIKELIHOOD ESTIMATORS (MLE)

Let $(t_1,\dots,t_n)$ be the set of n random lifetime from Pareto type I distribution with parameters θ and α.

The likelihood function is

$$L(\theta;t_1,t_2,\dots,t_n)=\prod_{i=1}^{n}f(t_n;\theta)$$

$$L(t_1,\dots,t_n|\theta)=\frac{\theta^n\alpha^{n\theta}}{\prod_{i=1}^{n}t_i^{\theta+1}}=\theta^n\alpha^{n\theta}e^{-(\theta+1)\sum\ln t}$$

$$=\theta^n e^{n\theta\ln\alpha}e^{-(\theta+1)\sum\ln t} \qquad (4)$$

Taking the logarithm for the likelihood function, so we get the function

$$\ln L(\theta;t_1,\dots,t_n)=n\ln\theta+n\theta\ln\alpha-(\theta+1)\sum_{i=1}^{n}\ln t_i$$

The partial derivative for the log-likelihood function, with respect to θ and then, equating to zero is

$$\frac{\partial[\ln L(\theta;t_1,\dots,t_n)]}{\partial\theta}=\frac{n}{\theta}+n\ln\alpha-\sum_{i=1}^{n}\ln t_i=0$$

Hence, the MLE of θ denoted by $\hat{\theta}_{ML}$ is

$$\hat{\theta}_{MLE}=\frac{n}{\sum_{i=1}^{n}\ln t_i-n\ln\alpha}=\frac{n}{\sum_{i=1}^{n}\ln\left(\frac{t_i}{\alpha}\right)} \qquad (5)$$

Since the maximum likelihood estimator is invariant and one to one mapping [4], the Maximum likelihood estimator of Reliability function $\hat{R}(t)_{MLE}$ well be

$$\hat{R}(t)_{MLE}=\left(\frac{\alpha}{t}\right)^{\hat{\theta}_{MLE}} \qquad (6)$$

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III. STANDARD BAYES ESTIMATOR

In this section, we used a two loss functions as following

A. Generalized Square Error Loss Function (GS)

Al-Nasser and Saleh (2006) suggested the Generalized square error loss function in estimating the scale parameter and the Reliability function for Weibull distribution, which introduced as follows[11]:

$$l_1(\theta,\hat\theta) = (\sum_{j=0}^{k} a_j\theta^j)(\hat\theta-\theta)^2 \quad , \quad k = 0,1,2,3,\dots \qquad (7)$$

Where, a_j $(j = 0,1,2,3,\dots k)$ is a constant.

B. Quadratic Loss Function (QLF)

The Quadratic loss function which is asymmetric loss function defined for the positive values of the parameter. The Quadratic loss function $L_2(\theta,\hat\theta)$ defined as follows: [5]

$$L_2(\theta,\hat\theta) = \left(\frac{\theta-\hat\theta}{\theta}\right)^2 \qquad (8)$$

C. Bayes Estimator under Jeffery Prior Information

Let us assume that θ has non-informative prior density defined as using Jeffrey prior information $g(\theta)$ which is given by[2]:

$$g_1(\theta) \propto \sqrt{I(\theta)}$$

Where $I(\theta)$ represents Fisher information which is defined as follows [8]:

$$I(\theta) = -nE\left(\frac{\partial^2 \ln f}{\partial\theta^2}\right) \quad , \text{ Hence,}$$

$$g_1(\theta) = b\sqrt{-nE\left(\frac{\partial^2 \ln f(t;\theta)}{\partial\theta^2}\right)} \qquad (9)$$

$$\frac{\partial \ln f}{\partial\theta} = \frac{1}{\theta} + \ln\alpha - \ln t$$

$$E\left(\frac{\partial^2 \ln f(t;\theta)}{\partial\theta^2}\right) = -\frac{1}{\theta^2}$$

After substitution into (9), we get

$$g_1(\theta) = \frac{b}{\theta}\sqrt{n} \quad , \quad \theta > 0 \qquad (10)$$

The posterior density function is

$$h_1(\theta|t_1,\dots,t_n) = \frac{g_1(\theta)L(\theta;t_1,\dots,t_n)}{\int_0^\infty g_1(\theta)L(\theta;t_1,\dots,t_n)\,d\theta}$$

$$h_1(\theta|t,\dots,t_n) = \frac{\theta^n e^{n\theta\ln\alpha} e^{-(\theta+1)\sum\ln x}\frac{c}{\theta}\sqrt{n}}{\int_0^\infty \theta^n e^{n\theta\ln\alpha} e^{-(\theta+1)\sum\ln x}\frac{c}{\theta}\sqrt{n}\,d\theta}$$

Where α is a constant.

$$h_1(\theta|t,\dots,t_n) = \frac{\theta^{n-1}e^{-\theta P}}{\int_0^\infty \theta^{n-1}e^{-\theta P}d\theta}$$

Hence, the posterior density function of θ with Jeffery prior will be

$$h_1(\theta|t,\dots,t_n) = \frac{P^n\theta^{n-1}e^{-\theta P}}{\Gamma(n)} \qquad (11)$$

The posterior density is recognized as the density of the Gamma distribution

$\theta \sim \text{Gamma}(n,P)$, with $E(\theta) = \frac{n}{P}$, $ver(\theta) = \frac{n}{P^2}$

Bayes estimator under Generalized square error loss function: Recall that, the Generalized square error loss function (GS) is:[9]

$$l_1(\hat\theta,\theta) = (a_0 + a_1\theta + \dots + a_k\theta^k)(\hat\theta-\theta)^2$$

Then, the Risk function under the generalized square error loss function is denoted by $R_{GS}(\hat\theta,\theta)$ is

$$R_{GS}(\hat\theta,\theta) = E[l_1(\hat\theta,\theta)] = \int_0^\infty l_1(\hat\theta,\theta)h_1(\theta|\underline{t})d\theta$$

$$R_{GS}(\hat\theta,\theta) = \int_0^\infty (a_0 + a_1\theta + \dots + a_k\theta^k)(\hat\theta^2 - 2\hat\theta\theta + \theta^2)h_1(\theta|\underline{t})d\theta$$

$$\begin{aligned} R_{GS}(\hat\theta,\theta) = {} & a_0\hat\theta^2 - 2a_0\hat\theta E(\theta|\underline{t}) + a_0E(\theta^2|\underline{t}) + a_1\hat\theta^2E(\theta|\underline{t}) \\ & -2a_1\hat\theta E(\theta^2|\underline{t}) + a_1E(\theta^3|\underline{t}) + \dots + a_k\hat\theta^2E(\theta^k|\underline{t}) \\ & -2a_k\hat\theta E(\theta^{k+1}|\underline{t}) + a_kE(\theta^{k+2}|\underline{t}) \end{aligned}$$

Taking the partial derivative for $R_{GS}(\hat\theta,\theta)$ with respect to $\hat\theta$ and setting it equal to zero yields:

$$\begin{aligned} \frac{\partial R_{GS}(\hat\theta,\theta)}{\partial\hat\theta} = {} & 2a_0\hat\theta - 2a_0E(\theta|\underline{t}) + 2a_1\hat\theta E(\theta|\underline{t}) - 2a_1E(\theta^2|\underline{t}) \\ & + \dots + 2a_k\hat\theta E(\theta^k|\underline{t}) - 2a_kE(\theta^{k+1}|\underline{t}) = 0 \end{aligned}$$

$$\hat\theta = \frac{a_0E(\theta|\underline{t}) + a_1E(\theta^2|\underline{t}) + \dots + a_kE(\theta^{k+1}|\underline{t})}{a_0 + a_1E(\theta|\underline{t}) + \dots + a_kE(\theta^k|\underline{t})} \qquad (12)$$

Since $\theta \sim \Gamma(n,P)$ and $E(\theta) = \frac{n}{P}$, $var(\theta) = \frac{n}{P^2}$

$$\hat\theta = \frac{a_0\frac{n}{P} + a_1\frac{(n+1)n}{P^2} + \dots + a_k\frac{(n+k)(n+k-1)\dots(n+1)n}{P^{k+1}}}{a_0 + a_1\frac{n}{P} + \dots + a_k\frac{(n+k-1)(n+k-2)\dots(n+1)n}{P^k}}$$

Therefore, the Bayes estimator for θ of Pareto distribution under generalized square error loss function with Jeffery prior denoted by $\hat\theta_{JGS}$ can be written as

$$\hat\theta_{JGS} = \frac{\sum_{j=0}^{k} a_j\frac{\Gamma(n+1+j)}{P^{j+1}\Gamma(n)}}{\sum_{j=0}^{k} a_j\frac{\Gamma(n+j)}{P^j\Gamma(n)}} \qquad (13)$$

Now, we can find the Bayes estimator of Reliability function under generalized square error loss function by two methods.

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1. ***using the probability density function of the parameter*** $\boldsymbol{\theta}$***:*** *[1]*

According to this method of obtaining the Bayes estimator for the reliability function using Jeffery prior under generalized square error loss function will be

$$\hat{R}(t)=\frac{a_0E(R(t)|\underline{t})+a_1E((R(t))^2|\underline{t})+\cdots+a_kE((R(t))^{k+1}|\underline{t})}{a_0+a_1E(R(t)|\underline{t})+\cdots+a_kE((R(t))^k|\underline{t})} \quad (14)$$

The m[th] moment of $R(t)$ can be found as follows

$$E((R(t))^m|\underline{t}) = \int_0^\infty (R(t))^m h_1(\theta|\underline{t})d\theta \quad (15)$$

$$E((R(t))^m|\underline{t}) = \int_0^\infty \frac{P^n\left(P-m\ln\left(\frac{\alpha}{t}\right)\right)^n\theta^{n-1}e^{-\theta\left(P-m\ln\left(\frac{\alpha}{t}\right)\right)}}{\left(P-m\ln\left(\frac{\alpha}{t}\right)\right)^n\Gamma(n)}d\theta$$

$$E((R(t))^m|\underline{t}) = \left\{\frac{P}{P-m\ln\left(\frac{\alpha}{t}\right)}\right\}^n \quad (16)$$

By putting m=1,2,…,k+1 in (16), and substituting into (14), we get,

$$\hat{R}(t)=\frac{a_0\left\{\frac{P}{P-\ln\left(\frac{\alpha}{t}\right)}\right\}^n+a_1\left\{\frac{P}{P-2\ln\left(\frac{\alpha}{t}\right)}\right\}^n+\cdots+a_k\left\{\frac{P}{P-(k+1)\ln\left(\frac{\alpha}{t}\right)}\right\}^n}{a_{0+}a_1\left\{\frac{P}{P-\ln\left(\frac{\alpha}{t}\right)}\right\}^n+\cdots+a_k\left\{\frac{P}{P-k\ln\left(\frac{\alpha}{t}\right)}\right\}^n}$$

Therefore, the Bayes estimator for the R (t) of Pareto distribution under generalized square error loss function with Jeffery Prior denoted by $\hat{R}(t)_{JGS}$ is:

$$\hat{R}(t)_{JGS} = \frac{\sum_{j=0}^k a_j\left\{\frac{P}{P-(j+1)\ln\left(\frac{\alpha}{t}\right)}\right\}^n}{\sum_{j=0}^k a_j\left\{\frac{P}{P-j\ln\left(\frac{\alpha}{t}\right)}\right\}^n} \quad (17)$$

2. ***Using the subsequent probability density function for the reliability function [7]***

The estimation of Bayes to the reliability function can be found using loss functions and rely on subsequent probability density function for the reliability function R (t), which can be found through the subsequent probability density function parameter θ using the style of the conversion through the relationship reliability function R (t) for parameter θ is done as follows :

$R(t) = \left(\frac{\alpha}{t}\right)^\theta \Rightarrow \ln R(t) = \theta\ln\left(\frac{\alpha}{t}\right)$, Hence

$$\theta = \frac{\ln R(t)}{\ln\left(\frac{\alpha}{t}\right)} \quad (18)$$

So, the subsequent probability density function for reliability function can be obtained as following:

$$\pi(R(t)|\underline{t}) = h_1(\theta)|J| \quad (19)$$

$$\pi(R(t)|\underline{t}) = h_1\left(\frac{\ln R(t)}{\ln\left(\frac{\alpha}{t}\right)}\right)|J| \quad (20)$$

$$|J| = \left|\frac{d\theta}{dR(t)}\right| = \frac{1}{\ln\left(\frac{\alpha}{t}\right)R(t)} \quad (21)$$

$$h_1(Z) = \frac{P^n}{\Gamma(n)}\left(\frac{\ln R(t)}{\ln\left(\frac{\alpha}{t}\right)}\right)^{n-1}e^{-(Z)P} \quad (22)$$

Now, substituting (21) and (22) into (20) given:

$$\pi(R(t)|\underline{t}) = \frac{P^n\left(\frac{\ln R(t)}{\ln\left(\frac{\alpha}{t}\right)}\right)^{n-1}e^{\left(\frac{\ln R(t)}{\ln\left(\frac{\alpha}{t}\right)}\right)P}}{\Gamma(n)\ln\left(\frac{\alpha}{t}\right)R(t)} \quad (23)$$

$$E((R(t))^m|\underline{t}) = \int_0^\infty (R(t))^m\pi(R(t)|t)dR(t) \quad (24)$$

After substituting (23) into (24), we can find the m[th] moment of R (t) as follows

$$E((R(t))^m|\underline{t}) = P^n\int_0^\infty (R(t))^m\left(\frac{\ln R(t)}{\ln\left(\frac{\alpha}{t}\right)}\right)^{n-1}\frac{e^{-\left(\frac{\ln R(t)}{\ln\left(\frac{\alpha}{t}\right)}\right)P}}{\Gamma(n)\ln\left(\frac{\alpha}{t}\right)R(t)}dR(t)$$

By using the transformation, $y = \frac{\ln R(t)}{\ln\left(\frac{\alpha}{t}\right)}$ which implies that,

$$\ln R(t) = y\ln\left(\frac{\alpha}{t}\right) \Rightarrow R(t) = e^{y\ln\left(\frac{\alpha}{t}\right)}$$

$\Rightarrow dR(t) = \ln\left(\frac{\alpha}{t}\right)e^{y\ln\left(\frac{\alpha}{t}\right)}dy$, Hence,

$$E((R(t))^m|\underline{t}) = P^n\int_0^\infty e^{ym\ln\left(\frac{\alpha}{t}\right)}y^{n-1}e^{-yT}\frac{\ln\left(\frac{\alpha}{t}\right)e^{y\ln\left(\frac{\alpha}{t}\right)}}{\Gamma(n)\ln\left(\frac{\alpha}{t}\right)e^{y\ln\left(\frac{\alpha}{t}\right)}}dy$$

$$= \frac{P^n}{\left\{P-m\ln\left(\frac{\alpha}{t}\right)\right\}^n}\int_0^\infty \frac{\left\{P-m\ln\left(\frac{\alpha}{t}\right)\right\}^n y^{n-1}e^{-y\left(P-m\ln\left(\frac{\alpha}{t}\right)\right)}}{\Gamma(n)}dy$$

$$E((R(t))^m|\underline{t}) = \left\{\frac{P}{P-m\ln\left(\frac{\alpha}{t}\right)}\right\}^n \quad (25)$$

Therefore, the Bayes estimator for the R (t) of Pareto distribution under Generalized Square Error Loss Function with Jeffery Prior denoted by $\hat{R}(t)_{JGS}$ is given by:

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$$\hat{R}(t)_{JGS} = \frac{\sum_{j=0}^{k} a_j \left\{\frac{P}{P-(j+1)\ln\left(\frac{\alpha}{t}\right)}\right\}^n}{\sum_{j=0}^{k} a_j \left\{\frac{P}{P-j\ln\left(\frac{\alpha}{t}\right)}\right\}^n} \quad (26)$$

Bayes estimator under Quadratic Loss function

Now, we derive Bayes estimator using Quadratic Loss function, where

$$l_2(\hat{\theta},\theta) = \left(\frac{\theta-\hat{\theta}}{\theta}\right)^2 = \left(1-\frac{\hat{\theta}}{\theta}\right)^2$$

The Risk function under the Quadratic Loss function is denoted by $R_Q(\hat{\theta},\theta)$, where

$$R_Q(\hat{\theta},\theta) = E(1-\frac{\hat{\theta}}{\theta})^2 = \int_0^\infty (1-\frac{\hat{\theta}}{\theta})^2 h_1(\theta|\underline{t})d\theta$$

Taking the partial derivative for the R_Q, with respect to $\hat{\theta}$ and then, equating to zero is:

$$\frac{\partial R_Q(\hat{\theta},\theta)}{\partial\hat{\theta}} = 2\int_0^\infty (1-\frac{\hat{\theta}}{\theta})\left(-\frac{1}{\theta}\right)h_1(\theta|\underline{t})d\theta = 0$$

$$\Rightarrow \hat{\theta}\int_0^\infty \frac{1}{\theta^2}h_1(\theta|\underline{t})d\theta - \int_0^\infty \frac{1}{\theta}h_1(\theta|\underline{t})d\theta = 0$$

$$\hat{\theta} = \frac{E(\frac{1}{\theta})}{E(\frac{1}{\theta^2})} \quad (27)$$

$$E\left(\frac{1}{\theta}\right) = \int_0^\infty \frac{P^n\theta^{n-2}e^{-\theta P}}{\Gamma(n)}d\theta = \frac{P}{(n-1)}$$

$$E\left(\frac{1}{\theta^2}\right) = \int_0^\infty \frac{P^n\theta^{n-3}e^{-\theta P}d\theta}{\Gamma(n)} = \frac{P^2}{(n-1)(n-2)}$$

Substituting into (27) gives

$$\hat{\theta}_{JQ} = \frac{\frac{P}{(n-1)}}{\frac{P^2}{(n-1)(n-2)}} = \frac{n-2}{P}$$

So, the Bayesian estimation of Reliability function under Quadratic loss function with Jeffery's prior

$$\hat{R}(t) = \frac{E\left\{\frac{1}{R(t)}\right\}}{E\left[\frac{1}{\{R(t)\}^2}\right]} \quad (28)$$

$$E\left(\frac{1}{R(t)}\right) = \int_0^\infty \frac{1}{R(t)}h_1(\theta|\underline{t})d\theta \quad (29)$$

$$E\left(\frac{1}{R(t)}\right) = \int_0^\infty e^{-\theta\ln(\frac{\alpha}{t})}\frac{P^n\theta^{n-1}e^{-\theta P}d\theta}{\Gamma(n)}$$

$$= \frac{P^n}{\left(P+\ln(\frac{\alpha}{t})\right)^n}\int_0^\infty \frac{\left(P+\ln(\frac{\alpha}{t})\right)^n\theta^{n-1}e^{-\theta\left(P+\ln(\frac{\alpha}{t})\right)}}{\Gamma(n)}d\theta$$

$$E\left(\frac{1}{R(t)}\right) = \frac{P^n}{\left\{P+\ln(\frac{\alpha}{t})\right\}^n} \quad (30)$$

$$E\left[\frac{1}{\{R(t)\}^2}\right] = \int_0^\infty e^{-2\theta\ln(\frac{\alpha}{t})}\frac{P^n\theta^{n-1}e^{-\theta P}}{\Gamma(n)}d\theta$$

$$= \frac{P^n}{\left\{P+2\ln(\frac{\alpha}{t})\right\}^n}\int_0^\infty \frac{\left(P+2\ln(\frac{\alpha}{t})\right)^n\theta^{n-1}e^{-\theta\left(P+2\ln(\frac{\alpha}{t})\right)}}{\Gamma(n)}d\theta$$

$$E\left[\frac{1}{\{R(t)\}^2}\right] = \frac{P^n}{\left\{P+2\ln(\frac{\alpha}{t})\right\}^n} \quad (31)$$

After substituting into(28), we get the Bayes estimator for the R(t) of Pareto distribution under Quadratic loss function with Jeffery Prior denoted by $\hat{R}(t)_{JQ}$ is:

$$\hat{R}(t)_{JQ} = \left\{\frac{P+2\ln\left(\frac{\alpha}{t}\right)}{P+\ln(\frac{\alpha}{t})}\right\}^n \quad (32)$$

D. Bayes Estimator under Exponential Prior Distribution

Assuming that θ has informative prior as Exponential distribution, which takes the following form [1]:

$$g_2(\theta) = \frac{1}{\lambda}e^{-\frac{\theta}{\lambda}} \quad , \ \theta > 0, \lambda > 0 \quad (33)$$

Since, the posterior distribution of θ is:

$$h_2(\theta|\underline{t}) = \frac{L(t_1,\dots,t_n|\theta)g_2(\theta)}{\int_0^\infty L(t_1,\dots,t_n|\theta)g_2(\theta)d\theta}$$

$$h_2(\theta|\underline{t}) = \frac{\theta^n e^{-(\theta+1)P}\frac{1}{\lambda}e^{-\frac{\theta}{\lambda}}}{\int_0^\infty \theta^n e^{-(\theta+1)P}\frac{1}{\lambda}e^{-\frac{\theta}{\lambda}}d\theta} \quad (34)$$

$$h_2(\theta|\underline{t}) = \frac{(P+\frac{1}{\lambda})^{n+1}\theta^n e^{-\theta(P+\frac{1}{\lambda})}}{\Gamma(n+1)}$$

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$$h_2(\theta|\underline{t}) = \frac{(P+\frac{1}{\lambda})^{n+1}\theta^{n}e^{-\theta(P+\frac{1}{\lambda})}}{\Gamma(n+1)}$$

Bayes Estimator under Generalized Square Error Loss Function

To obtain the Bayes Estimator using Generalized Square Error Loss Function (GS), we have

$\theta \sim \Gamma\left(n+1, P+\frac{1}{\lambda}\right)$, with:

$$E(\theta) = \frac{n+1}{P+\frac{1}{\lambda}}, \; var(\theta) = \frac{n+1}{\left(P+\frac{1}{\lambda}\right)^2}$$

$$E(\theta^k) = \frac{(n+k)(n+k-1)\ldots.(n+1)}{\left(P+\frac{1}{\lambda}\right)^K} \qquad (35)$$

After substituting into generalized square error loss function (12), we get

$$\hat{\theta} = \frac{a_0\left\{\frac{n+1}{P+\frac{1}{\lambda}}\right\} + a_1\left\{\frac{(n+2)(n+1)}{\left(P+\frac{1}{\lambda}\right)^2}\right\} + \cdots + a_k\left\{\frac{(n+k)(n+k-1)\ldots..(n+1)}{(P+\frac{1}{\lambda})^{k+1}}\right\}}{a_0 + a_1\left\{\frac{n+1}{P+\frac{1}{\lambda}}\right\} + \cdots + a_k\left\{\frac{(n+k)(n+k-1)\ldots.(n+1)}{(P+\frac{1}{\lambda})^{k}}\right\}}$$

So, the Bayes estimator of θ under generalized square error loss function denoted by $\hat{\theta}_{EGS}$ is

$$\hat{\theta}_{EGS} = \frac{\sum_{j=0}^{k} a_j\left\{\frac{\Gamma(n+2+j)}{(P+\frac{1}{\lambda})^{j+1}\,\Gamma(n+1)}\right\}}{\sum_{j=0}^{k} a_j\left\{\frac{\Gamma(n+1+j)}{(P+\frac{1}{\lambda})^{j}\,\Gamma(n+1)}\right\}} \qquad (36)$$

From (12) we get, we can find the estimator of Reliability function under Generalized Square Error loss function:

$$\hat{R}(t) = \frac{a_0E(R(t)|\underline{t}) + a_1E\left((R(t))^2|\underline{t}\right) + \cdots + a_kE\left((R(t))^{k+1}|\underline{t}\right)}{a_0 + a_1E(R(t)|\underline{t}) + \cdots + a_kE\left((R(t))^{k}|\underline{t}\right)} \qquad (37)$$

Since $R(t) = \left(\frac{\alpha}{t}\right)^{\theta} = e^{\theta\ln(\frac{\alpha}{t})}$

$$E\left((R(t))^m|\underline{t}\right) = \int_0^{\infty}(R(t))^m h_2(\theta|\underline{t})d\theta \qquad (38)$$

$$= \frac{\left(\frac{1}{\lambda}+P\right)^{n+1}}{\left\{\frac{1}{\lambda}+P-m\ln(\frac{\alpha}{t})\right\}^{n+1}}\int_0^{\infty}\frac{\left\{\frac{1}{\lambda}+P-m\ln(\frac{\alpha}{t})\right\}^{n+1}\theta^{n}e^{-\theta\left(\frac{1}{\lambda}+P-m\ln(\frac{\alpha}{t})\right)}d\theta}{\Gamma(n+1)}$$

$$E\left((R(t))^m|\underline{t}\right) = \left\{\frac{\frac{1}{\lambda}+P}{\frac{1}{\lambda}+P-m\ln\left(\frac{\alpha}{t}\right)}\right\}^{n+1} \qquad (39)$$

After substituting into (37), we get the Bayes estimator of $R(t)$ under Generalized Square Error loss function denoted by $\hat{R}(t)_{EGS}$ as

$$\hat{R}(t)_{EGS} = \frac{\sum_{j=0}^{k} a_j\left\{\frac{\frac{1}{\lambda}+P}{\frac{1}{\lambda}+P-(j+1)\ln\left(\frac{\alpha}{t}\right)}\right\}^{n+1}}{\sum_{j=0}^{k} a_j\left\{\frac{\frac{1}{\lambda}+P}{\frac{1}{\lambda}+P-j\ln\left(\frac{\alpha}{t}\right)}\right\}^{n+1}} \qquad (40)$$

***Bayes Estimator under Quadratic loss function**:*

To obtain the Bayes Estimator under Quadratic loss function, we have:

$$E\left(\frac{1}{\theta}|\underline{t}\right) = \int_0^{\infty}\frac{1}{\theta}h_2(\theta|\underline{t})d\theta = \frac{(P+\frac{1}{\lambda})}{n}\int_0^{\infty}\frac{(P+\frac{1}{\lambda})^{n}\,\theta^{n-1}e^{-\theta(P+\frac{1}{\lambda})}d\theta}{\Gamma(n)}$$

$$E\left(\frac{1}{\theta}|\underline{t}\right) = \frac{(P+\frac{1}{\lambda})}{n} \qquad (41)$$

$$E\left(\frac{1}{\theta^2}|\underline{t}\right) = \int_0^{\infty}\frac{1}{\theta^2}h_2(\theta|\underline{t})d\theta \qquad (42)$$

$$E\left(\frac{1}{\theta^2}|\underline{t}\right) = \frac{(P+\frac{1}{\lambda})^2}{n(n-1)}\int_0^{\infty}\frac{(P+\frac{1}{\lambda})^{n-1}\,\theta^{n-2}e^{-\theta(P+\frac{1}{\lambda})}d\theta}{\Gamma(n-1)} = \frac{(P+\frac{1}{\lambda})^2}{n(n-1)}$$

After substituting into (27), the Bayes estimator of θ under Quadratic loss function with Exponential prior, denoted by $\hat{\theta}_{EQ}$ is:

$$\hat{\theta}_{EQ} = \frac{n-1}{(P+\frac{1}{\lambda})}$$

Bayesian estimation of Reliability function under Quadratic loss function with exponential prior is obtained according to (27), therefore,

$$\hat{R}(t) = \frac{E\left(\frac{1}{R(t)}\right)}{E\left\{\frac{1}{(R(t))^2}\right\}} \qquad (43)$$

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$$E\left(\frac{1}{R(t)}\right) = \int_0^\infty \frac{1}{R(t)} h_2(\theta|t)\, d\theta \qquad (44)$$

$$= \frac{\left(P+\frac{1}{\lambda}\right)^{n+1}}{\left(P+\frac{1}{\lambda}+\ln(\frac{\alpha}{t})\right)^{n+1}} \int_0^\infty \frac{\left(P+\frac{1}{\lambda}+\ln(\frac{\alpha}{t})\right)^{n+1} \theta^n e^{-\theta\left(P+\frac{1}{\lambda}+\ln(\frac{\alpha}{t})\right)}}{\Gamma(n+1)} d\theta$$

$$E\left(\frac{1}{R(t)}\right) = \frac{\left(P+\frac{1}{\lambda}\right)^{n+1}}{\left(P+\frac{1}{\lambda}+\ln(\frac{\alpha}{t})\right)^{n+1}} \qquad (45)$$

$$E\left(\frac{1}{(R(t))^2}\right) = \int_0^\infty \frac{e^{-2\theta\ln(\frac{\alpha}{t})} \left(P+\frac{1}{\lambda}\right)^{n+1} \theta^n e^{-\theta\left(P+\frac{1}{\lambda}\right)}}{\Gamma(n+1)} d\theta$$

$$E\left(\frac{1}{(R(t))^2}\right) = \frac{\left(P+\frac{1}{\lambda}\right)^{n+1}}{\left(P+\frac{1}{\lambda}+2\ln(\frac{\alpha}{t})\right)^{n+1}} \qquad (46)$$

After substituting into (43), the Bayes estimator for the R(t) of Pareto distribution under Quadratic loss function with exponential prior, denoted by $\hat{R}(t)_{EQ}$ is:

$$\hat{R}(t)_{EQ} = \left(\frac{P+\frac{1}{\lambda}+2\ln\left(\frac{\alpha}{t}\right)}{P+\frac{1}{\lambda}+\ln\left(\frac{\alpha}{t}\right)}\right)^{n+1} \qquad (47)$$

IV. SIMULATION STUDY

We generated L=2500 samples of size n = 20, 50, and 100 to represent small, moderate and large sample sizes from Pareto distribution with different values of the shape parameter (θ = 0.5, 1.5, and 2.5), with scale parameter $\alpha = 1, 1.4$. The scale parameter λ of Exponential prior is (λ =1.5, 3) and assuming the values of k, a_0, a_1 and a_2 in Generalized Square Error loss function to be: K= 1 with $a_0 = 0.5,\ a_1 = 2$, K= 2 with $a_0 = 0.5,\ a_1 = 2,\ a_2 = 3$ Monte – Carlo simulation study is performed to compare the methods of estimation by using mean square Errors (MSE's), as follows:

$$\text{MES}\,(\hat{R}(t)) = \frac{1}{L}\sum_{i=1}^{L}\left(\hat{R}_i(t) - R(t)\right)^2$$

Where: L is the number of replications, $\hat{R}(t_i)$ is the estimators of R(t), at the i^{th} replication. Then, compute the integral mean squares error (IMSE's) which is defined as distance between the estimate value of the reliability function and actual value of reliability fonction given by équation [3]:

$$IMSE\left(\hat{R}(t)\right) = \frac{1}{L}\sum_{i=1}^{L}\left[\frac{1}{n_t}\sum_{j=1}^{n_t}\left(\hat{R}_i(t_j) - R(t_j)\right)^2\right] = \frac{1}{n_t}\sum_{j=1}^{n_t} MSE\left(\hat{R}(t_j)\right)$$

Where n_t is the random limits of t_i , using t = 1.5, 1.8, 2.1, 2.4, 2.7, 3.

In this section, we have summarized and discussed the simulation results that are conducted to examine and compare the performance of the estimators for reliability function according to their IMSE:

The results show clearly, that, the increasing of the scale parameter α will decrease the values of IMSE for all estimators and with all cases.

- From table (1) when $\theta = 0.5, \alpha = 1, 1.4$, the performance of Bayes estimator under Generalized Squared error loss function when k=2 with exponential prior ($\lambda = 0.5$) is the best comparing to other estimators for all sample sizes. Followed by Bayes estimator under Generalized Squared error loss function when k=1 with exponential prior ($\lambda = 0.5$) for all sample size.

- The results in tables (2), (3) when $\theta = 1.5, 2.5, \alpha = 1, 1.4$, show that, Bayes estimator under Quadratic loss function with exponential prior (λ = 0.5) is the best comparing to other estimators for all sample sizes, followed by Bayes estimator under Generalized Squared error loss function when k=2 with exponential prior ($\lambda = 3$)

- On the other hand, using exponential prior with small value of λ (λ=0.5) is more appropriate than using exponential prior with relatively large value of λ (λ=3) with small values of θ, for all sample sizes.

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Table (1): IMSE Values for Different Estimators of Reliability Function of Pareto Distribution when $\theta = 0.5$

Estimator \ n		20		50		100	
		$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$
MLE		0.0040446	0.0021093	0.0014703	0.0007484	0.0007168	0.0003621
BJ(GS1), K=1		0.0036366	0.0019295	0.0014116	0.0007234	0.0007021	0.0003560
BJ(GS2), K=2		0.0035263	0.0018682	0.0013950	0.0007145	0.0006978	0.0003538
BJ(Qu)		0.0048125	0.0023994	0.0015736	0.0007865	0.0007419	0.0003621
BE(GS1) K=1	$\lambda = 0.5$	0.0032582	0.0017190	0.0013540	0.0006932	0.0006879	0.0003465
	$\lambda = 3$	0.0039314	0.0021333	0.0014562	0.0007542	0.0007139	0.0003630
BE(GS2) K=2	$\lambda = 0.5$	0.0031653	0.0016674	0.0013385	0.0006850	0.0006837	0.0003604
	$\lambda = 3$	0.0037608	0.0020468	0.0014295	0.0007414	0.0007070	0.0003607
BE(Qu)	$\lambda = 0.5$	0.0042584	0.0021157	0.0015062	0.0007524	0.0007265	0.0003636
	$\lambda = 3$	0.0054695	0.0027423	0.0016742	0.0008377	0.0007674	0.0003824
Best Estimator		BE(GS2) $\lambda = 0.5$	BE(GS2) $\lambda = 0.5$	BE(GS2) $\lambda = 0.5$	BE(GS2) $\lambda = 0.5$	BE(GS2) $\lambda = 0.5$	BE(GS2) $\lambda = 0.5$

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Table (2): IMSE Values for Different Estimators of Reliability Function of Pareto Distribution when $\theta = 1.5$

Estimator \ n		20		50		100	
		$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$
MLE		0.0060542	0.0052179	0.0024023	0.0002003	0.0012040	0.0009937
BJ(GS1), K=1		0.0058798	0.0047814	0.0023836	0.0001938	0.0011981	0.0009766
BJ(GS2), K=2		0.0060767	0.0047952	0.0024215	0.0001943	0.0012077	0.0009776
BJ(Qu)		0.0082058	0.0067932	0.0027369	0.0002233	0.0012894	0.0010510
BE(GS1) K=1	$\lambda = 0.5$	0.0062537	0.0044932	0.0024665	0.0001897	0.0012179	0.0009639
	$\lambda = 3$	0.0054469	0.0045802	0.0023047	0.0001902	0.0011783	0.0009676
BE(GS2) K=2	$\lambda = 0.5$	0.0067379	0.0047430	0.0025584	0.0001945	0.0012481	0.0009763
	$\lambda = 3$	0.0055510	0.0045335	0.0023273	0.0001896	0.0011840	0.0009658
BE(Qu)	$\lambda = 0.5$	0.0052673	0.0042101	0.0022718	0.0001843	0.0011704	0.0009524
	$\lambda = 3$	0.0083384	0.0069288	0.0027778	0.0002269	0.0013020	0.0010617
Best Estimator		BE(Qu), $\lambda = 0.5$	BE(Qu), $\lambda = 0.5$	BE(Qu), $\lambda = 0.5$	BE(Qu), $\lambda = 0.5$	BE(Qu), $\lambda = 0.5$	BE(Qu), $\lambda = 0.5$

Table (3): IMSE Values for Different Estimators of Reliability Function of Pareto Distribution when $\theta = 2.5$

Estimator \ n		20		50		100	
		$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1$	$\alpha = 1.4$
MLE		0.0038279	0.0048701	0.0015517	0.0019273	0.0007823	0.0009652
BJ(GS1), k=1		0.0042687	0.0048432	0.0016358	0.0019328	0.0008030	0.0009659
BJ(GS2), k=2		0.0045315	0.0050195	0.0016802	0.0019661	0.0008139	0.0009743
BJ(Qu)		0.0051197	0.0065207	0.0017773	0.0021894	0.0008425	0.00103271
`BE(GS1) K=1	$\lambda = 0.5$	0.0076865	0.0074016	0.0022770	0.0024322	0.0009681	0.00109466
	$\lambda = 3$	0.0038383	0.0044111	0.0015639	0.0018588	0.0007849	0.0009471
BE(GS2) K=2	$\lambda = 0.5$	0.0083752	0.0080390	0.0023902	0.0025496	0.0009960	0.0011250
	$\lambda = 3$	0.0040566	0.0045520	0.0016035	0.0018872	0.0007949	0.0009544
BE(Qu)	$\lambda = 0.5$	0.0035096	0.0042679	0.0015009	0.0018325	0.0007679	0.0009387
	$\lambda = 3$	0.0048390	0.0061238	0.0017384	0.0021398	0.0008337	0.0010217
Best Estimator		BE(Qu), $\lambda = 0.5$	BE(Qu), $\lambda = 0.5$	BE(Qu), $\lambda = 0.5$	BE(Qu), $\lambda = 0.5$	BE(Qu), $\lambda = 0.5$	BE(Qu), $\lambda = 0.5$