ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 4, Issue 2, August 2014

Similarity Solution for Partial Differential Equation of Fractional Order

Hetal Choksi and M. G. Timol Department of mathematics
Veer Narmad South Gujarat University, Surat.

***Abstract*— Using Finite Lie group of scaling transformation, the similarity solution is derived for partial differential equation of fractional order α. The reduce similarity equation will be an ordinary differential equation of fractional order with new independent variable known as similarity variable. Further similarity solution for some particular cases of fractional differential equation like diffusion equation, wave equation and the fractional diffusion-wave equation are also derived similarity solutions are well agreed to those found in literature.**

Index Terms—**Lie Group, Scaling Transformation, Similarity equation, Fractional diffusion-wave equation.**

I. INTRODUCTION

The fractional calculus is one of the most accurate tools to refine description of natural phenomena. Recently there has been considerable interest in the theory and application of fractional differential equations due to their numerous applications in many fields of applied science and engineering important phenomena in finance, viscoelasticity, material science, electromagnetic and electro chemistry [Barkai et al. [1], Meerschaert et al. [2], Mainardi [3], Tadjeran [4], King [5]] are well described by differential equation of fractional order. Recently the applications of fractional signals and systems to control theory was exclusively given by Magin et al. [6] various applications of fractional calculus like image processing are found in the volume that edited by Machilo et al. [7]. These applications show the necessity and advance importance fractional calculus in inter necessity and advance importance fractional calculus in inter disciplinary sciences. So far there have been several fundamental works on these advance topics of fractional derivative and fractional differential equation, return back, Oldham and Spanier (1974)[8], Miller and Ross [9], Podlubny [10], Kilbas , Srivstava and Trujillo [11] and others [Samko et al. [12], Caponetto et al.[13], Diethelm [14]].Machado et al.[15] have published a review on article recent history of fractional calculus. A recent development in the theory of abstract differential equation with fractional derivative is given by Hernandez et al. [16]. These works form an introduction to the theory of fractional differential equation along with, somewhere, construction of differential equation of fractional order and provide systematic understanding of fractional calculus such as the existence and uniqueness of solution, some analytical method for solving fractional differential equation like Green function method, inverse transformation method , Laplace transform method, Power series method, etc. In the literature so far no method exists to find exact solution for nonlinear fractional differential equation. Only approximate solution is found in some cases that are derived using linearization for perturbation method. This short coming has motivates us to construct possible efficient exact solution method for nonlinear fractional differential equation, It is worth to note that similarity method is only the possible method that yield exact solution for nonlinear partial differential equation [Hansen ([17], Bluman[18]], etc.We apply similarity method based on one parameter family of scaling group of transformation which is also known as finite Lie group method to derive exact solution of partial differential equation of fractional order that considered in this paper. We give theory in Brief of symmetry group of scaling transformation. First we would like to note that some well-known partial differential equation of fractional order like one dimensional time fractional diffusion wave equation which is successfully used for modeling the relevant physical process [for example , Caputo[19], Gionaand Roman[20], Hilfer [21], Mainardi [22], Metzler et al.[23], Pipkin[24], Podlubny [25].

A. SIMILARITY SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATION

To find the general solution of most of the fractional differential equation which are usually non-linear partial differential equation of science and engineering is too difficult. Practical application of this equations demands particular solutions with pre-defined specific initial and boundary conditions. If fractional differential equation of arbitrary order is invariant under a group of transformation, then among its solution may be some of that are their own images under transformation. [In general the image of solution is another solution.]These invariants solutions are only a class of exact solution and are generally easier to calculate then other solutions; often they may be calculated by Solving an ordinary fractional differential equation. If these invariant solutions are so described situations are practical interest, they may be provide us which simple answers to seemingly difficult question. In this section we confine our attention to certain fractional differential equation in one depended variable, say u and two in depended variable called x and t , that are invariant to following one parameter family of stretching group.

$\bar{t} = \lambda^{\beta} t, \quad \bar{x} = \lambda x, \quad \bar{u} = \lambda^{\alpha} u$ ……….. (a)

Here λ is group parameter that labels different transformation of group p and q are family parameter that label different

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 4, Issue 2, August 2014

groups of the family. The fractional differential equation consider here are invariant to every group in the family, but not every pair of values of p and q is admissible ; For each fractional differential equation the family parameters p and q are coupled by linear constrained.

$Mp+Nq=L$ ………………. (b)

Where M,N and L are fixed constant to be determined by the structure of the particular the fractional differential equation. Thus only one of the family parameters p and q in a particular problem are determine by the boundary and initial conditions. Each solution u(x,t) of a partial differential equation of fractional order represents some short of surface S in (u, x, t)-space. If the partial differential equation of fractional order is invariant under a group G, then the image surface of S, composed of the images ($\bar{u}$, $\bar{x}$, $\bar{t}$)of the points (u, x, t) of S, generally represents another solution u(x, t) itself is invariant under the transformations of G, then the surface S is its own image. Let the surface S be represented by the equation

F (u, x, t) =0 …………… (1)

If its own image, then

F ($\bar{u}$, $\bar{x}$, $\bar{t}$)=0 ……….. (2)

as well. Equation (2) can be written

$F(\lambda^{\alpha}u,\ \lambda x,\ \lambda^{\beta}t)=0$ …… (3)

If we differentiate equation (3) with respect to λ and then set λ=1, we obtain the linear partial differential equation of fractional order

$F(\lambda^{\alpha}u,\ \lambda x,\ \lambda^{\beta}t)=0$ …… (4)

Whose characteristic equations are

$\frac{du}{\alpha u}=\frac{dx}{x}=\frac{dt}{\beta t}=\frac{dF}{0}$………… (5)

Three independent integrals of equations of above equations are

$F,\ \eta_1(t,x,u)=xt^{-\frac{1}{\beta}}\ and\ \eta_2(t,x,u)=ut^{-\frac{\alpha}{\beta}}$…… . (6)

The most general solution of equation (6) is F=f(x,y), where f is any arbitrary function. Since the surface S is designated by f(x,y)=F=0, we can solve this last equation for η_1 in terms of x and write: y=y(x) is an arbitrary function of x only. Written in terms of u,x,t, the relation y=y(η) become

$u=t^{\frac{\alpha}{\beta}}y(xt^{-\frac{1}{\beta}})$ …………..(7)

Equation (7) is that type of transformation which is use to convert P.D.E. of fractional into O.D.E. in many cases & this is most general form. Here is worth to note that if for given P.D.E. of fractional order the type of transformation given by equation (7) does not work then in that case we can say that Similarity solution for the equation under consideration does not exist.

B. Final Stage

We consider following partial differential equation of fractional order [15]

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=D\frac{\partial^{2}u(x,t)}{\partial x^{2}},\quad x>0,t>0,D>0,\quad \alpha\geq 1,\ \ldots\ldots\ldots(8)$$

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\begin{cases}\frac{\partial^{n}u(x,t)}{\partial t^{n}}, & \alpha=n\in N\\ \frac{1}{\binom{n}{\alpha}}\frac{\partial^{n}}{\partial t^{n}}\int_0^t(t-s)^{n-\alpha-1}u(x,s)ds, & n-1<\alpha<n\in N\end{cases}\ \ldots\ldots\ (9)$$

It is interesting to note that α=1 equation (8) will be that standard differential equation & for α=2. It will be standard wave equation & hence for arbitrary value of α equation (8)is known fractional diffusion, wave equation & hence equation (8) diffusion wave equation. This equation is obtained by replacing the first or second order time derivative in the diffusion or wave equation respectively, by a generalized derivative of order$\alpha\geq$1, defined in the sense of the Riemann-Liouville fractional calculus:[26], [27]Set of equations (8) & (9) already appeared in texts on physics and mathematics. Mathematical aspects of the boundary value problems for this equation and for more general ones and their applications in physics have been treated in papers by Engler[28], Fujita[29], Mainardi[10-12], Pruss[31], Saichev and Zaslavski[32], Schneider and Wyss[33]., and Wyss[34]. In a series of papers (see [12] and the references there) the two basic boundary value problems for the fractional diffusion wave equation (1≤α≤2) were considered.

(a) Cauchy problem: u(x,0 +) = g(x), $-\infty<x<+\infty$; u$(\mp\infty,t)=0,\ t>0,$

(b) Signaling problem: u(x,0 +) =0, $x>0$; u(0 +, t) = h(t), $u(+\infty,t)=0,t>0.$

By using integral transforms the Green's functions $g_c(x,t,\alpha)\ and\ g_s(x,t,\alpha)$ for these problems were expressed in terms of some special functions with the similarity argument as derived above (7) (with β=$\frac{2}{\alpha}$);$\eta=xt^{-\frac{\alpha}{2}}/\sqrt{D}$ $(x>0,t>0)$.We will explain this fact in our article and determine by using the method of group analysis that equation (8) is in fact invariant under a symmetry group of scaling transformations. In the general case ($\alpha\notin N$) one cannot use the chain rule for the operation of differentiation to get a reduced equation for the scale-in-variant solutions of (8) as in the case of partial differential equations. In spite of this we will transform equation (8) into an ordinary differential equation of fractional order with the new independent variable $z=xt^{-\frac{\alpha}{2}}$. The derivative then is an Erdelyi-Kober derivative depending on a parameter α.For $\alpha=1\ and\ \alpha=2$ this reduced equation corresponds to the ordinary differential equations well known in the literature (see [35]). Here first we determine symmetry group of transformation for equation (8) and here we have the transformation by (a). The invariants of scaling transformation under which equation (8) is invariant yields

$$\eta_1(t,x,u)=xt^{-\frac{1}{\beta}}\ ,\ \eta_2(t,x,u)=ut^{-\frac{\alpha}{\beta}},\ldots(10)$$

Equation (10) yields required similarity transformations for equation (8). It is interesting to note that similarity transformations (10) and (6) are equivalent. Further note that η_1&η_2 given in equation (10) is obtained from the condition of invariance of equation (8) under the group transformation (a) for the cases $\alpha=n\in N$& for n-1< $\beta<n$ (n∈ N)

Now in the case $\alpha=n\in N$ we get

$$\frac{\partial^{\alpha}\bar{u}(\bar{t},\bar{x})}{\partial x^{\alpha}}=\lambda^{n\beta}\frac{\partial^{\alpha}\bar{u}(\bar{t},\bar{x})}{\partial\bar{x}^{\alpha}}$$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 4, Issue 2, August 2014

And for n-1< $\beta < n$ (n∈ N) using same variable substitute we get

$$\frac{\partial^{\beta}\bar{u}(\bar{t},\bar{x})}{\partial x^{\beta}} = \frac{1}{\Gamma n-\alpha}\frac{\partial^{n}}{\partial x^{n}}\int_{0}^{x}(t-s)^{n-s-1}u(\bar{x},\bar{s})ds$$

$$= \frac{\lambda^{n\beta}}{\Gamma n-\alpha}\frac{\partial^{n}}{\partial \bar{x}^{n}}\int_{0}^{\bar{t}}(\bar{t}-\tau)^{n-\alpha-1}\bar{u}(\bar{t},\tau)d\tau$$

$$=\lambda^{\alpha\beta}\frac{\partial^{\alpha}\bar{u}(\bar{t},\bar{x})}{\partial x^{\alpha}}.$$

And hence finally we have

$$\frac{\partial^{\alpha}\bar{u}}{\partial \bar{t}^{\alpha}} - D\frac{\partial^{2}\bar{u}}{\partial x^{2}} = \lambda^{\alpha\beta}\frac{\partial^{\alpha}\bar{u}}{\partial \bar{t}^{\alpha}} - D\lambda^{2}\frac{\partial^{2}\bar{u}}{\partial \bar{x}^{2}} = 0,$$

$$if\ \beta = \frac{2}{\alpha}$$

By analogy with the partial differential equation of fractional order, we use the transformation

$u(t,x) = v(\eta)$, $\eta = xt^{-\frac{\alpha}{2}}$ .... (11)

To determine the scale-invariant, solutions of the partial differential equation of fractional order (8). From (11), it is easy to see that, the partial derivative, u_x, is given in terms of derivatives of v by

$u_x = t^{-\frac{\alpha}{2}}v'(\eta)$ And $u_{xx} = v''(\eta)t^{-\alpha}$ ....(12)

The partial derivative of fractional order $\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}$, α> 0 of the function u(x, t) =v (η), η=x$t^{-\frac{\alpha}{2}}$ is given by the relation

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = t^{-\alpha}\left(p_{\frac{2}{\alpha}}^{1-\alpha,\alpha}v\right)(\eta), \eta > 0$$

With the Erdlyi-Kober fractional differential operator $p_{\beta}^{\tau,\alpha}$ of order α>0 which is same to that of define by

$$\left(p_{\beta}^{\tau,\alpha}g\right)(z) = \Pi_{j=0}^{\eta-1}\left(\tau + j - \frac{1}{\beta}z\frac{d}{dz}\right)\left(K_{\beta}^{\tau+\alpha,\eta-\alpha}g\right)(z), \quad (13)$$

$$\eta = \begin{cases}[\alpha]+1, & \alpha \notin N\\ \alpha, & \alpha \in N,\end{cases}$$

Where

$$\left(K_{\beta}^{\tau,\alpha}g\right)(z) = \begin{cases}\frac{1}{\Gamma\alpha}\int_{1}^{\infty}(u-1)^{\alpha-1}u^{-(\tau+\alpha)}g\left(zu^{\frac{1}{\beta}}\right)du, & \alpha > 0,\\ g(z) & \alpha = 0\end{cases}$$

With the Erdelyi-Kober fractional differential operator$p_{\frac{2}{\alpha}}^{1-\alpha,\alpha}$.

Now we substitute the expressions (11) into the partial differential equation of fractional order (8), and we find

$$t^{-\alpha}\left(p_{\frac{2}{\alpha}}^{1-\alpha,\alpha}v\right)(\eta) = t^{-\alpha}Dv''(\eta), z > 0,\ D > 0.$$

$$\left(p_{\frac{2}{\alpha}}^{1-\alpha,\alpha}v\right)(\eta) = Dv''(\eta), z > 0,\ \alpha \geq 1, \text{.............. (14)}$$

With the Erdelyi-Kober fractional differential operator $p_{\frac{2}{\alpha}}^{1-\alpha,\alpha}$.Following the definition of Erdelyi-Kober fractional differential operator (13), in the case $\alpha = n \in N$ the reduced equation (14) for the scale-invariant solutions is a linear ordinary differential equation of order max {2, n}.

Case $\alpha = 1$ this equation takes the form

$$\left(p_{2}^{0,1}v\right)(z) = -\frac{1}{2}zv'(z) = Dv''(z),$$

And in the case $\alpha = 2$ the form

$$\left(p_{1}^{-1,2}v\right)(z) = 2zv'(z) + z^{2}v''(z) = Dv''(z).$$

Thus we get second order ordinary differential equation which is also known as scale in variant solution of diffusion equation of fractional order given by (8). The general solution of above equation will be

V (z)= $c_1v_1(z) + c_2v_2$

And v_1 $and\ v_2$ are some function of α and D.

We have not discussed the general solution in detail as our aim is to derive similarity solution of partial differential equation of fractional order using scaling group transformation method.

II. CONCLUSION

Similarity solution for partial differential equation of fractional order is obtained using group transformation technique it is observed that group method are powerful tool as they do not based on the linear operator, superposition or any other linear solution technique therefore these methods are applicable to both linear and non—linear differential models and partial differential equations of fractional order too

REFERENCES

[1] E. Barkai, R. Metzler, and j. Klafter, (2000). From continuous time random walks to the fractional fokker-planck equation, Physics Review E, 61, 132-138.

[2] M. M. Meerschaert, D.A. Benson, H. P. Scheffler, and Becker-Kern, P. (2002). Governing equations and solutions of anomalous random walk limits, Physics Review E, 66, 102-105.

[3] F.Mainardi, (2010). Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, Imperial College Press, London.

[4] C.Tadjeran (2007) , and Meerschaert, M.M. (2007). A second order accurate numerical method for the two-dimensional fractional diffusion equation, Journal of Computational Physics, 220, 813- 823.

[5] King (1990)[5]

[6] R.Magin et al. (2011), Ortigueira M.D., Podlubny, I. and Trujillo, J.J. (2011). On the fractional signals and systems, Signal Processing, 91, 350-371

[7] J.T.Machado, A.C.J.Luo, R.S. Barbosa, R.S., Silva, M.F. and L.B. Figueiredo, (2011). Nonlinear Science and Complexity, Springer, New York

[8] Oldham and Spanier (1974), Oldham, K.B. and Spanier, J. (1974). The Fractional Calculus: Theory and Application of Different ion and Integration to Arbitrary order, Academic Press, California.

[9] Miller and Ross (1993), Miller, K. and Ross, B. (1993). An Introduction to the Fractional Calculus and Fractional Elsevier, Amstrdam Differential Equations, Wiley and Sons, New

[10] Podlubny, I. (1999). Fractional Differential Equations, Academic Press, California.

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 4, Issue 2, August 2014

[11] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006). Theory and Applications of Fractional Differential Equations,

[12] Samko, S.G., Kilbas, A.A. and Marichev, O.I. (1993). Fractional Integrals and Derivatives; Theory and Applications, Gordan and Breach, Amsterdam.

[13] Caponetto, R., Dongola, G., Fortuna, L. and Petrás, I. (2010). Fractional Order Systems: Modeling and Control Applications, World Scientific Publishing Company, Singapore

[14] Diethelm, K. (2010). The Analysis of Fractional Differential Equations, Springer, New York.

[15] Machado, J. T., Kiryakova, V. and Mainardi, F. (2010). Recent history of fractional calculus,

[16] Communication in Nonlinear Science and Numerical Simulation, 16, 1140-1153.

[17] Podlubny, I. (1999). Fractional Differential Equations, Academic Press, California.

[18] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006). Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies

[19] Samko, S.G., Kilbas, A.A. and Marichev, O.I. (1993). Fractional Integrals and Derivatives; Theory and Applications, Gordan and Breach, Amsterdam.

[20] Caponetto, R., Dongola, G., Fortuna, L. and Petrás, I. (2010). Fractional Order Systems: Modeling and Control Applications, World Scientific Publishing Company, Singapore

[21] K.Diethelm (2010). The Analysis of Fractional Differential Equations, Springer, New York.

[22] Machado J. T.Kiryakova, V. and F. Mainardi (2010). Recent history of fractional calculus, Communication in Nonlinear Science and Numerical Simulation, 16, 1140-1153.

[23] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, Chaos Solitons Fractals 7 (1996), 1461-1477.

[24] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation, Appl.Math. Lett. 9, No. 6(1996), 23-28.

[25] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, in ''Fractals and Fractional Calculus in Continuum Mechanics'' _A. Carpinteri and F. Mainardi, Eds), pp. 291-348, Springer-Verlag, Vienna, 1997.

[26] J. Pr¨uss, ''Evolutionary Integral Equations and Applications,'' Birkha¨user, Basel, 1993.

[27] Saichev and G. Zaslavsky, Fractional kinetic equations: solutions and applications, Chaos 7, No. 4(1997), 753F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena, Chaos Solitons Fractals 7 (1996), 1461-1477.

[28] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation, Appl.Math. Lett. 9, No. 6(1996), 23-28.

[29] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, in ''Fractals and Fractional Calculus in Continuum Mechanics'' _A. Carpinteri and F. Mainardi, Eds), pp. 291-348, Springer-Verlag, Vienna, 1997.

[30] J. Pr¨uss, ''Evolutionary Integral Equations and Applications,'' Birkha¨user, Basel, 1993.

[31] Saichev and G. Zaslavsky, Fractional kinetic equations: solutions and applications, Chaos 7, No. 4(1997), 753764.