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ON TERNARY CUBIC EQUATION

$(x-h)^2 + (y-k)^2 + 4\,(33z^2 - 4 - \alpha^2) = 6\,(x-h)(y-h)z$

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Abstract we obtain the non-trivial integral solutions for the ternary cubic equation

$(x-h)^2 + (y-k)^2 + 4\,(33z^2 - 4 - \alpha^2) = 6\,(x-h)(y-h)z$

.

A few interesting relations among the solutions are presented.

Index Terms: Ternary Cubic, integral solutions, Pell's form, nasty numbers

Notations

Oblong number of rank $n = obl_n = n(n+1)$

Tetrahedral number of rank $n = Tet_n = \frac{n(n+1)(n+2)}{6}$

Triangular number of rank $n = t_{3,n} = \frac{n(n+1)}{2}$

Polygonal number of rank n with sides m $= t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$

Square pyramidal number of rank n $= Sqp_n = \frac{n(n+1)(2n+1)}{6}$

Pentagonal pyramidal number of rank $n = Pp_n = \frac{n^2(n+1)}{2}$

Star number = $ST_n = 6n\,(n-1) + 1$

Stella Octangula number = $St.oct_n = n\left(2n^2 - 1\right)$

$$K = Kynea\ number\ of\ rank\ n = \left(2^n + 1\right)^2 - 2$$

4D figurate number whose generating polynomial is a square of side of length 4DF = $\frac{n^4 - n^2}{12}$

I. INTRODUCTION

Diophantine equations have an unlimited field for research of their variety [1,2,6, 7,8,9]. In particular, one may refer [3-5]. Wherein the ternary cubic Diophantine equations are analyzed for the non-trivial integral solutions. These results have motivated us to search for non-trivial integral solutions of their varieties of ternary cubic Diophantine equation. This paper concerns with the problem of determining non-trivial integral solutions of the equation with three unknowns given by $(x-h)^2 + (y-k)^2 + 4\,(33z^2 - 4 - \alpha^2) =$

$6\,(x-h)(y-h)z$

explicit integral solutions of the above equation are presented. A few interesting relations among the solutions are obtained.

II. METHOD OF ANALYSIS

The ternary cubic equation under consideration is

$$(x-h)^2 + (y-k)^2 + 4\,(33z^2 - 4 - \alpha^2) = 6\,(x-h)(y-h)z \qquad (1)$$

Taking $x - h = u + v$ (2)

$y - k = u - v$ (3)

We get

$$v^2\,(6z+2) + 132z^2 - 16 - 4\alpha^2 = u^2(6z-2) \qquad (4)$$

Again taking the transformation

$u = X + 6z + 2$

$v = X + 6z - 2$

and apply in (4) we get

$$X^2 = 3z^2 + \alpha^2 \qquad (5)$$

It is well known that the general form of the integral solutions (x_n, z_n) $n = 0,1,2,\ldots$ of the Pellian equation.

$x^2 = Az^2 + 1$ is represented by

$$\left(x_n + \sqrt{A}\ z_n\right) = \left(x_0 + \sqrt{A}\ z_0\right)^{n+1};\ n = 0,1,2,\ldots \qquad (6)$$

Where (x_0, z_0) is the smallest positive inter solution

These having the solutions of (x_n, z_n) of (6), the general form of integral solutions (x_n, z_n) for $x^2 = Az^2 + \alpha^2$ is

$$x_n = \frac{\alpha}{2}\left[\left(\tilde{x}_0 + \sqrt{A}\ z_0\right)^{n+1} + \left(x_0 - \sqrt{A}\ z_0\right)^{n+1}\right] \qquad (7)$$

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$$z_n = \frac{\alpha}{2\sqrt{A}}\left[\left(\tilde{x}_0 + \sqrt{A}\, z_0\right)^{n+1} - \left(x_0 - \sqrt{A}\, z_0\right)^{n+1}\right] \quad (8)$$

Case I Choose $\alpha = 1$

Now from (5), the solutions of X_n and z_n are as follows

$$X_n = \frac{1}{2}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] \quad (9)$$

$$z_n = \frac{1}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right] \quad (10)$$

Thus the solutions of *x*, *y*, *z* are

$$x_n = \left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] + \frac{6}{\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right] + h \quad (11)$$

$$y_n = 4 + k \quad (12)$$

$$z_n = \frac{1}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right] \quad (13)$$

Some examples for the solutions of *x*, *y*, *z* are presented below

x_n	z_n	$x_n = 2x_n + 12z_n + h$	y_n	z_n
2	1	16+h	4+k	1
7	4	62+h	4+k	4
26	15	232+h	4+k	15
97	56	866+h	4+k	56
362	209	3232+h	4+k	209
1351	780	12062+h	4+k	780
5042	2911	45016+h	4+k	2911
18817	10864	168002+h	4+k	10864
70226	40545	626992+h	4+k	40545
262087	151316	2339966+h	4+k	151316
978122	564719	8732872+h	4+k	564719

Recurrence Relations

$x_{n+2} - 4x_{n+1} + x_n + 2h = 0$

$y_n = 4 + k$

$z_{n+2} - 4z_{n+1} + z_n = 0$

Properties

1. $X_n^2 - 1 = 3\ times\ a\ perfect\ square$
2. $2(X_n^2 - 1)$ is a nasty number
3. $x_n - 2X_n - 12x_n \equiv 0$ (mod h)
4. $y_n - y_{n-1} = 0$
5. $y_n - 4 \equiv 0\ (mod\ k)$
6. y_n^2 is a perfect square
7. $6\ (y_n - k)$is a nasty number
8. $x_n - h \equiv 0\ (mod\ 2)$
9. $u_n - v_n \equiv 0\ (mod\ 4)$
10. $u_n + v_n \equiv 0\ (mod\ 2)$
11. $x_{n+2} - 4x_{n+1} + x_n \equiv (mod h)$
12. $(x_n - 2X_n - 12z_n + 1)^2 - obl_h - h - 1 = 0$
13. $(x_n - 2X_n - 12z_n + 1)^3 - 6t_{3,h} - 1 \equiv 0\ (mod\ h)$
14. $3[(x_n - 2X_n - 12z_n + 1)^3 - 2Pen_h - 3h - 1]$ is a nasty number
15. $y_n^2 - 2\,t_{3,k} - 16 \equiv 0\ (mod\ 7)$
16. $y_n^3 - ST_k - 63 - k^3 \equiv 0\ (mod\ 54)$
17. $(x_n - 2X_n - 12z_n + 1)y_n - hk - k - 4 \equiv (mod 4)$
18. $y_n^3 - 6\,Tet_k - 3k^2 - 8 \equiv 0 (mod 10)$
19. $ky_n^2 - k^3 - 8\ obl_k \equiv 0 (mod\ 8)$
20. $y_n^3 - ST_k - 63 - 54k$ is a cubic integer

Case 2 Choose $\alpha = a^2$

Then consider the equation

$X^2 = 3z^2 + a^4$

Then the corresponding solutions of (5) are given by

$$X_n = \frac{a^4}{2}\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] \quad (14)$$

$$z_n = \frac{a^2}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right] \quad (15)$$

Then the solutions of *x*, *y*, *z* are

$$x_n = a^2\left[\left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}\right] + \frac{6a^2}{\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right] + h \quad (16)$$

$$y_n = 4 + k \quad (17)$$

$$z_n = \frac{a^2}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right] \quad (18)$$

Properties

1. $X_n^2 - 3z_n^2$ is a quartic integer
2. $X_n^2 - a^4 = 3\ times\ a\ perfect\ square$
3. $2(X_n^2 - a^4)$ is a nasty number
4. $X_n^2 - 3Z_n^2 \equiv 0$ (mod a)
5. $y_n - k$ is a perfect square
6. $y_n - 4 \equiv 0\ (mod\ k)$
7. $x_n - h \equiv 0\ (mod\ 2)$
8. $u_n - v_n \equiv 0\ (mod\ 4)$
9. $.u_n + v_n \equiv 0\ (mod\ 2)$
10. $6(X_n^2 - 3z_n^2) = (6a^2)$ times a nasty number
11. $2\lfloor ky_n^2 - 16t_{3,k} - 8k\rfloor$ is a nasty number
12. $ky_n^2 - 2Pen_k - 16t_{3,k} \equiv 0\ (mod\ k)$

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13. $2y_n^3 - st.oct_k + 24\,obl_k - 128 \equiv 0 \pmod{73}$

14. $2(y_n + 2)^3 - 6\,SqP_n - 9k^2 - 16 \equiv 0 \pmod{23}$

15 $6[2(y_n + 2)^3 - 6\,SqP_n - 23k - 16]$ is a nasty number

Case 3 **Choose $\alpha = 2^{n+1}a^2$**

Then consider the equation

$X^2 = 3z^2 + 2^{n+2}a^4$

Then the corresponding solutions of (5) are given by

$$X_n = 2^n a^2 \left[(2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}\right] \quad (19)$$

$$z_n = \frac{2^n a^2}{\sqrt{3}} \left[(2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}\right] \quad (20)$$

Then the solutions of *x*, *y*, *z* are

$$x_n = 2^{n+1}a^2\left[(2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1}\right] + 2^{n+1}\left(\frac{6a^2}{\sqrt{3}}\right)\left[(2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}\right] + h \quad (21)$$

$$y_n = 4 + k \quad (22)$$

$$z_n = \frac{2^n a^2}{\sqrt{3}}\left[(2+\sqrt{3})^{n+1} - (2-\sqrt{3})^{n+1}\right] \quad (23)$$

Properties

1. $X_n^2 - 3z_n^2 \equiv 0 \pmod{a}$

2. $2(X_n^2 - 2^{n+2}a^4)$ is a nasty number

3. $X_n^2 - 3z^2 - 4K + 4(2^{n+1} - 1) = 0$ when a = 1

4. $k^2y_n^2 - 12\,4DF - 16\,Pen_k - 9k^2 = 0$

5. $6(k^2y_n^2 - 12\,4DF - 16\,Pen_k)$ is a nasty number

6. $(x_n - 2X_n - 12z_n + 1)^3 - 3\,Obl_h - 1$ is a cubic integer

7. $(x_n - 2X_n - 12z_n + 1)^3 - 6\,3t_{3,h} \equiv 0 \pmod{h+1}$

8. $2(x_n - 2X_n - 12z_n + 1)^3$
$-St.oct_h - 12t_{3,h} - 2 \equiv 0 \pmod{h}$

9. $3k^2y_n^2 - 4ST_k - 3k^2 \equiv 0 \pmod{72}$

10. $3k^2y_n^2 - 4ST_k - 3k^4$ is 12 times a nasty number

11. $y_n - 4 \equiv 0 \pmod{k}$

12. $x_n - h \equiv 0 \pmod{2}$

13. $u_n - v_n \equiv 0 \pmod{4}$

14. $u_n + v_n \equiv 0 \pmod{2}$

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