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Topics On the Diagonal Matrix Functions

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Abstract: *Let $\Omega \subset R^n$ be a bounded domain with smooth boundary i.e. $\partial\Omega \in C^{\infty}$. In this paper we consider the matrix function $a(x) \in C^n(\overline{\Omega}; End\ C^l)$. Assume that for every $x \in \overline{\Omega}$, the matrix $a(x)$ has l simple eigenvalues $\mu_1(x), \cdots, \mu_l(x)$. Then the eigenvalues of the matrix $a(x)(x \in \overline{\Omega})$ may be enumerated in such a way that $\mu_j(x) \in C(\overline{\Omega})(j = 1, \cdots, l)$. There exists a matrix function $U(x) \in C^n(\overline{\Omega}; End\ C^l)$ such that $U^{-1}(x) \in C^n(\overline{\Omega}; End\ C^l)$ and $a(x) = U(x)\Lambda(x)U^{-1}(x)$, where $\Lambda(x)$ is the diagonal matrix: $\Lambda(x) = diag\{\mu_1(x), \cdots, \mu_l(x)\}$ At the same time $\mu_j \in C^n(\overline{\Omega})(j = 1, \cdots, l)$.* [2]

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I. INTRODUCTION

We consider the matrix function $a(x) \in C^n(\overline{\Omega}; End\ C^l)$. Assume that for every $x \in \overline{\Omega}$, the matrix $a(x)$ has l simple eigenvalues $\mu_1(x), \cdots, \mu_l(x)$. Then the eigenvalues of the matrix $a(x)(x \in \overline{\Omega})$ may be enumerated in such a way that $\mu_j(x) \in C(\overline{\Omega})(j = 1, \cdots, l)$, then the following theorem on The diagonalizing matrix functions holds

Theorem on The diagonalizing matrix functions There exists a matrix function

$$U(x) \in C^n(\overline{\Omega}; End\ C^l)$$

such that $U^{-1}(x) \in C^n(\overline{\Omega}; End\ C^l)$ and

$$a(x) = U(x)\Lambda(x)U^{-1}(x), (1)$$

where $\Lambda(x)$ is the diagonal matrix:

$$\Lambda(x) = diag\{\mu_1(x), \cdots, \mu_l(x)\}$$

At the same time $\mu_j \in C^n(\overline{\Omega})(j = 1, \cdots, l)$.

Let $x_0 \in \overline{\Omega}$. We first construct the matrix function

$$U(x) \in C^n(V_{x_0}; End\ C^l)$$

for a sufficiently small neighborhood $V_{x_0} \subset \overline{\Omega}$ of x_0 satisfying (1) in V_{x_0} and the inclusion

$$U^{-1}(x) \in C^n(V_{x_0}; End\ C^l).$$

Let $r \in \{1, \cdots, l\}$ be a fixed index. We introduce the matrix

$$Q(x) = 1/(2\pi i)\int_{\gamma_\varepsilon} (a(x) - zI)^{-1} dz \quad (2)$$
$$|x - x_0| < \varepsilon', x \in \overline{\Omega}$$

Here $\gamma_\varepsilon = \{z \in C : |z - \mu_r(x_0)| = \varepsilon\}$ is a contour oriented counterclockwise. Denote

$$V_\varepsilon = \{z \in C : |z - \mu_r(x_0)| < \varepsilon\},$$
$$W_{\varepsilon'} = \{x \in \overline{\Omega} : |x - x_0| < \varepsilon'\},$$
$$\mu_i(W_{\varepsilon'}) = \{\mu_i(x) : x \in W_{\varepsilon'}\}.$$

For sufficiently small numbers $\varepsilon, \varepsilon'$ we have

$$\mu_i(W_{\varepsilon'}) \cap \mu_j(W_{\varepsilon'}) = \varnothing\ (i \neq j)$$
$$\mu_i(W_{\varepsilon'}) \cap V_\varepsilon = \varnothing\ \ (i \neq r)$$
$$\mu_r(W_{\varepsilon'}) \subset W_\varepsilon$$

We denote by the symbol tr the trace of a matrix or an operator. Considering the following equality

$$tr\int_{\gamma_\varepsilon} z(a(x) - zI)^{-1} dz = \sum_{j=1}^{l}\int_{\gamma_\varepsilon} z(\mu_j(x) - z)^{-1} dz,$$

we obtain

$$\mu_r(x) = 1/(2\pi i) tr\int_{\gamma_\varepsilon} z(a(x) - zI)^{-1} dz.(3)$$

Since $a(x) \in C^n(\overline{\Omega}; End\ C^l)$, it follows that

$$\mu_j(x) \in C^n(\overline{\Omega}) \quad (j = 1, \cdots, l).$$

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Let $y_\nu(x) = (y_{1\nu}(x), \cdots, y_{l\nu}(x))$ be the eigenvectors of the matrix $a(x) = (a_{ij}(x))_{i,j=1}^{l}$ corresponding to the eigenvalues $\mu_\nu(x)$, i. e.

$$\sum_{j=1}^{l} a_{ij}(x) y_{j\nu}(x) = \mu_\nu(x) y_{i\nu}(x) \ (i = 1\cdots, l).$$

It is easy to verify that the matrix $U(x) = (y_{ij}(x))_{i,j=1}^{l}$ satisfies the equalities

$$(a(x)U(x))_{pq} = \sum_{\nu=1}^{l} a_{p\nu}(x) y_{\nu q}(x) = \mu_q(x) y_{pq}(x),$$

$$(U(x)\Lambda(x))_{pq} = \sum_{\nu=1}^{l} y_{p\nu}(x)\delta_{\nu q}\mu_\nu(x) = \mu_q(x) y_{pq}(x)$$

where $\delta_{\nu q}$ is the Kronecker delta. Therefore, $a(x)U(x) = U(x)\Lambda(x)$. Since the columns of the matrix $U(x)$ consists of the linearly independent eigenvectors of the matrix $a(x)$ we have

$$det\, U(x) \neq 0 \ (x \in W_{\varepsilon'}).(4)$$

From the following equality

$$(a(x) - zI)^{-1} =$$
$$U(x)(\Lambda(x) - zI)^{-1} U^{-1}(x), \ (x \in W_{\varepsilon'})$$

we obtain

$$Q(x) = 1/(2\pi i) U(x) (\int_{\gamma_\varepsilon} (\Lambda(x) - zI)^{-1} dz) U^{-1}(x)$$
$$= U(x) T_r U^{-1}(x) \qquad (5)$$

$$T_r = diag\{\delta_{1r}, \cdots, \delta_{lr}\}$$

From this equality, it is easy to see that the image of the operator $Q(x): C^l \to C^l$ is 1-dimensional i.e. from $T_r = diag\{\delta_{1r}, \ldots, \delta_{lr}\}$ it follows that the linear set $\alpha(x) = \{q(x)h : h \in C^l \ , x \in W_{\varepsilon'}\}$ is one dimensional and contains the eigenvectors $y_r(x)$, $(x \in W_{\varepsilon'})$. Since the matrix $Q(x)$ acts by the formula

$$Q(x)h =< h, \phi_r(x) >_{C^l} y_{r(x)}, (h \in C^1, x \in W_{\varepsilon'}), (6)$$

where $\phi_r(x) \in C^l, t \in W_{\varepsilon'}$.

Setting $h = a(x)h_1, (h_1 \in C^l)$ and keeping in mind the equalities

$$Q(x)a(x)h_1 = a(x)Q(x)h_1$$
$$=< a(x)h_1, \phi_r(x) >_{C^l} y_r(x)$$
$$=< h_1, a(x)S^* \phi_r(x) >_{C^l} y_r(x)$$
$$= \mu_r(x) < h, \phi_r(x) >_{C^l} y_r(x),$$

for arbitrary $h_1 \in C^l$, we obtain

$$a(x)^* \phi_r(x) = \overline{\mu_r(x)} \phi_r(x).$$

According to (4), $tr\, Q(x) = tr\, T_r = 1$. Hence $< y_r(x), \phi_r(x) >_{C^l} = 1$. Using (6) it is easy to obtain the entries of the matrix $Q(x)$:

$$(Q(x))_{ij} = y_{ir}(x) \overline{\phi_{jr}(x)},$$

where $\phi_{jr}(x) \ (j = 1, \cdots, l)$ are the entries of the vector $\phi_r(x)$:

$$\phi_r(x) = (\phi_{1r}(x), \cdots, \phi_{lr}(x)).$$

From here and from (2) it follows that

$$y_{ir}(x) \overline{\phi_{jr}(x)} \in C^n(W\varepsilon') \ \ (i, j = 1, \cdots, l).(7).$$

Replacing ε', if necessary, by a smaller number we may obtain an index $w \in \{1, \cdots, l\}$, such that $\phi_{wr}(x) \neq 0 \ \ (t \in W_{\varepsilon'})$. Next, replacing $y_r(x), \ \phi_r(x)$ if necessary, with

$$\phi_{wr}(x)^{-1} y_r(x), \phi_{wr}(x)^{-1} \phi_r(x)$$

respectively, we may assume, without loss of generality that $\phi_{wr}(x) = 1, \ (x \in W_{\varepsilon'})$. Substituting $j = w$ in (2.6) we have

$$y_{ir}(x) \in C^n(W_{\varepsilon'}) \ \ (i = 1, \cdots, l).$$

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According to (3.3) we have

$$U^{-1}(x) \in C^n(W_{\varepsilon'}; End\ C^l).$$ Let $F_1, F_2 \subset \overline{\Omega}$ be two closed balls,

$$F_j = \overline{\Omega} \cap F_j, (j = 1,2), \lambda(F_j) > 0$$

$(\lambda = Lebesgue\ measure)$. As above construct the matrix functions

$$U_j(x) \in C^n(F_j\ ; End\ C^l)\ \ (j = 1,2)$$

such that

$$U_j^{-1}(x) \in C^n(F_j; End\ C^l)\ \ (j = 1,2)$$

$$a(x) = U_j(x)\Lambda(x)U_j^{-1}(x)\ \ (t \in F_j,\ j = 1,2)$$

We will construct a matrix function

$$U(x) \in C^n(F_1 \cup F_2\ \ ; End\ C^l)$$

such that

$$U^{-1}(x) \in C^n(F_1 \cup F_2; End\ C^l)$$

$$a(x) = U(x)\Lambda(x)U^{-1}(x)\ \ (x \in F_1 \cup F_2)$$

The columns of the matrices

$$U_1(x), U_2(x)\ (x \in F)$$

consist of the eigenvectors of the matrix $a(x)$ and hence are collinear. Consequently,

$$U_1(x) = U_2(x)\Omega(x),\ \ \Omega(x) = diag\{\Omega_1(x), \cdots, \Omega_l(x)\},$$

where $\omega_j(x), \omega_j^{-1}(x) \in C^n(F_1 \cap F_2)\ \ (j = 1, \cdots, l)$. We extend the function $\omega_j(x)\ '\ \ (x \in F_1 \cap F_2),\ \ (j = 1, \cdots, l)$ to the function $\tilde{\omega}_j(x) \in C^n(F_2)$ such that $\tilde{\omega}_j^{-1}(x) \in C^n(F_2)\ (j = 1, \cdots, l)$. Put;

$$\tilde{\Omega}(x) = diag\{\tilde{\omega}_1(x), \cdots, \tilde{\omega}_l(x)\}\ \ \ \ (x \in F_2).$$

It is easy to see that the matrix function;

$$U(x) = \begin{cases} U_1(x) & if\ \ x \in F_1 \\ U_2(x)\tilde{\Omega}(x) & If\ \ x \in F_2 \end{cases}$$

satisfies the above stated conditions. The proof of the Lemma is finished by applying the sewing method.

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