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Thermoelastic Analysis on a Circular Plate Subjected to Distributed Annular Heat Supply

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Abstract- This paper deals with the general problem of determining the quasi-static thermal stress in a thin circular plate with boundary conditions of radiation type subjected to distributed annular heat source along the circumference of a circle over the upper face. The lower face is kept at zero temperature and circular edge is thermally insulated. Numerical estimates for heating medium are obtained in terms of Bessel's functions and depicted graphically.

Key words: Quasi-static, transient response, circular plate, annular heat source, temperature distribution, thermal stress

I. INTRODUCTION

The research on the prediction of stresses in circular plate with distributed heat supply has never ceased because of the importance of their basic structures in numerous civil, mechanical, electrical and computer engineering applications. However, it is evident from the list of existing literature that the stresses influenced by annular heat supply have not been studied theoretically despite of various recognized advantages. Earlier Nowacki [4] has determined steady-state thermal stresses in a circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and circular edge. Roy Choudhari [5] has succeeded in determining the quasi-static thermal stresses in a circular plate subjected to transient temperature along the circumference of upper face with lower face at zero temperature and the fixed circular edge thermally insulated. Wankhede [7] has determined the quasi-static thermal stresses in circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature. Recently Noda et. al. [1] have considered a circular plate and discussed the transient thermo elastic-plastic bending problem, making use of the strain increment theorem. Khobragade et. al. [8] have studied the problem of partially distributed the heat supply of a thin circular disc applying finite Hankel and finite Fourier transform with Dirichlet type boundary conditions. In a recent work Varghese et. al. [2] has studied the inverse thermo elastic problem of a thin annular plate with third kind boundary conditions using Integral transform techniques. In the present paper an attempt is made to study the theoretical solution of a thermo elastic problem to determine the temperature distribution, stress functions and small deflection of a thin circular plate with boundary conditions of radiation type and subjected to known partially distributed annular heat supply occupying the space

$$D=\{(x,y,z)\in R^3: \quad 0\le (x^2+y^2)^{1/2}\le a, \; -h/2\le z\le h/2\}.$$

II. FORMULATION OF THE ROBLEM: GOVERNING EQUATIONS

Consider a circular plate of radius a and thickness h subjected to partially distributed heating of a concentric annular region of the upper face. The plate is kept at zero initial temperature. The annular region $D_1 : R_1 < r < R_2$ of the upper face is subjected to temperature distribution as follows:

$$T(r,z,t)\big|_{z=h/2}=\frac{Q_0}{\lambda}[H(r-R_1)-H(r-R_2)]\,g(t) \qquad (1)$$

where

$$H(\hat{x})=\begin{cases}1, & \hat{x}\ge 0,\\ 0, & \hat{x}<0;\end{cases}$$

The differential equation of the displacement function $\phi(r,z,t)$, where $r=(x^2+y^2)^{1/2}$, for the heating is given as [1]

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}=(1+\nu)a_t\,T \qquad (2)$$

with

$$\phi=0 \text{ at } r=a \text{ for all time } t \qquad (3)$$

where ϕ is the displacement component, ν and a_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the circular plate respectively and $T(r,z,t)$ is the heating temperature of the plate at time t satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\kappa}\frac{\partial T}{\partial t} \qquad (4)$$

where $\kappa = K/\rho c$ is the thermal diffusivity of the material of the plate, K is the conductivity of the medium, c is its specific heat and ρ is its calorific capacity, subject to the boundary conditions having medium $0\le r\le a$ and $-h/2\le z\le h/2$ as

$$M_r(T,\bar{k}_1,\bar{k}_2,a)=F_1(z,t) \qquad (5)$$

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$$M_z(T, 0, 1, -h/2) = F_2(z,t), \quad (6)$$

$$M_z(T, 0, 1, h/2) = (Q_0/\lambda) f(r) g(t) \quad (7)$$

being:

$$M_\vartheta(\chi, \bar{k}, \bar{\bar{k}}, \$) = (\bar{k}\chi + \bar{\bar{k}}\chi')_{\vartheta=\$}$$

and

$$f(r) = \begin{cases} 1, & R_1 < r < R_2, \\ 0, & 0 < r < R_1,\ R_2 > r > a; \end{cases} \quad (8)$$

where $f(r) = H(r - R_1) - H(r - R_2)$, the prime (´) denotes differentiation with respect to ϑ, radiation constants are $\bar{k}_1$ and $\bar{k}_2$ on the curved surfaces of the plate respectively. The functions $F_1(z,t)$ and $F_2(z,t)$ are known constants and they are set to be zero here as in other literatures, so as to obtain considerable mathematical simplicities.

The stress distribution components σ_{rr} and $\sigma_{\theta\theta}$ of the plate for the heating and the cooling processes are given by [4]

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial \phi}{\partial r} \quad (9)$$

$$\sigma_{\theta\theta} = -2\mu \frac{\partial^2 \phi}{\partial r^2} \quad (10)$$

where μ is the Lame's constants, while each of the stress functions σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the circular plate in the plane state of stress.

The deflection $\omega(r,t)$ of the circular plate satisfying the differential equation [2] as

$$D\nabla_1^4 \omega = -\frac{\nabla_1^2 M_T}{(1-\upsilon)} \quad (11)$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \quad (12)$$

and the resultant thermal momentum as

$$M_T(r,t) = a_t\, E \int_{-h/2}^{h/2} z\, T(r, z, t)\, dz \quad (13)$$

Further assuming that the edge of the circular plate is fixed and clamped

$$\omega = \frac{\partial \omega}{\partial r} = 0 \text{ at } r = a \quad (14)$$

Thus, the equations (4) to (14) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE HEAT CONDUCTION EQUATION

DETERMINATION OF THE TEMPERATURE DISDTRIBUTION $T(r, z, t)$

To obtain the expression for temperature $T(r, z, t)$, assume

$$T(r,z,t) = \sum_{n=1}^{\infty} A_n\, J_0(\alpha_n r) \sinh\{\gamma_n [z + (h/2)]\}\, g(t) \quad (15)$$

where $\alpha_1, \alpha_2, \ldots$ are the positive roots of the equation $\alpha \bar{k}_1 J_0'(\alpha a) + \bar{k}_2 J_0(\alpha a) = 0$, $J_n(x)$ is the Bessel function of the first kind of order n and

$$J_n'(\alpha x) = \frac{d}{d(\alpha x)} J_n(\alpha x)$$

Substituting equation (15) in differential equation (4), one obtains

$$\gamma_n{}^2 = \alpha_n{}^2 + \frac{1}{\kappa}\frac{g'(t)}{g(t)}, \quad n = 1, 2 \quad (16)$$

Also assume that

$$f(r) = \sum_{n=1}^{\infty} B_n\, J_0(\alpha_n r) \quad (17)$$

Multiplying equation (17) by $r\, J_0(\alpha_n r)$ on both sides and integrating from 0 to a, one obtains from the theory of Bessel function

$$B_n \int_0^a r\,[J_0(\alpha_n r)]^2\, dr = \int_0^a r\, f(r)\, J_0(\alpha_n r)\, dr$$

$$= \int_{R1}^{R2} r\, J_0(\alpha_n r)\, dr$$

and using equation (7), we get

$$B_n = \frac{2\,[R_2 J_1(\alpha_n R_2) - R_1 J_1(\alpha_n R_1)]}{\alpha_n\, a^2 [(J_0(\alpha_n a))^2 + (J_1(\alpha_n a))^2]} \quad (18)$$

Substituting equation (15) into the equation of equation (7) and using equations (5), one obtain

$$A_n = \frac{2 Q_0\, \bar{k}_1\, [R_2 J_1(\alpha_n R_2) - R_1 J_1(\alpha_n R_1)]}{a^2\, \lambda\, (J_0(\alpha_n a))^2 [\bar{k}_1 \alpha_n + \bar{k}_2]\, \sinh\{\gamma_n h\}} \quad (19)$$

Substituting equation (18) into equation (14), an axially symmetric temperature field is obtained as

$$T(r,z,t) = \frac{2 Q_0\, \bar{k}_1}{a^2\, \lambda} \sum_{n=1}^{\infty} J_0(\alpha_n r)\, \Psi_n\, g(t) \quad (20)$$

where

$$\Psi_n = \frac{[R_2 J_1(\alpha_n R_2) - R_1 J_1(\alpha_n R_1)]}{(J_0(\alpha_n a))^2 [\bar{k}_1 \alpha_n + \bar{k}_2]} \times \frac{\sinh\{\gamma_n [z + (h/2)]\}}{\sinh\{\gamma_n h\}}$$

IV. DETERMINATION OF DISPLACEMENT FUNCTION

For obtaining the displacement function $\phi(r, z, t)$, we substitute temperature distribution equation (20) into equation (2) and using the result

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) J_0(\alpha_n r) = -\alpha_n^2\, J_0(\alpha_n r) \quad (21)$$

one obtains

$$\phi(r,z,t) = -\frac{2 Q_0 (1+v) a_t \bar{k}_1}{a^2 \lambda} \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r) \Psi_n}{\alpha_n{}^2}\, g(t) \quad (22)$$

V. DETERMINATION OF STRESS FUNCTION

Substituting the equation (22) in equation (9) and (10), we get the required expression for thermal stresses as [4]

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$$\sigma_{rr} = -\frac{4\mu Q_0 (1+v) a_t \bar{k}_1}{a^2 \lambda} \frac{1}{r} \sum_{n=1}^{\infty} \frac{J_1(\alpha_n r) \Psi_n}{\alpha_n} g(t) \qquad (23)$$

$$\sigma_{\theta\theta} = -\frac{2\mu Q_0 (1+v) a_t \bar{k}_1}{a^2 \lambda}$$

$$\times \sum_{n=1}^{\infty} [J_0(\alpha_n r) - J_2(\alpha_n r)] \Psi_n g(t) \qquad (24)$$

VI. DETERMINATION OF QUASI-STATIC SMALL DEFLECTION

We assume that

$$\varpi(r,t) = \sum_{n=1}^{\infty} C_n(t) [J_0(\alpha_n r) - J_0(\alpha_n a)] \qquad (25)$$

Substituting equation (20) in equation (13) and using equation (21), one obtains

$$M_T(r,t) = \frac{Q_0 a_t E \bar{k}_1}{a^2 \lambda} \sum_{n=1}^{\infty} J_0(\alpha_n r) \Omega_n g(t) \qquad (26)$$

From equation (11), one obtains the $C_n(t)$ by substituting the equations (25) and (26) and using equation (21) as

$$C_n(t) = \frac{Q_0 a_t E \bar{k}_1 \alpha_n^2 J_0(\alpha_n r) \Omega_n g(t)}{D a^2 \lambda (1-\upsilon) [\alpha_n^4 J_0(\alpha_n r) - J_0(\alpha_n a)]} \qquad (27)$$

Finally in order to obtain the required thermal deflection $\varpi(r,t)$, substitute the $C_n(t)$ from equation (27) into equation (25) we get

$$\varpi(r,t) = \frac{Q_0 a_t E \bar{k}_1}{D a^2 \lambda (1-\upsilon)}$$

$$\times \sum_{n=1}^{\infty} \frac{\alpha_n^2 J_0(\alpha_n r) \Omega_n [J_0(\alpha_n r) - J_0(\alpha_n a)]}{[\alpha_n^4 J_0(\alpha_n r) - J_0(\alpha_n a)]} g(t) \qquad (28)$$

where

$$\Omega_n = \frac{[R_2 J_0(\alpha_n R_2) - R_1 J_0(\alpha_n R_1)]}{(J_0(\alpha_n a))^2 \gamma_n^2 [\bar{k}_1 \alpha_n + \bar{k}_2]}$$

$$\times \frac{[h\gamma_n (1+\cosh\{\gamma_n h\}) - 2\sinh\{\gamma_n h\}]}{\sinh\{\gamma_n h\}}$$

VII. SPECIAL CASE

Set

$$g(t) = 1 - e^{-\omega t} \qquad (29)$$

and

$$A = \frac{2 Q_0 \bar{k}_1}{a^2 \lambda}, \quad B = -\frac{2 Q_0 (1+v) a_t \bar{k}_1}{a^2 \lambda},$$

$$C = \frac{Q_0 a_t E \bar{k}_1}{D a^2 \lambda (1-\upsilon)} \qquad (30)$$

Substituting the value of (29) in the equations (20), to (24) and (28) one obtains

$$\frac{T(r,z,t)}{A} = \sum_{n=1}^{\infty} J_0(\alpha_n r) \Psi_n (1 - e^{-\omega t}) \qquad (31)$$

$$\frac{\phi(r,z,t)}{B} = \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r) \Psi_n}{\alpha_n^2} (1 - e^{-\omega t}) \qquad (32)$$

$$\frac{\sigma_{rr}}{B} = 2\mu \frac{1}{r} \sum_{n=1}^{\infty} \frac{J_1(\alpha_n r) \Psi_n}{\alpha_n} (1 - e^{-\omega t}) \qquad (33)$$

$$\frac{\sigma_{\theta\theta}}{B} = \mu \sum_{n=1}^{\infty} [J_0(\alpha_n r) - J_2(\alpha_n r)] \Psi_n (1 - e^{-\omega t}) \qquad (34)$$

$$\frac{\varpi(r,t)}{C} = \sum_{n=1}^{\infty} \frac{\alpha_n^2 J_0(\alpha_n r) \Omega_n [J_0(\alpha_n r) - J_0(\alpha_n a)]}{[\alpha_n^4 J_0(\alpha_n r) - J_0(\alpha_n a)]} (1 - e^{-\omega t}) \qquad (35)$$

In particular case we also set

$$g(t) = \sin(\omega t) \qquad (36)$$

for investigating the transient response on a circular plate subjected to the radiation flux varying as a sinusoidal time function and illustrated graphically.

VIII. NUMERICAL RESULTS, DISCUSSION AND REMARKS

To interpret the numerical computation we consider material properties of low carbon steel, which can be used for medium duty shafts, studs, pins, distributor cams, cam shafts, and universal joints having mechanical and thermal properties $\kappa = 13.97$ $\upsilon = 0.29$, $\lambda = 51.9$ and $a_t = 14.7$. With the general convention that the thickness of the thin circular plate is taken $\le$ (diameter / 40) as $h = 0.1$, with radius $a = 1$ and radiation constant $\bar{k}_1 = \bar{k}_2 = 0.86$. $\alpha_n = 1.2558$, 4.0795, 7.1557, 10.2710, 13.3983, 16.5312, 19.6668, 22.8040, 25.9423, 29.0812, 32.2207, 35.3605, 38.5007, 41.6411, 44.7816, 47.9223, 51.0631, 54.2040, 57.3449, 60.4859 are the positive roots of the transcendental equation $\alpha \bar{k}_1 J_0'(\alpha) + \bar{k}_2 J_0(\alpha) = 0$ for $a = 1$. In the foregoing analysis will be illustrated by the numerical results shown in Figure 1 to 5. Figure 1 depicts the distributions of the temperature increment $T(r,z,t)$ verse radius at different values of time with $z = 0.1$. It shows that heat gain follows the sinusoidal nature crossing the inner core with increase of radius up to the outer region of radiation flux. But it was observed that while approaching towards the inner boundary region of radiation flux temperature increment increases slowly in the first quadrant having sinusoidal nature and then decrease slowly following the initial nature. The physical meaning emphasis for this phenomenon is that there is reduction in the rate of heat propagation before radius $r = 0.5$ up to $r = 0.75$ leading to compressive radial stress at inner part and expand more on outer due to partially distributed annular heat supply. The difference in both the results i.e. with $e^{-\omega t}$ and $\sin(\omega t)$ lies with the peaks that can be clearly observed in figure 1. Figure 2 depicts the displacement function and it is noteworthy that it is in agreement with the boundary condition (3) and attains maximum at the center and zero at the outer

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edge. Figure 3 and 4 shows the distributions of the radial and axial thermal stresses at different value of time. The radial stresses are smaller than axial stresses from figures 5-6. Figure 7 shows the variation of displacement function with same aforementioned parameter versus time with $z = 0.1$.

IX. CONCLUSION

In this problem, we modify the conceptual idea proposed by Roy Choudhari [5] for circular plate and investigated the results for temperature distributions, displacement, stress function and deflection subjected to annular heat supply. The thermo elastic behavior is examined with the help of arbitrary heat supply on the upper surface. The series solution converges provided we take sufficient number of terms in the series. Any particular case can be derived by assigning suitable values to the parameters and functions in the series expressions. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications.

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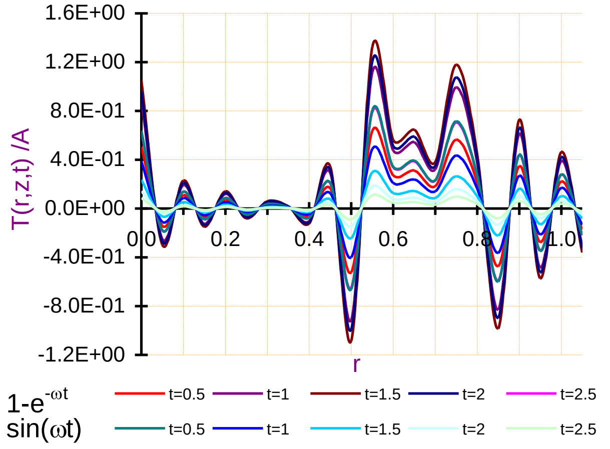

Fig (1): Distribution of the temperature versus radius for z = 0.1 and different values of time

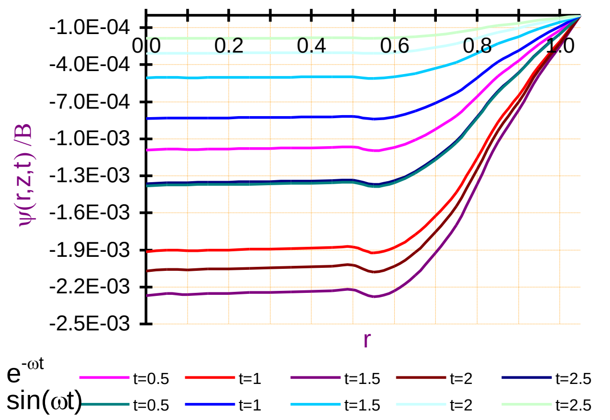

Fig (2).Distribution of the displacement function versus radius for z = 0.1 and different values of time

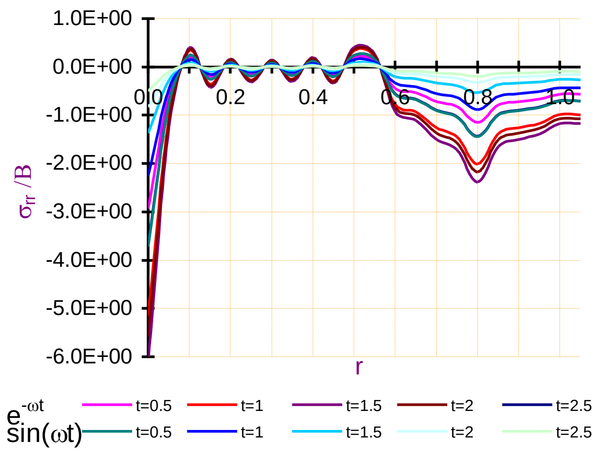

Fig (3).Distribution of the radial stress versus radius for z = 0.1 and different values of time

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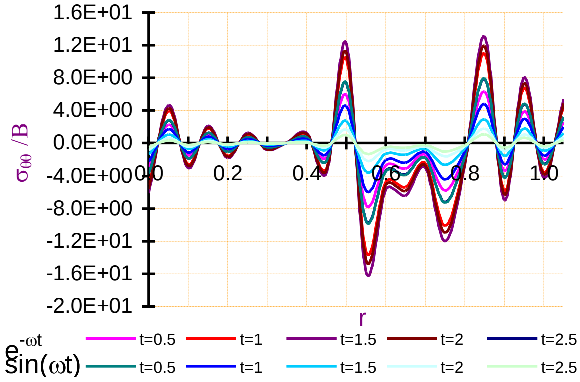

Fig (4).Distribution of the axial stress versus radius for z = 0.1 and different values of time

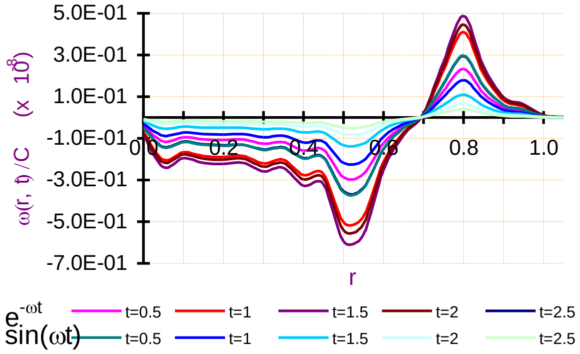

Fig (5).Deflection subjected to annular heat supply versus radius for z = 0.1 and different values of time

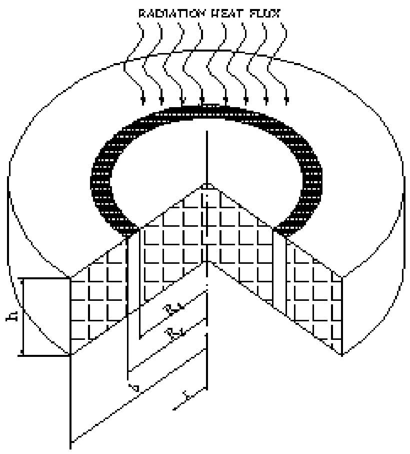

Fig (6).Configuration of thin circular plate subjected to annular radiation flux

AUTHOR BIOGRAPHY

Dr. N.W. Khobragade For being M.Sc in statistics and Maths he attained Ph.D. He has been teaching since 1986 for 27 years at PGTD of Maths, RTM Nagpur University, Nagpur and successfully handled different capacities. At present he is working as Professor. Achieved excellent experiences in Research for 15 years in the area of Boundary value problems and its application. Published more than180 research papers in reputed journals. Fourteen students awarded Ph.D Degree and four students submitted their thesis in University for award of Ph.D Degree under their guidance.