ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 4, Issue 1, July 2014

# Integral Functionals of the Nadaraya-Watson Regression Function

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***Abstract - In present paper the problem of statistical estimation of the nonlinear integral functional of a regression function is discussed. For the regression function and its derivatives are taken well known Nadaraya-Watson estimator. The problem is naturally considered in the Sobolev space. As an estimator for this function is proposed the plug-in estimator. There are proved theorems about consistency and asymptotically normality. The order of the convergence is determined. The general methodology is used for some special cases. There is solved the estimation problem of Fisher's information and Shannon entropy for Nadaraya-Watson's regression function.***

***Index Terms*-Nadaraya-Watson estimation, regression function, integral functional.**

## I. INTRODUCTION

In the present paper we investigate the integral functional of a regression function and its derivatives. In our investigation we use the Nadaraya-Watson Regression Function introduced and studied in [1, 2]. The study of functionals of a probability distribution density or of a regression function and its derivatives is an interesting task attracts an active interest on the part of researchers (see e.g. [3-9]). There are detailed studies of functionals of a probability distribution density function and its derivatives (see [5-8] and the references therein). Investigations of functionals of a regression function and its derivatives are more modest ([3,4]). Let $a(t)$ denotes the regression function, then we may consider, say, the particular cases:

$$I_1(a) = \int_{-\infty}^{\infty} a^2(t)\,dt,\ I_2(a) = \int_{-\infty}^{\infty} \frac{\left(a'(t)\right)^2}{a(t)}\,dt,$$

$$I_3(a) = \int_{-\infty}^{\infty} \left(a(t)\right)^3 dt,\ I_4(a) = \int_{-\infty}^{\infty} a(t)\log a(t)\,dt.$$

Related problems were studied in the above-mentioned works [3,4]. Our approach in this paper is based on the derivation of a representation theorem which we further use to obtain the results connected with asymptotic properties, in the particular with consistency and the central limit theorem. The statement of the problems and the discussion were inspired by [5].

Let's consider a regression model of the form:

$$Y(t) = a(t) + \varepsilon(t), \qquad (1)$$

where $t \in [0,1]$, $\varepsilon(\cdot)$ is noise with $E\varepsilon(t) = 0$, $E\varepsilon^2(t) = \sigma^2 < \infty$, $Y(t)$ is a observed random function, and $a(t)$ is an unknown regression function. Suppose that we have $n$ numbers:

$$0 \le t_1 \le t_2 \le \cdots \le t_n \le 1,$$

where each $t_k$, $k = 1, 2, \dots, n$ depends on $n$ and $\max_i |t_i - t_{i-1}| = O\left(\frac{1}{n}\right)$. We have $n$ observations:

$$Y(t_1), Y(t_2), \dots, Y(t_n).$$

The estimator of the unknown function $a(t)$ was introduced by Nadaraya E. A. [1] and WatsonG. S. [2] and defined by the expression:

$$\hat{a}_n(t) = \sum_{i=1}^{n} \frac{W\left(\frac{t-t_i}{h_n}\right)}{\sum_{k=1}^{n} W\left(\frac{t-t_k}{h_n}\right)} \cdot Y(t_i), \qquad (2)$$

where, $\{h_n,\ n = 1, 2, \dots\}$ is a sequence of positive numbers monotonically tending to zero. $W(t)$ is the function with probability density properties.

For simplicity, we introduce the designation

$$\frac{W\left(\frac{t-t_i}{h_n}\right)}{\sum_{k=1}^{n} W\left(\frac{t-t_k}{h_n}\right)} \equiv \alpha(t, t_i, h_n)$$

and Nadaraya-Watson regression function will be rewritten as follows:

$$\hat{a}_n(t) = \sum_{i=1}^{n} \alpha(t, t_i, h_n) \cdot Y(t_i).$$

The following equations are true:

$$\sum_{i=1}^{n} \alpha(t, t_i, h_n) = 1, \qquad \sum_{i=1}^{n} \alpha^{(k)}(t, t_i, h_n) = 0$$

(Here and in the following it means $k$-th derivative by the variable $t$).

Here is introduced the estimator of the k-th derivative of the regression function $a^{(k)}(t)$ as formula

$$\hat{a}_n^{(k)}(t) = \frac{1}{h_n^k} \cdot \sum_{i=1}^{n} \alpha^{(k)}(t, t_i, h_n) \cdot Y(t_i), \qquad (3)$$

for all $k = 0, 1, 2, \dots, m$. It was assumed that $\hat{a}_n^{(0)}(t) \doteq \hat{a}_n(t)$.

Let $\varphi: \mathbb{R}^{m+2} \to \mathbb{R}$ be a continuous bounded function. Consider an integral functional of the form:

$$I(a) = \int_{-\infty}^{\infty} \varphi\left(t, a(t), a'(t), \dots, a^{(m)}(t)\right) dt. \qquad (4)$$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 4, Issue 1, July 2014

We have the selection $(t_i, Y_i),\ i = 1, 2, \dots, n$. This means that

$$Y_i = Y(t_i) = a(t_i) + \varepsilon(t_i). \qquad (5)$$

To estimate $I(a)$ we use the plug-in estimator, i.e. consider the functional:

$$I(\hat{a}_n) = \int_{-\infty}^{\infty} \varphi\left(t, \hat{a}_n(t), \hat{a}_n'(t), \dots, \hat{a}_n^{(m)}(t)\right) dt.$$

## II. REPRESENTATION THEOREM

Our consideration is based on a representation theorem which will lead to obtaining the results we are interested in. Let us list the conditions which the considered variables are supposed to satisfy:

Conditions on $a$:

$(a1)$ The function $a = a(t)$ is defined and continuous on [0, 1] and takes its values in the interval $[-k, k]$;

$(a2)$ $a = a(t)$ has continuous derivatives up to order $m$ inclusive;

$(a3)$ For any $i = 0, 1, \dots, m$, $a^{(i)}(t)$ takes its values in $[-k, k]$ and $a^{(i)}(\cdot) \in L_1([0, 1])$.

Conditions on $\varepsilon_k$:

$(\varepsilon 1)$ Random values $\varepsilon_k,\ k = 1, 2, \dots$ are independent, bounded and equally distributed;

$(\varepsilon 2)$ $E\varepsilon_k = 0,\ D\varepsilon_k^2 = \sigma^2 < \infty$;

For brevity, we will use notation for $\varphi = \varphi(x, x_0, \dots, x_m) \in C_b^2(R^{m+2})$ function:

$$\frac{\partial \varphi}{\partial x_i} = \varphi_{(i)},\ i = 0, 1, \dots, m$$

and $\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \varphi_{(ij)},\ i, j = 0, 1, \dots, m.$

Conditions on $\varphi$:

$(\varphi 1)$ The function $\varphi: \mathbb{R}^{m+2} \to \mathbb{R}$ is continuous, bounded, integrable and has bounded continuous derivatives up to second order, inclusive, in some convex domain $A$ which contains the domain $\mathbb{R} \times [-k, k]^{m+1}$;

$(\varphi 2)$ All first and second derivatives of the function $\varphi$ are uniformly bounded in the domain $A$ by a constant $C_\varphi > 0$.

By this conditions, for the function $\varphi$ we have for all $i, j = 0, 1, \dots, m$:

$$\sup\{|\varphi_{(i,j)}|(s, s_0, s_1, \dots, s_m): (s, s_0, s_1, \dots, s_m) \in A\} \le C_\varphi \quad (6)$$

Conditions on $W$:

$(W1)$ $\int_{-\infty}^{\infty} W(t)\, dt = 1$;

$(W2)$ Function $W(t)$ has the compact support $[-\tau, \tau]$ and $W(-\tau) = W(\tau) = 0$;

$(W3)$ $W(t)$ has continuous derivatives up to order $m \ge 1$;

$(W4)$ There exists the constants $0 < C_{1W}, C_{2W} < \infty$, for which

$\inf_{t \in R} |W^{(i)}(t)| \ge C_{1,W}$,

$\sup_{t \in R} |W^{(i)}(t)| \le C_{2,W}$,

for all $i = 0, 1, \dots, m$;

$(W5)$ For any $i = 0, 1, \dots, m$, $W^{(i)} \in L_1([-\tau, \tau])$.

Conditions on $h_n$:

$(h_n 1)$ $\frac{\sqrt{\max(|\log h_n|, \log\log n)}}{\sqrt{n h_n^{0,5+m}}} \to 0$, as $n \to \infty$.

In the following we will need to estimate $|\alpha^{(k)}(t, t_i, h_n)|$ and let's show, that the inequality

$$|\alpha^{(k)}(t, t_i, h_n)| \le \frac{1}{n \cdot h_n^k} \cdot C_k$$

is true for all $k = 0, 1, \dots, m$, where $0 < C_k < \infty$ are constants.

For $k = 0$ this inequality is true:

$$|\alpha^{(0)}(t, t_i, h_n)| = \left| \frac{W\left(\frac{t - t_i}{h_n}\right)}{\sum_{k=1}^{n} W\left(\frac{t - t_k}{h_n}\right)} \right|$$

$$\le \frac{C_{2,W}}{n \cdot C_{1,W}} \equiv \frac{1}{n \cdot h_n^0} \cdot C_0.$$

For $k = 1$ this inequality is also true:

$$|\alpha^{(1)}(t, t_i, h_n)| =$$

$$= \frac{1}{h_n} \cdot \left| \frac{W'\left(\frac{t-t_i}{h_n}\right) \sum_{k=1}^{n} W\left(\frac{t-t_k}{h_n}\right) - W\left(\frac{t-t_i}{h_n}\right) \sum_{k=1}^{n} W'\left(\frac{t-t_k}{h_n}\right)}{\left(\sum_{k=1}^{n} W\left(\frac{t-t_k}{h_n}\right)\right)^2} \right|$$

$$\le \frac{1}{h_n} \cdot \frac{\left|W'\left(\frac{t-t_i}{h_n}\right) \cdot \sum_{k=1}^{n} W\left(\frac{t-t_k}{h_n}\right)\right|}{\left(\sum_{k=1}^{n} W\left(\frac{t-t_k}{h_n}\right)\right)^2} +$$

$$+ \frac{1}{h_n} \cdot \frac{\left|W\left(\frac{t-t_i}{h_n}\right) \cdot \sum_{k=1}^{n} W'\left(\frac{t-t_k}{h_n}\right)\right|}{\left(\sum_{k=1}^{n} W\left(\frac{t-t_k}{h_n}\right)\right)^2} \le$$

$$\le \frac{n \cdot C_{2,W}^2 + n \cdot C_{2,W}^2}{h_n \cdot (n \cdot C_{1,W})^2} = \frac{1}{n \cdot h_n^1} \cdot \frac{2C_{2,W}^2}{C_{1,W}^2} \equiv \frac{1}{n \cdot h_n^1} \cdot C_1.$$

By the same way we will obtain, that for all $k = 0, 1, \dots, m$ this inequality is also true:

$$|\alpha^{(k)}(t, t_i, h_n)| \le \frac{1}{n \cdot h_n^k} \cdot C_k.$$

And if $C_\alpha$ is the largest value of $C_k$, for all $k = 0, 1, \dots, m$, we get

$$|\alpha^{(k)}(t, t_i, h_n)| \le \frac{1}{n \cdot h_n^k} \cdot C_\alpha.$$

Denote by $a_n(t)$ mathematical expectation of $\hat{a}_n(t)$:

$$a_n(t) = E\hat{a}_n(t) = E\left(\sum_{i=1}^{n} \alpha(t, t_i, h_n) \cdot Y(t_i)\right) = \sum_{i=1}^{n} \alpha(t, t_i, h_n) \cdot a(t_i).$$

Then we obtain,

$$a_n^{(k)}(t) = E\hat{a}_n^{(k)}(t) = \frac{1}{h_n^k} \cdot \sum_{i=1}^{n} \alpha^{(k)}(t, t_i, h_n) \cdot a(t_i).$$

Let we show that there also exist expressions $I(a)$, $I(a_n)$ and $I(\hat{a}_n)$ and they are finite.

Using the Taylor formula for any point $(s, s_0, s_1, \dots, s_m) \in A$ and some $\tilde{s}_i \in A$ we can write

$$|\varphi|(s, s_0, s_1, \dots, s_m) =$$
$$= \left| \sum_{i=0}^{m} \varphi_{(i)}(s, 0, 0, \dots, 0) s_i + \frac{1}{2} \sum_{i,j=0}^{m} \varphi_{(i,j)}(s, \tilde{s}_0, \tilde{s}_1, \dots, \tilde{s}_m) s_i s_j \right|.$$

Accordingly, there exists a constant $C$ such that

$$|\varphi|(s, s_0, s_1, \dots, s_m) \le C \left( \sum_{i=0}^{m} |s_i| + \sum_{i=0}^{m} |s_i|^2 \right).$$

Hence it follows that for any bounded measurable functions $f_0(t), f_1(t), \dots, f_m(t)$ from $L_1(R)$ we have

$$\int_{-\infty}^{\infty} |\varphi|(t, f_0(t), f_1(t), \dots, f_m(t)) dt < \infty. \qquad (7)$$

And therefore $I(a)$ exists.

The conditions which are imposed on the function $W$ ensure boundness and membership in $L_1(R)$, Then condition $(W4)$ and (6)-(7) imply the finiteness of both variables $I(a_n)$ and $I(\hat{a}_n)$, for any $n \in \mathbb{N}$.

By the Taylor formula we can write

$$I(\hat{a}_n) - I(a_n) = S_n(h_n) + R_n, \qquad (8)$$

Where, for any $h_n > 0$, $S_n(h_n)$ is the sum of independent random variables:

$$S_n(h_n) = \sum_{i=0}^{m} \int_0^1 \varphi_{(i)} \left( t, a_n(t), a_n'(t), \dots, a_n^{(m)}(t) \right) \times \left( \hat{a}_n^{(i)}(t) - a_n^{(i)}(t) \right) dt. \qquad (9)$$

A remainder $R_n$ has the form:

$$R_n = \frac{1}{2} \cdot \sum_{i,j=1}^{m} \int_0^1 \varphi_{(ij)} \left( \tilde{b}_m(t) \right) \times \left( \hat{a}_n^{(i)}(t) - a_n^{(i)}(t) \right) \times \left( \hat{a}_n^{(j)}(t) - a_n^{(j)}(t) \right) dt. \qquad (10)$$

Where $\tilde{b}_m(t)$ is a point on the straight line connecting the points $\left( t, \hat{a}_n(t), \hat{a}_n'(t), \dots, \hat{a}_n^{(m)}(t) \right)$ and $\left( t, a_n(t), a_n'(t), \dots, a_n^{(m)}(t) \right)$.

Let us estimate the remainder $R_n$. Applying the standard procedure, from (7) and (10) we obtain:

$$|R_n| \le C_\varphi \int_0^1 \sum_{i=0}^{m} \left( \hat{a}_n^{(i)}(t) - a_n^{(i)}(t) \right)^2 dt. \qquad (11)$$

Let $W_m^2$ denote the Sobolev space of functions having a square-integrable continuous and bounded second derivative with the norm $\|g\|_m = \sqrt{\sum_{i=0}^{m} \int_0^1 |g^{(i)}(t)|^2 dt}$ and the scalar product $(g_1, g_2)_m = \sqrt{\sum_{i=0}^{m} \int_0^1 \left( g_1^{(i)}(t) \cdot g_2^{(i)}(t) \right) dt}$.

Denote $r_n(m) = \|\hat{a}_n - a_n\|_m^2$, then we can write

$$|R_n| \le C_\varphi r_n(m). \qquad (12)$$

Assume,

$$U_k = U_k(t) = \alpha(t, t_i, h_n) \cdot [Y(t_k) - a(t_k)], \qquad k = 1, 2, \dots, n,$$

where $a(t_k) = EY(t_k)$. Then

$$\sum_{k=1}^{n} U_k = \sum_{k=1}^{n} \alpha(t, t_i, h_n) \cdot [Y(t_k) - a(t_k)] =$$
$$= \sum_{k=1}^{n} \alpha(t, t_i, h_n) \cdot Y(t_k) - \sum_{k=1}^{n} \alpha(t, t_i, h_n) \cdot a(t_k)$$
$$= \hat{a}_n(t) - a_n(t).$$

Therefore, $r_n(m) = \|\sum_{k=1}^{n} U_k\|_m^2$. (13)

Let us estimate the norm of one of the summands $U_k$ in (13) for each $k = 1, 2, \dots, n$. We obtain

$$\|U_k\|_m = \left( \sum_{i=0}^{m} \int_0^1 |U_k^{(i)}(t)|^2 dt \right)^{\frac{1}{2}} =$$
$$= \left( \sum_{i=0}^{m} \int_0^1 \left| (\alpha(t, t_i, h_n) \cdot [Y(t_k) - a(t_k)])^{(i)} \right|^2 dt \right)^{\frac{1}{2}} =$$
$$= \left( \sum_{i=0}^{m} \int_0^1 \left| \alpha^{(i)}(t, t_i, h_n) \cdot [Y(t_k) - a(t_k)] \right|^2 dt \right)^{\frac{1}{2}} =$$
$$= |Y(t_k) - a(t_k)| \cdot \left( \sum_{i=0}^{m} \int_0^1 \left| \alpha^{(i)}(t, t_i, h_n) \right|^2 dt \right)^{\frac{1}{2}}$$
$$\le \frac{|\varepsilon_k| C_\alpha}{n} \cdot \left( \sum_{i=0}^{m} \int_0^1 \left| \frac{1}{h_n^i} \right|^2 dt \right)^{\frac{1}{2}}$$
$$= \frac{|\varepsilon_k| C_\alpha}{n} \cdot \left( \sum_{i=0}^{m} \frac{1}{h_n^{2i}} \right)^{\frac{1}{2}} = \frac{|\varepsilon_k| C_\alpha}{n} \cdot \left( \frac{1 - h_n^{2m+2}}{h_n^{2m}(1 - h_n^2)} \right)^{\frac{1}{2}}$$
$$\le L \cdot \frac{1}{n h_n^m} = M_m \sim O\left( \frac{1}{n h_n^m} \right) \qquad (14)$$

For sufficiently large $L > 0$.

To estimate $r_n(m)$ we use the McDiarmid'sine quality, which we give here for convenience (for details see [10]).

McDiarmid's Inequality: Let $H(t_1, t_2, \dots, t_k)$ be a real function such that for each $i = 1, 2, \dots, k$ and some $c_i$, the supremum in $t_1, t_2, \dots, t_k, t$ of the difference $|H(t_1, t_2, \dots, t_{i-1}, t_i, t_{i+1}, \dots, t_k) - H(t_1, t_2, \dots, t_{i-1}, t, t_{i+1}, \dots, t_k)| \le c_i$

If $X_1, X_2, \dots, X_k$ are independent random variables taking values in the domain of the function $H(t_1, t_2, \dots, t_k)$ then for every $\varepsilon > 0$

$$P\{|H(X_1, X_2, \dots, X_k) - EH(X_1, X_2, \dots, X_k)| > \varepsilon\} \le 2\exp\left( -\frac{2\varepsilon^2}{\sum_{i=1}^{k} c_i^2} \right).$$

Let us apply McDiarmid's inequality for the function

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 4, Issue 1, July 2014

$$H(U_1, U_2, \dots, U_n) = \left\| \sum_{k=1}^{n} U_k \right\|_m .$$

We have:

$$\left| H(U_1, U_2, \dots, U_{i-1}, U_x, U_{i+1}, \dots, U_n) - H(U_1, U_2, \dots, U_{i-1}, U_y, U_{i+1}, \dots, U_n) \right|$$

$$= \left| \left\| \sum_{k=1, k\neq i}^{n} U_k + U_x \right\|_m - \left\| \sum_{k=1, k\neq i}^{n} U_k + U_y \right\|_m \right| \leq \|U_x\|_m + \|U_y\|_m$$

And as $c_k$ we take $c_k \equiv 2M_m,\ k = 1, \dots, n$ . From (14), for any $\delta > 0$ we obtain:

$$P\left\{ \left| \left\| \sum_{k=1}^{n} U_k \right\|_m - E\left\| \sum_{k=1}^{n} U_k \right\|_m \right| > \delta \right\} \leq$$

$$\leq 2\exp\left( -\frac{2\delta^2}{\sum_{k=1}^{n} 4M_m^2} \right) = 2\exp\left( -\frac{2\delta^2}{n \cdot 4M_m^2} \right) =$$

$$= 2\exp\left( -\frac{2\delta^2 n^2 h_n^{2m}}{n \cdot 4L^2} \right) = 2\exp\left( -\frac{\delta^2 n h_n^{2m}}{2L^2} \right).$$

We substitute here $\delta = \frac{2L\sqrt{\log n}}{\sqrt{n h_n^m}}$ and we have:

$$P\left\{ \left| \left\| \sum_{k=1}^{n} U_k \right\|_m - E\left\| \sum_{k=1}^{n} U_k \right\|_m \right| > \delta \right\} \leq$$

$$\leq 2\exp\left( -\frac{4L^2 \log n \cdot n h_n^{2m}}{n h_n^{2m} \cdot 2L^2} \right) = 2\exp(-2\log n) = \frac{2}{n^2}.$$

By the Borelli-Cantelli lemma, we can write

$$\left\| \sum_{k=1}^{n} U_k \right\|_m = E\left\| \sum_{k=1}^{n} U_k \right\|_m + O\left( \frac{\sqrt{\log n}}{\sqrt{n h_n^m}} \right). \qquad (15)$$

Using the Jensen's inequality

$$\left( E\left\| \sum_{k=1}^{n} U_k \right\|_m \right)^2 \leq E\left\| \sum_{k=1}^{n} U_k \right\|_m^2 =$$

$$= E \sum_{k=1}^{n} \sum_{i=0}^{m} \int_0^1 \left| \alpha^{(i)}(t, t_i, h_n) \cdot [Y(t_k) - a(t_k)] \right|^2 dt$$

$$= \sum_{k=1}^{n} \sum_{i=0}^{m} \int_0^1 E\left| \alpha^{(i)}(t, t_i, h_n) \cdot [Y(t_k) - a(t_k)] \right|^2 dt$$

$$\leq C_\alpha^2 \cdot \sum_{k=1}^{n} \sum_{i=0}^{m} \int_0^1 \frac{1}{n^2 \cdot h_n^{2i}} \cdot E[Y(t_k) - a(t_k)]^2 dt$$

$$\leq \frac{C_W^2 \sigma^2}{n} \cdot \sum_{i=0}^{m} \frac{1}{h_n^{2i}} = \frac{C_\alpha^2 \sigma^2}{n} \cdot \frac{1 - h_n^{2m+2}}{h_n^{2m}(1 - h_n^2)} \leq K \cdot \frac{1}{n h_n^{2m}}, \qquad (16)$$

From (12), (13), (15) and (16) we conclude that $R_n = O\left( \frac{\log n}{n h_n^{2m}} \right)$.

Therefore the following statement is true.

Theorem 1.Assume that conditions $(a1) - (a3), (\varepsilon 1) - (\varepsilon 3), (\varphi 1) - (\varphi 2), (W1) - (W5)$ and $(h1)$ are fulfilled.Then representation (8) is true and the remainder with probability 1 has the order

$$R_n = O\left( \frac{\log n}{n h_n^{2m}} \right). \qquad (17)$$

### III. CONSISTENCY

In this section of the paper we use Theorem 1 to prove that the estimator $I(\hat{a}_n)$ is consistent. Theorem 2.Let the conditions of Theorem 1 be fulfilled. If the positive sequence $(h_n)_{n=1}^{\infty},\ 0 < h_n < 1$,is chosen so that

$$\frac{\log n}{n h_n^{2m}} \to 0, \quad \text{as} \quad n \to \infty \qquad (18)$$

then with probability 1 we have

$$I(\hat{a}_n) \to I(a). \qquad (19)$$

Proof: By Theorem 1 and formula (8)

$$I(\hat{a}_n) - I(a_n) = S_n(h_n) + R_n, \qquad (20)$$

where $R_n = o(1)$ and

$$S_n(h_n) = \sum_{i=0}^{m} \int_0^1 \varphi_{(i)}\left(t, a_n(t), a_n'(t), \dots, a_n^{(m)}(t)\right) \times \left( \hat{a}_n^{(i)}(t) - a_n^{(i)}(t) \right) dt.$$

By condition $(a1)$:

$$\left\{ \left(t, a_n(t), a_n'(t), \dots, a_n^{(m)}(t)\right) : t \in [0,1] \right\} \subset [0,1] \times [-k, k]^{m+1}.$$

This and condition $(\varphi 2)$ imply that there exists a constant $C_\varphi > 0$, such that

$$\sup\left\{ \left|\varphi_{(i)}\right|(t, t_0, t_1, \dots, t_m) : (t, t_0, t_1, \dots, t_m) \in [0,1] \times [-k, k]^{m+1} \right\} \leq C_\varphi.$$

We can write:

$$ES_n(h_n) = 0,$$

$$DS_n(h_n) = ES_n^2(h_n) \leq$$

$$\leq C_\varphi^2 \cdot \sum_{i=0}^{m} \int_0^1 E\left[\hat{a}_n^{(i)}(t) - a_n^{(i)}(t)\right]^2 dt$$

$$\leq C_\varphi^2 \cdot \sum_{i=0}^{m} \int_0^1 \left| \sum_{k=0}^{n} \alpha^{(i)}(t, t_i, h_n) \cdot E[Y(t_k) - a(t_k)] \right|^2 dt$$

$$\leq C_\varphi^2 C_\alpha^2 \sigma^2 \cdot \sum_{i=0}^{m} \left( \frac{1}{n h_n^i} \right)^2 \sim C \cdot \frac{1}{n h_n^{2m}} \to 0 \qquad (21)$$

because $\frac{\log n}{n h_n^{2m}} \to 0$ and so $S_n(h_n) \to 0$ as $n \to \infty$.

We can write

$$Ea_n^{(k)}(t) = \int_{-\tau}^{\tau} W(u) a^{(k)}(t)(t - u h_n) du + O\left( \frac{1}{n h_n^k} \right). \qquad (22)$$

Hence we make the following conclusions:

i) for conclusion (17), $\frac{1}{n h_n^k}$ tends to zero for any $k = 0, 1, \dots, m$;

ii) $Ea_n^{(k)}(t) \to a^{(k)}(t)$ as $n \to \infty$.

Summarizing the above discussion, we ascertain that if $n \to \infty$ then

$$I(a_n) = \int_0^1 \varphi\left(t, a_n(t), a_n'(t), \dots, a_n^{(m)}(t)\right) dt \to$$

$$\to \int_0^1 \varphi\left(t, a(t), a'(t), \ldots, a^{(m)}(t)\right) dt = I(a)$$

Since $I(\hat{a}_n) - I(a_n) = o(1)$, we conclude that $I(\hat{a}_n) - I(a) \to 0$ a.e. The theorem is proved.

## IV. CENTRAL LIMIT THEOREM

Using our representation theorem we can obtain the limit distribution property for the integral functional

$$I(\hat{a}_n) = \int_0^1 \varphi\left(t, \hat{a}_n(t), \hat{a}_n'(t), \ldots, \hat{a}_n^{(m)}(t)\right) dt.$$

Consider the difference

$$I(\hat{a}_n) - I(a_n) = S_n(h_n) + R_n, \qquad (8)$$

Where for any $h_n > 0$, $S_n(h_n)$ is the sum of independent random variables

$$S_n(h_n) = \sum_{i=0}^{m} \int_0^1 \varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right)$$
$$\times \left(\hat{a}_n^{(i)}(t) - a_n^{(i)}(t)\right) dt. \qquad (9)$$

$R_n$ is a remainder having the form:

$$R_n = \frac{1}{2} \cdot \sum_{i,j=1}^{m} \int_0^1 \varphi_{(ij)}\left(\tilde{b}_m(t)\right) \times \left(\hat{a}_n^{(i)}(t) - a_n^{(i)}(t)\right)$$
$$\times \left(\hat{a}_n^{(j)}(t) - a_n^{(j)}(t)\right) dt. \qquad (10)$$

Clearly,

$$ES_n(h_n) = 0 \text{ and } ER_n \to 0 \text{ as } n \to \infty. \qquad (23)$$

Moreover

$$E\left(S_n(h_n)\right)^2 =$$
$$= \sigma^2 \sum_{i=0}^{m} \left(\int_0^1 \varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right) dt\right)^2 \qquad (24)$$

and $\mathrm{Var} R_n \to 0$ as $n \to \infty$.

Using appropriate conditions, we have to prove that the variable $\sqrt{n}\left(I(\hat{a}_n) - I(a_n)\right)$ is asymptotically normal and calculate the limiting variance. For this, according to the theorem and formulas (8), (23) and (24), we have to show the asymptotic normality of the variable $\sqrt{n} S_n(h_n)$. As follows from (10), in this case it suffices to study this property for the variables:

$$d_k = Y(t_k) \cdot \sum_{i=0}^{m} \int_0^1 \alpha^{(i)}(t, t_j, h_n)$$
$$\times \varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right) dt \qquad (25)$$

It can be easily verified that

$$Ed_k = a(t_k) \cdot \sum_{i=0}^{m} \int_0^1 \alpha^{(i)}(t, t_j, h_n)$$
$$\times \varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right) dt \qquad (26)$$

Thus we consider the sequence of independent random variables:

$$f_n(k) = \alpha(n,k)\left(Y(t_k) - a(t_k)\right) = \alpha(n,k)\varepsilon_k,$$

Where

$$\alpha(n,k) = \sum_{i=0}^{m} \int_0^1 \alpha^{(i)}(t, t_j, h_n)$$
$$\times \varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right) dt.$$

Let consider the sum $S_n(h_n) = \sum_{k=1}^{n} \alpha(n,k)\varepsilon_k$.

Let $F_{k,n}$ be the probability distribution function of a random variable $\alpha(n,k)\varepsilon_k$ and $F_\varepsilon$ be the distribution function of a random variable - $\varepsilon_k$. The Lindeberg's condition is written in the form $\forall \delta > 0$, $\lim_{n\to\infty} L_n(\delta) = 0$ where

$$L_n(\delta) = \frac{\sum_{j=1}^{n} \int x^2 J\left(|x| \geq \delta\sigma\left(\sum_{k=1}^{n} \alpha^2(n,k)\right)^{1/2}\right) dF_{k,n}(x)}{\sigma^2 \sum_{k=1}^{n} \alpha^2(n,k)},$$

here $J(A)$ is the indicator function of the set $A$. It is easy to see that

$$L_n(\delta) \leq \frac{1}{\sigma^2} \max_{1\leq j\leq n} \int x^2 J(|x| \geq \delta\sigma v(n,j)) dF_\varepsilon,$$

where

$$v(n,j) = \frac{|\alpha(n,j)|}{\left(\sum_{j=1}^{n} \alpha^2(n,j)\right)^{1/2}}.$$

It remains to show that $\max_{1\leq j\leq n} v(n,j) \to 0$ as $n \to \infty$. But since

$$\max_{1\leq j\leq n} |\alpha(n,j)| = O\left(\frac{1}{n h_n^m}\right),$$

we have $\max_{1\leq j\leq n} v(n,j) = O\left(\frac{1}{\sqrt{n}}\right)$.

Thus the Lindeberg's condition is fulfilled and we can conclude that the theorem is valid.

Theorem 3: Let the conditions of Theorem 1 be fulfilled. Then if $h_n \to 0$ and $\sqrt{n} h_n^{m+1} \to \infty$ as $n \to \infty$, we have

$$\sqrt{n}\left(I(\hat{a}_n) - I(a_n)\right) \to_d N(0, r^2),$$

where

$$r^2 = \sigma^2 \cdot \sum_{i=0}^{m} \left(\int_0^1 \varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right) dt\right)^2.$$

## V. APPLICATIONS

Let us consider of the integral functional

$$I_1(a) = \int_0^1 a^2(t)\, dt.$$

Then $\varphi(t, x_0, x_1, \ldots, x_m) = x_0^2$ for $x_0 \in [-b, b] \supset [-k, k], b > 0$.

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 4, Issue 1, July 2014

Thus $r^2 = 4\sigma^2 \left(\int_0^1 a(t)\,dt\right)^2$ and, using the Conditions $h_n \to 0$ and $\sqrt{n}h_n \to \infty$ as $n \to \infty$, we have the convergence

$$\sqrt{n}\left(I(\hat{a}_n) - I(a_n)\right) \to_d N(0, r^2).$$

For the functional

$$I_2(a) = \int_0^1 \frac{\left(a'(t)\right)^2}{a(t)} dt$$

we obtain $\varphi(t, x_0, x_1, \dots, x_m) = \frac{x_1^2}{x_0}$. Then, assuming that $t \in [0,1] \Rightarrow a(t) \in [a, b], b > a > 0$, we have

$$r^2 = \sigma^2 \cdot \left( -\int_0^1 \left( \frac{\left(a'(t)\right)^2}{\left(a(t)\right)^2} - \frac{2a'(t)}{a(t)} \right) dt \right)^2 =$$

$$= \sigma^2 \left( \frac{a'(1)}{a(1)} \log a(1) - \frac{a'(0)}{a(0)} \log a(0) - \frac{1}{2} \cdot \frac{\left(a'(1)\right)^2}{\left(a(1)\right)^2} + \frac{1}{2} \cdot \frac{\left(a'(0)\right)^2}{\left(a(0)\right)^2} \right)^2$$

For $h_n \to 0$ and $\sqrt{n}h_n^2 \to \infty$ as $n \to \infty$, we have the convergence

$$\sqrt{n}\left(I(\hat{a}_n) - I(a_n)\right) \to_d N(0, r^2).$$

Let us consider the functional

$$I_3(a) = \int_{-\infty}^{\infty} \left(a(t)\right)^s dt, s > 1.$$

Then $\varphi(t, x_0, x_1, \dots, x_m) = x_0^s$, for $x_0 \in [-b, b] \supset [-k, k], b > 0$. Therefore

$$r^2 = s^2\sigma^2 \left( \int_0^1 a^{s-1}(t)\,dt \right)^2$$

And for the condition $h_n \to 0$ and $\sqrt{n}h_n \to \infty$ as $n \to \infty$, we have the convergence

$$\sqrt{n}\left(I(\hat{a}_n) - I(a_n)\right) \to_d N(0, r^2).$$

Let us now take the functional

$$I_4(a) = \int_{-\infty}^{\infty} a(t) \log a(t)\,dt.$$

Then for some sufficiently large $b \geq k > 0$, if $0 < x_0 \leq b$ we have $\varphi(t, x_0, x_1, \dots, x_m) = \varphi(x_0) = x_0 \log x_0$. Let us extend the definition of the function $\varphi$ by defining $\varphi(x) = 0$ for $-b \leq x \leq 0$. Assume that $t \in [0,1] \Rightarrow a(t) \in [a, b], b > a > 0$ and $b \geq b$. Then

$$r^2 = \sigma^2 \left( \int_0^1 a(t)\left(1 + \log a(t)\right) dt \right)^2$$

And for the condition $h_n \to 0$ and $\sqrt{n}h_n \to \infty$ as $n \to \infty$, we have the convergence

$$\sqrt{n}\left(I(\hat{a}_n) - I(a_n)\right) \to_d N(0, r^2).$$

## VI. ITERATED LOGARITHM LAW

Applying the well-known iterated logarithm law from Kuelb's (paper [11]), we ascertain that the following statement is true.

Theorem 4. If the sequence $h_n$ is chosen so that

$$R_n = o\left(\sqrt{\frac{\log\log n}{n}}\right),$$

then

$$\limsup_{n\to\infty} \pm \frac{\sqrt{n}\left(I(\hat{a}_n) - I(a_n)\right)}{\sqrt{2\log\log n}} = r.$$

Indeed, it can be easily verified that

$$\limsup_{n\to\infty} \pm \frac{\sqrt{n}\left(I(\hat{a}_n) - I(a_n)\right)}{\sqrt{2\log\log n}}$$

$$= \limsup_{n\to\infty} \pm \frac{\sqrt{n}\left(\alpha(n,k)Y(t_k) - \alpha(n,k)a(t_k)\right)}{\sqrt{2\log\log n}} = r.$$

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ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 4, Issue 1, July 2014

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