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# Interior Thermoelastic Problem of a Semi infinite Circular Beam due to Heat Generation

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***Abstract-This paper is concerned with an inverse transient thermoelastic problem to determine the temperature distribution, unknown temperature gradient, displacement and thermal stresses on the curved surface of a thick semi-infinite circular beam occupying the space*** $D: 0 \le r \le a, 0 \le z \le \infty$. ***The governing heat conduction equation has been solved by using Marchi-Zgrablich transforms and Fourier cosine transform techniques. The results are obtained in terms of Bessel's function in the form of infinite series.***

**KEY WORDS & PHRASES: - Unsteady state thermo elastic problem, thermal stresses, thick circular beam, Michelle's function.**

## I. INTRODUCTION

The direct problems of the thermo elasticity in a thin circular plate have been considered by Nowacki [2], Wankhede [5] has determined the quasi- static thermal stresses in a circular plate subjected to arbitrary temperature on the upper face with the lower face at zero temperature and the fixed circular edge thermally insulated. Noda et.al [3] discussed an analytical method for an inverse problem of three dimensional transient thermo elasticity in a transversely isotropic solid. Tanigawa et.al [3] has studied the theoretical analysis thermoelastoplastic deformation of plate subject to partially distributed heat supply. Khobragade et.al [6, 7] solved an inverse unsteady state thermo elastic problem of a thin circular plate in Marchi-Fasulo transform domain.

In this article, we analyzed inverse thermo elastic problem of temperature and thermal stresses of thick, semi-infinite circular beam due to heat generation. The governing heat conduction equation has been solved by using Marchi-Zgrablich and Fourier Cosine transform techniques. The result presented here will be more useful in engineering applications.

## II. STATEMENT OF THE PROBLEM

Consider a thick circular beam occupying the space D : 0 ≤ r≤*a*, 0≤ z< ∞. The material is homogeneous and isotropic. The differential equation governing the displacement potential function $\phi$ (r ,z ,t) is given by [ Noda et. al. (3) ]

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \left[\frac{1+\upsilon}{1-\upsilon}\right]\alpha_t T \qquad (1)$$

Where, ν and $\alpha_t$ are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate and T is temperature of the plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \chi(r,z,t) = \frac{1}{k}\frac{\partial T}{\partial t} \qquad (2)$$

Subject to initial condition

$$T(r,z,0) = f(r,z) \qquad (3)$$

And boundary conditions are

$$\left[T(r,z,t) + k_1 \frac{\partial T(r,z,t)}{\partial r}\right]_{r=0} = g_1(z,t) \qquad (4)$$

$$\left[T(r,z,t) + k_2 \frac{\partial T(r,z,t)}{\partial r}\right]_{r=a} = g_2(z,t) \qquad (5)$$

$$\left[\frac{\partial T(r,z,t)}{\partial z}\right]_{z=0} = f_1(r,t) \qquad (6)$$

$$\left[\frac{\partial T(r,z,t)}{\partial z}\right]_{z=\infty} = f_2(r,t) \quad ,0 \le r \le a, \quad t > 0 \qquad (7)$$

Where k is the thermal diffusivity of the material of the plate. The displacement function in the cylindrical co-ordinate system are represented by the Goodier thermoelastic function $\phi$ and Love's function L as [3]:

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 L}{\partial r \partial z} \qquad (8)$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2} \qquad (9)$$

In which Goodier thermoelastic potential [3] must satisfy the equation

$$\nabla^2 \phi = \left(\frac{1+\nu}{1-\nu}\right) a_t T$$

The Love's function [8] must satisfy

$$\nabla^2(\nabla^2 L) = 0 \qquad (10)$$

Where,

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$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The component of stresses are represented by the use of the potential $\phi$ and Love's function L as

$$\sigma_{rr} = 2G\left\{\left[\frac{\partial^2\phi}{\partial r^2} - \nabla^2\phi\right] + \frac{\partial}{\partial z}\left[\upsilon\nabla^2 L - \frac{\partial^2 L}{\partial r^2}\right]\right\} \quad (11)$$

$$\sigma_{\theta\theta} = 2G\left\{\left[\frac{1}{r}\frac{\partial\phi}{\partial r} - \nabla^2\phi\right] + \frac{\partial}{\partial z}\left[\upsilon\nabla^2 L - \frac{1}{r}\frac{\partial^2 L}{\partial r^2}\right]\right\} \quad (12)$$

$$\sigma_{zz} = 2G\left\{\left[\frac{\partial^2\phi}{\partial z^2} - \nabla^2\phi\right] + \frac{\partial}{\partial z}\left[(z-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right]\right\} \quad (13)$$

$$\sigma_{rz} = 2G\left\{\frac{\partial^2\phi}{\partial r\partial z} + \frac{\partial}{\partial r}\left[(1-\upsilon)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right]\right\} \quad (14)$$

The equations (1) to (14) constitute the mathematical formulation of the problem under consideration.

## III. SOLUTION OF THE PROBLEM

Applying finite Marchi-Zgrablich transform defined in [9] to the equations (2) and using equations (4), (5) one obtains

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \chi(r,z,t) = \frac{1}{k}\frac{\partial T}{\partial t} \quad (15)$$

By using the operational property of finite Marchi-Zgrablich transform, we get

$$\frac{\partial^2 \overline{T}}{\partial z^2} - \mu_n^2\overline{T} + \overline{\chi} = \frac{1}{k}\frac{\partial \overline{T}}{\partial t} + g(z,t) \quad (16)$$

Again, applying Fourier cosine transform to the equation (14), we get

$$\frac{d\overline{T}_c^*}{dt} + kp^2\overline{T}_c^* = \phi_1^* + \overline{\chi}_1^* \quad (17)$$

where

$$\overline{\chi}_1^* = k\overline{\chi}_c^* \quad and\ \phi_1^* = k\mu - k\mu_n^2\overline{T}_c^* - kg_c^*$$

Equation (15) is a linear equation whose solution is given by

$$\overline{T}^*(n,z,t) = e^{-kp^2t}\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2t'}dt' + Ce^{-kp^2t}$$

Using (3) ,we get

$$C = F^*(m,n)$$

Thus, we have,

$$\overline{T}^*(n,z,t) = e^{-kp^2t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2t'}dt' + \overline{F}^*(m,n)\right]$$

(18)

Applying inversion of Fourier cosine transform and Marchi-Zgrablich transform [9] to theequation (18), one obtains

$$T(r,z,t) = \sum_{n=1}^{\infty}\frac{1}{C_n}\left\{e^{-kp^2t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2t'}dt' + \overline{F}^*(m,n)\right]\right\} \times S_0(k_1,k_2,\mu_n r)$$

(19)

This is the desired solution of the given problem.Let us assume Love's function L , which satisfy condition (10)as

$$L(r,z) = \sum_{n=1}^{\infty}\frac{1}{C_n}\psi S_0(k_1,k_2,\mu_n r) \quad (20)$$

Where,

$$\psi = e^{-kp^2t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2t'}dt' + \overline{F}^*(m,n)\right]$$

The displacement potentialis given by

$$\phi = A\sum_{n=1}^{\infty}\frac{1}{C_n}\psi S_0(k_1,k_2,\mu_n r)[\psi + B(t)] \quad (21)$$

where, $A = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t$

$$B(t) = e^{-kp^2t}\left[\int_0^t\left(\phi_1^* + \overline{\chi}_1^*\right)e^{-kp^2t'}dt' + \overline{F}^*(m,n)\right]dt$$

## IV. DETERMINATION OF THERMOELASTIC DISPLACEMENT

Substituting the equations (18) and (19) in the equation (8) one obtains

$$u_r = A\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0{}'(k_1,k_2,\mu_n r)[\psi + B(t)] - \sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0{}'(k_1,k_2,\mu_n r) \quad (22)$$

$$u_z = 2(1-\upsilon)\left[\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}\psi S_0{}'(k_1,k_2,\mu_n r) + \frac{1}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0{}'(k_1,k_2,\mu_n r)\right] \quad (23)$$

## V. DETERMINATION OF STRESS FUNCTIONS

Substituting the values from the equation (20) and (21) in the equations (11) to (14) we get,

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$$\sigma_{rr} = 2G\left\{\begin{array}{l}\left[\begin{array}{l} A\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}S_0{}''(k_1,k_2,\mu_n r)[\psi + B(t)] \\ -A\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}S_0{}''(k_1,k_2,\mu_n r)[\psi + B(t)] - \\ \frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0{}'(k_1,k_2,\mu_n r)[\psi + B(t)] \end{array}\right] + \\ +\frac{\partial}{\partial z}\left[\begin{array}{l} \upsilon\left[\begin{array}{l} A\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}S_0{}''(k_1,k_2,\mu_n r) \\ +\frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0{}'(k_1,k_2,\mu_n r) \end{array}\right] \\ -\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}S_0{}''(k_1,k_2,\mu_n r) \end{array}\right] \end{array}\right\} \quad (24)$$

$$\sigma_{\theta\theta} = 2G\left\{\begin{array}{l} \frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0{}'(k_1,k_2,\mu_n r)[\psi + B(t)] \\ \left[\begin{array}{l} A\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}S_0{}''(k_1,k_2,\mu_n r)[\psi + B(t)] + \\ \frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0{}'(k_1,k_2,\mu_n r)[\psi + B(t)] \end{array}\right] + \\ +\frac{\partial}{\partial z}\left[\begin{array}{l} \upsilon\left[\begin{array}{l} A\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}\psi S_0{}''(k_1,k_2,\mu_n r) \\ +\frac{1}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0{}'(k_1,k_2,\mu_n r) \end{array}\right] \\ -\frac{1}{r}\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}\psi S_0{}''(k_1,k_2,\mu_n r) \end{array}\right] \end{array}\right\} \quad (25)$$

$$\sigma_{zz} = 2G\left\{\begin{array}{l} \left[\begin{array}{l} A\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}S_0{}''(k_1,k_2,\mu_n r)[\psi + B(t)] \\ +\frac{A}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}S_0{}'(k_1,k_2,\mu_n r)[\psi + B(t)] \end{array}\right] + \\ (1-\upsilon)\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}\psi S_0{}''(k_1,k_2,\mu_n r) \\ +\frac{(1-\upsilon)}{r}\sum_{n=1}^{\infty}\frac{\mu_n}{C_n}\psi S_0{}'(k_1,k_2,\mu_n r) \end{array}\right\} \quad (26)$$

$$\sigma_{rz} = 2G\left[\begin{array}{l} (1-\upsilon)\sum_{n=1}^{\infty}\frac{\mu_n{}^3}{C_n}\psi S_0'''(k_1,k_2,\mu_n r) \\ -\frac{(1-\upsilon)}{r^2}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu_n{}^2}{C_n}\psi S_0{}''(k_1,k_2,\mu_n r) \end{array}\right] \quad (27)$$

Where,

$$A = \left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha_t \ and \ \psi$$

$$= e^{-kp^2 t}\left[\int_0^t \left(\overline{\phi}_1^* + \overline{\chi}_1^*\right) e^{kp^2 t'} dt' + \overline{F}^*(m,n)\right]$$

$$B(t) = \int \psi dt$$

## VI. MATERIAL PROPERTIES

The numerical calculation has been carried out for an Aluminum (pure) circular plate with the material properties as

Density ρ =169 lb/ft $^3$

Specific heat = 0.208 Btu/lb OF

Thermal conductivity K = 15.9 x $10^6$Btu/(hr. ft OF)

Thermal diffusivity α = 3.33 ft $^2$/hr.

Poisson ratio ν = 0.35

Coefficient of linear thermal expansion

$\alpha_t$ = 12.84 x $10^{-6}$1/F

Lame constantμ = 26.67

Young's modulus of elasticity E = 70G Pa

## VII. DIMENSIONS

The constants associated with the numerical calculation are taken as

Radius of the disk b = 2ft

Thickness of the circular disk h = 0.2 ft.

## VIII. CONCLUSION

In this study, we develop the analysis for the temperature field by introducing the methods of the Marchi- Zgrablich and Fourier cosine transform techniques and determined the expression for temperature, displacement and thermal stresses of a semi-infinite, thick circular beam with known boundary conditions which is useful to design of structure or machines in engineering applications.

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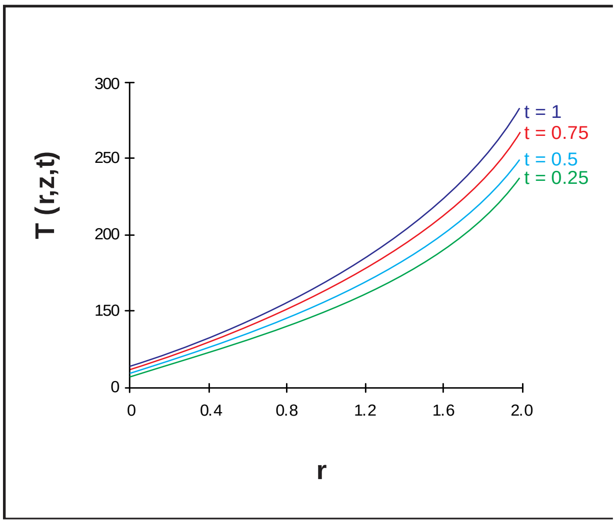


**Fig. 1. Temperature distribution vs radius**

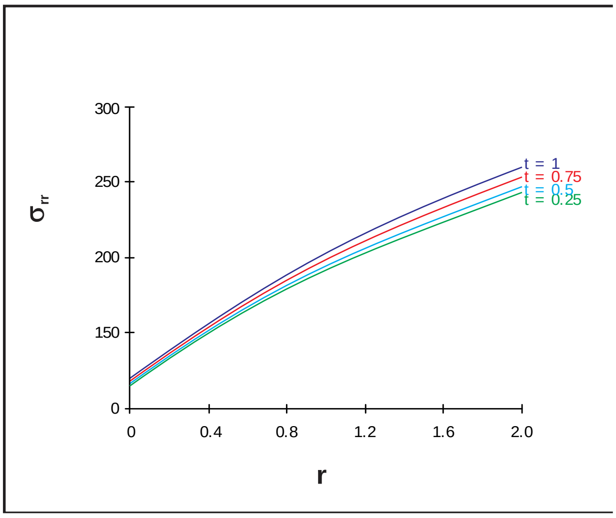


**Fig. 2. Radial Stresses vs radius**

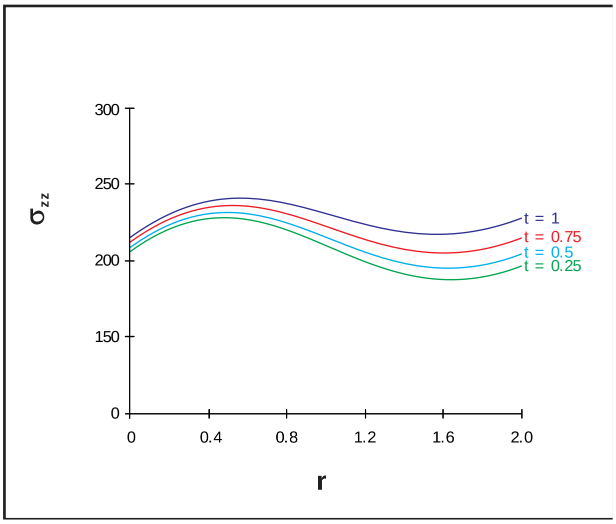


**Fig. 3. Axial Stresses vs radius**

## AUTHOR BIOGRAPHY

**Dr. N.W. Khobragade** for being M.Sc in statistics and Maths he attained Ph.D. He has been teaching since 1986 for 27 years at PGTD of Maths, RTM Nagpur University, Nagpur and successfully handled different capacities. At present he is working as Professor. Achieved excellent experiences in Research for 15 years in the area of Boundary value problems and its application. Published more than180 research papers in reputed journals. Fourteen students awarded Ph.D Degree and four students submitted their thesis in University for award of Ph.D Degree under their guidance.