ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

Solution of Game Theory Problems by KKL Method

N. W. Khobragade, N. K. Lamba and P. G. Khot*
Department of Mathematics, MJP Educational Campus,
* Department of Statistics, MJP Educational Campus
RTM Nagpur University, Nagpur 440 033, India.

Abstract -***In this paper, a new approach to the solution of Game theory problem is suggested, which is based on the iterative procedure so called*** *KKL method*. ***(Where KKL is formed from the first letter of author's surname). The proposed KKL method is computationally more effective and easy as compared to the traditional simplex method.***

Key words and Phrases: KKL Method, Optimum Solution, Game theory.

I. INTRODUCTION

Game theory attempts to study decision-making in the situations where two or more intelligent and the rational opponents are involved under conditions of conflict and cooperation. The approach of the game theory is to seek to determine a rival's most profitable counter-strategy to one's own 'best' moves and to formulate the appropriate defensive measures.

Game theory is the formal study of conflict and cooperation. Game theoretic concepts apply whenever the actions of several agents are interdependent. These agents may be individuals, groups, firms, or any combination of these. The concepts of game theory provide a language to formulate structure, analyze, and understand strategic scenarios.

In practical life, it is required to take decision in a competing situation when there are two or more opposite parties with conflicting interests and the outcome is controlled by the decision of the all parties concerned. Such problems occur frequently in the economics, Business, Administration Sociology, Political Science and Military training. It is in this context that the game theory was developed in the twentieth century. However the mathematician treatment of the Game Theory was made available only in 1944, when John-Von-Newmann and the Oscar Morgenstern published their article 'Theory of the Game and Economics behavior' The Von-Newmann's approach to solve the Game theory problems was based on the maximum losses. Most of the problems can be handled by this principle. In the present paper, an attempt has been made to solve the game theory problems by the KKL method.

II. ALGORITHM OF KKL METHOD

Step 1. For the $m \times n$ rectangular game when either m or n or both are greater than equal to three , KKL linear programming approach is as follows:-

Let the two person zero sum game be defined as follows:

Player A has m course of action $(A_1, A_2, \dots\dots\dots, A_m)$ and player B has n course of the action $(B_1, B_2, \dots\dots\dots, B_n)$. The pay-off to the player A if he selects strategy A_i and player B select B_j is a_{ij} . Mixed strategy for player A is defined by the probabilities $p_1, \dots\dots\dots . p_m$,

where $p_1 + \dots\dots\dots + p_m = 1$ and

mixed strategy for player B is defined by $q_1, \dots\dots\dots . q_n$

where $q_1 + \dots\dots\dots + q_n = 1$.

Let the game can be defined as a linear programming problem as given below:

Player A

Minimize $Z = \frac{1}{v}$ or $y_1 + y_2 + \dots\dots\dots . y_n$

Subject to the constraints:

$$a_{11}y_1 + a_{12}y_2 \dots\dots\dots \dots\dots + a_{1n}y_n \ge 1$$
$$a_{21}y_1 + a_{22}y_2 \dots\dots\dots \dots\dots + a_{2n}y_n \ge 1$$
$$\dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots$$
$$\dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots$$
$$a_{m1}y_1 + a_{m2}y_2 \dots\dots\dots \dots\dots + a_{mn}y_m \ge 1$$

Player B

Maximize $Z = \frac{1}{v}$ or $x_1 + x_2 + \dots\dots\dots . x_n$

Subject to the constraints:

$$a_{11}x_1 + a_{12}x_2 \dots\dots\dots \dots\dots + a_{1n}x_n \le 1$$
$$a_{21}x_1 + a_{22}x_2 \dots\dots\dots \dots\dots + a_{2n}x_n \le 1$$
$$\dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots$$
$$\dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots\dots\dots \dots$$
$$a_{m1}x_1 + a_{m2}x_2 \dots\dots\dots \dots\dots + a_{mn}x_m \le 1$$

The steps for the computation of the optimal solution are as follows:

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

Step 2**:** Formulate the linear programming model of the real world problem that is obtained a mathematical representation of the problems objective function and constraints as stated below.

Maximize $M = c_1x_1 + c_2x_2 + \dots\dots\dots + c_nx_n$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n \le b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n \le b_2$$

....

....

$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n \le b_m$$

$$x_1, x_2, x_3, \dots\dots\dots .., x_n \ge 0$$

If the objective function is minimized, then convert it into a problem of maximizing by using the rule

Minimum M= -(Maximum (-M))

all $bi's$, $i = 1,2,\dots\dots, m$ must be non negative. If any one of bi is negative, multiply corresponding inequality by (-1), So as to get all $bi's$, $i = 1,2,\dots\dots, m$ non-negative.

Step 3: Convert all in equations of the constraints into the equations by introducing slack variables in the left hand side of constraints and assign a zero coefficient to the corresponding variable in the objective function. Thus we can reformulate the problem in terms of equation as follows: -

Maximize $M = c_1x_1 + c_2x_2 + \dots\dots\dots + c_nx_n$

$$+ 0p_1 + 0p_2 + \dots\dots + 0p_m$$

Subject to constraints:

$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n + p_1 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n + p_2 = b_2$$

....

....

$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n + p_m = b_m$$

where $x_1, x_2, x_3, \dots\dots\dots .., x_n \ge 0$

and $p_1, p_2, p_3, \dots\dots\dots .., p_m \ge 0$

Step 4**:** An initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = \dots\dots\dots . = x_n = 0$. Thus we get $p_1 = b_1$, $p_2 = b_2$, …………, $p_m = b_m$.

Step 5: For computational, efficiency and simplicity, the initial basic feasible solution, the constraint of the standard linear programming problem as well as the function can be displayed in a tabular form, called the new simplex tableau as solve below:-

NEW KKL TABLEAU:

Row(R)	Basic	z	y_1	…	y_n	p_1	...	p_m	R.H.S
Initial (R^*)	z	1	C_1	…	c_n	0	…	0	0
First (R_1)	p_1	0	a_{11}	…	a_{1n}	1	…	0	b_1
Second (R_2)	p_2	0	a_{21}	…	a_{2n}	0	…	0	b_2
…	…	…	…	…	…	…	…	…	…
m^{th} Row (R_m)		0	a_{m1}	…	a_{mn}	0	…	1	bm

The interpretation of data of the above tableau is given as under: -

(i). The Initial row (R^*), called the objective row of the new simplex table indicates the Coefficients of variables (m+n) in the objective function.

(ii).From 1^{st} to $m^{th\ row}$ is called constraint row of the table represents the coefficient of the constraints.

(iii). The first column labeled (R) denotes the m rows also known as objective column, the second column labeled 'Basic variable' points out the basic variables, and in the initial KKL tableau these variable are the slack variables, the third column indicates the coefficient of 'z' whose value for the first row is 1 and for all other rows it is zero.

(iv). The body matrix (under non-basic variables) in the initial KKL tableau consists of the decision variable in the constraint set.

(v). The Identity matrix in the initial KKL tableau represents the coefficients of the slack variables that have been added to the original inequality to make them equations.

Step 6**:** Test the Solution for optimality: Examine the initial row (R^*) of the above KKL tableau.

(i). If all the elements in the initial row (R^*) are positive then the current solution is optimal.

(ii). If there exists some negative number, the current solution can be improved by entering the column, which contain less negative term, let it be y_r where p_2. Then (a). Select the maximum positive term in the corresponding column of y_r, let it be a_{rj}, $1 \le j \le n$, which is our new pivot element and find out the corresponding identity element, let it be lies in the column p_k where $1 \le k \le m$, next drop this column p_k.

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

(b). Repeat the above process till all the coefficient of the Initial Row (R^*) becomes positive, if any one coefficient of the row R_1 is negative then repeat the above process to find the optimum solution.

III. STATEMENT OF THE PROBLEM

Problem 1. Two companies A and B are competing for the same product. Their different strategies are given in the following pay-off matrix:

$$\text{Company A} \overset{\text{Company B}}{\begin{bmatrix} 3 & -4 & 2 \\ 1 & -3 & -7 \\ -2 & 4 & 7 \end{bmatrix}}$$

Use KKL method to determine the best strategies for both players.

Solution. The given game does not posses the saddle point, and its value lies from –2 and +3, Thus adding a constant k=3 to all the elements of pay-off matrix to matrix, then the pay-off matrix become:

$$\text{Company A} \overset{\text{Company B}}{\begin{bmatrix} 6 & -1 & 5 \\ 4 & 0 & -4 \\ 1 & 7 & 10 \end{bmatrix}}$$

Let the strategies of the two players be

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix},$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$$

Where $p_1 + \dots\dots\dots + p_m = 1$ and $q_1 + \dots\dots\dots + q_n = 1$

Maximize $Z = \frac{1}{v}$ or $y_1 + y_2 + y_3$

Subject to the constraints:

$$6y_1 - y_2 + 5y_3 \le 1$$

$$4y_1 - 4y_3 \le 1$$

$$y_1 + 7y_2 + 10y_3 \le 1 \quad \text{and}$$

$$y_j \ge 0, \ (j = 1,2,3)$$

Minimize $Z = \frac{1}{v}$ or $x_1 + x_2 + x_3$

Subject to the constraints:

$$6x_1 + 4x_2 + x_3 \ge 1$$

$$-x_1 + 7x_3 \ge 1$$

$$5x_1 - 4x_2 + 10x_3 \ge 1 \quad \text{and} \quad x_j \ge 0, \ (j = 1,2,3)$$

By introducing slack variables $p_1 \ge 0$, $p_2 \ge 0$ and $p_3 \ge 0$ respectively, the set of constraints of the LPP are converted into the system of equations; the iterative 'New KKL tableau" are:

Initial table:

Row	Basic	z	y1	y2	y3	p1	p2	p3	RHS
R_1	z	1	-1	-1	-1	0	0	0	0
R_2	p_1	0	6	-1	5	1	0	0	1
R_3	p_2	0	4	0	-4	0	1	0	1
R_4	p_3	0	1	7	10	0	0	1	1

Where p_1, p_2 and p_3 denotes the slack variables and R_1, R_2, R_3, R_4 represents the first, second, third and fourth row respectively. From the above table it clear that the least negative coefficient of z lies in column y_1, y_2 and y_3 which is (–1) (either we can choose any one), let consider y_2 (since the no. of iteration to obtain the solution becomes less), which will enters in the basis, pivot element is 7(arbitrary in column y_2) and the corresponding identity element lies in the p_3 column, hence we will drop column vector p_3.

First Iteration**:** Introduce y_2 and drops p_3

Row	Basic	z	y1	y2	y3	p1	p2	p3	RHS
R_1	z	1	-6/7	0	3/7	0	0	**1/7**	**1/7**
R_2	p_1	0	43/7	0	45/7	1	0	**1/7**	**8/7**
R_3	p_2	0	4	0	-4	0	1	**0**	**1**
R_4	y_2	0	1/7	1	10/7	0	0	**1/7**	**1/7**

Since y_1 in the initial row (R_1) is only one negative i.e. it enters the basis and corresponding pivot element is **(43/7)**, next drops the corresponding identity element which lies in column p_1.

Second Iteration**:** Introduce y_1 and drops p_1

Row	Basic	z	y1	y2	y3	p1	p2	p3	RHS
R_1	z	**1**	**0**	**0**	57/43	6/43	0	7/43	91/301
R_2	y_1	**0**	**1**	**0**	45/43	7/43	0	1/43	8/43
R_3	p_2	**0**	**0**	**0**	-352/43	-28/43	1	-4/43	11/43
R_4	y_2	**0**	**0**	**1**	55/43	-1/43	0	6/43	5/43

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

Since all the elements in the initial row are positive then the current solution is optimal, and its value is $x_1 = 75/4$, $x_2 = 50/4$ and

$$\frac{1}{v} = x_1 + x_2 + x_3 = \frac{8}{43} + 0 + \frac{5}{43}$$

Therefore the value of the game for the modified matrix,

$$v = \frac{43}{13},$$

Since $\frac{q_i}{v} = y_j$, $q_i = v\,y_j$

$$q_1 = v\,y_1 = \frac{8}{43} \times \frac{43}{13} = \frac{8}{13},$$

$$q_2 = v\,y_2 = \frac{5}{43} \times \frac{43}{13} = \frac{5}{13} \text{ and } q_3 = v\,y_3 = 0$$

Company $A's$ best strategies appear in the initial row R_1, under p_1, p_2 and p_3 respectively with positive signs.

Therefore $x_1 = \frac{6}{43}$, $x_2 = 0$ and $x_3 = \frac{7}{43}$

Thus

$$p_1 = x_1 v = \frac{6}{43} \times \frac{43}{13} = \frac{6}{13},\ p_2 = x_2 v = 0 \text{ and}$$

$$p_3 = x_3 v = \frac{7}{43} \times \frac{43}{13} = \frac{7}{13}$$

Hence optimal strategies for company A: - are (6/13, 0, 7/13),
for company B:- are (8/13 , 5/13 ,0) and the value of game is :- 43/13-3 = 4/13.

Problem 2**.** Solve the following game by Using KKL technique:

Player B

$$\text{Player A} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix}$$

Solution. Add a suitable constant to make all the entries of the above payoff matrix to ensure them all positive. Thus, adding the constant k=4 to each element, we get the payoff matrix:

Player B

$$\text{Player A} \begin{bmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{bmatrix}$$

Let the strategies of the two players be

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix},$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$$

where $p_1 + \dots\dots + p_m = 1$ and $q_1 + \dots\dots + q_n = 1$

Maximize $Z = \frac{1}{v}$ or $x_1 + x_2 + x_3$

Subject to the constraints:

$5x_1 + 7x_2 + 10x_3 \geq 1$

$3x_1 + 9x_2 + 6x_3 \geq 1$

$7x_1 + x_2 + 2x_3 \geq 1$ and

$x_j \geq 0$, $(j = 1,2,3)$

Minimize $Z = \frac{1}{v}$ or $y_1 + y_2 + y_3$

Subject to the constraints:

$5y_1 + 3y_2 + 7y_3 \leq 1$

$7y_1 + 9y_2 + y_3 \leq 1$

$10y_1 + 6y_2 + 2y_3 \leq 1$

and $y_j \geq 0$, $(j = 1,2,3)$

where $x_j = \frac{p_j}{u}$ and $\frac{y_j}{v}$, $(j = 1,2,3)$; u is minimum expected gain to A and v is the minimum expected loss to B.

By introducing slack variables $p_1 \geq 0$, $p_2 \geq 0$ and $p_3 \geq 0$ respectively, the set of constraints of the LPP are converted into the system of equations; the iterative KKL 'New simplex tableau" are:

Initial table**:**

Row	Basic	z	y_1	y_2	y_3	p_1	p_2	p_3	RHS
R^*	z	1	-1	-1	-1	0	0	0	0
R_1	p_1	0	5	3	7	1	0	0	1
R_2	p_2	0	7	9	1	0	1	0	1
R_3	p_3	0	10	6	2	0	0	1	1

Where p_1, p_2 and p_3 denotes the slack variables and R^*, R_1, R_2, R_3 represents the initial, first, second and third row respectively. From the above table it clear that the least negative coefficient of z lies in column y_1, y_2 and y_3 which is (–1) (either we can choose any one), let consider y_3 (since the no. of iteration require to obtain the solution becomes less), which will enters in the basis, pivot element is 7(arbitrary in column y_3) and the corresponding identity element lies in the p_1 column, hence we will drop column vector p_1 .

First Iteration**:** Introduce y_3 and drops p_1

Row	Basic	z	y_1	y_2	y_3	p_1	p_2	p_3	RHS
R^*	z	**1**	**-2/7**	**-4/7**	**0**	**1/7**	**0**	**0**	**1/7**
R_1	y_3	**0**	**5/7**	**3/7**	**1**	**1/7**	**0**	**0**	**1/7**
R_2	p_2	**0**	**44/7**	**60/7**	**0**	**-1/7**	**1**	**0**	**6/7**
R_3	p_3	**0**	**60/7**	**36/7**	**0**	**-2/7**	**0**	**1**	**5/7**

Since y_1 in the initial row (R^*) is next least negative i.e. it enters the basis and corresponding pivot element is **60/7**, next drops the corresponding identity element which lies in column p_3.

Second Iteration**:** Introduce y_1 and drops p_3, we get

Row	Basic	z	y_1	y_2	y_3	p_1	p_2	p_3	RHS
R^*	z	**1**	**0**	**-2/5**	**0**	**2/15**	**0**	**1/30**	**1/6**
R_1	y_3	**0**	**0**	**0**	**1**	**1/6**	**0**	**-1/12**	**1/12**
R_2	p_2	**0**	**0**	**24/5**	**0**	**1/15**	**1**	**-11/15**	**1/3**
R_3	y_1	**0**	**1**	**3/5**	**0**	**-1/30**	**0**	**7/60**	**1/12**

Since y_2 in the initial row (R_1) is next least negative i.e. it enters the basis and corresponding pivot element is **3/5**, next drops the corresponding identity element which lies in column y_1 .

Third Iteration**:** Introduce y_2 and drops y_1 .

Row	Basic	z	y_1	y_2	y_3	p_1	p_2	p_3	RHS
R^*	z	**1**	**2/3**	**0**	**0**	**1/9**	**0**	**I/9**	**2/9**
R_1	y_3	**0**	**0**	**0**	**1**	**1/6**	**0**	**-1/12**	**1/12**
R_2	p_2	**0**	**-8**	**0**	**0**	**1/3**	**1**	**-5/3**	**-1/3**
R_3	y_2	**0**	**5/3**	**1**	**0**	**-1/18**	**0**	**7/36**	**5/36**

Since all the elements in the initial row (R^*) are positive then the current solution is optimal,

and its value is $x_2 = 5/36$, $x_3 = 1/12$ and

$$\frac{1}{v} = x_1 + x_2 + x_3 = 0 + 5/36 + 1/12 = 2/9$$

Therefore the value of the game for the modified matrix,

$v = \frac{9}{2}$,

Since $\frac{q_i}{v} = y_j$, $q_i = v\, y_j$

$q_1 = v\, y_1 = 0$, $q_2 = v\, y_2 = \frac{5}{36} \times \frac{9}{2} = \frac{5}{8}$ and

$q_3 = v\, y_3 = \frac{1}{12} \times \frac{9}{2} = \frac{3}{8}$

Company $A's$ best strategies appear in the initial row R^*, under p_1 , p_2 and p_3 respectively with positive signs.

Therefore $x_1 = \frac{1}{9}$, $x_2 = 0$ and $x_3 = \frac{1}{9}$

Thus $p_1 = x_1\, v = \frac{1}{9} \times \frac{9}{2} = \frac{1}{2}$, $p_2 = x_2\, v = 0$ and

$p_3 = x_3\, v = \frac{1}{9} \times \frac{9}{2} = \frac{1}{2}$

Hence optimal strategies for company A are (0, 5/8, 3/8),

for company B are (1/2 , 0 ,1/2) and the value of game is (9/2-4)= 1/2.

IV. CONCLUSION

We observed that the solution of Game Theory problem has been obtained by KKL technique very easily and requires less or at the most equal number of iterations than traditional simplex method. This technique is very useful to apply on numerical problems, reduces the labour work, gives more accuracy and improved optimum solution. Therefore this KKL method is more powerful in solving Game Theory problems as compare to traditional simplex method.

REFERENCES

[1] Brown G: Iterative Solution of Games by Fictitious Play," in Activity Analysis of Production and Allocation, John Wiley & Sons, New York, 1951.

[2] B¨orgers' (1994): `Weak Dominance and Approximate Common Knowledge,' Journal of Economic Theory 64(1) 265- 277)

[3] Fudenberg D and Levine D: The Theory of Learning in Games, Boston: MIT Press, 1998.

[4] O'Neill B: Non metric Test of the Minimax Theory of Two-person Zero-sum Games," Proceedings of the National Academy of Sciences, 84, (1987), 2106-2109.

[5] Rasmussen, E.: Games and information: an introduction to game theory.

[6] Skyrms, B. (1996): Evolution of the Social Contract. Cambridge University Press: Cambridge.

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

[7] Stinchcombe, M. B. (2001): General Normal Form Games. Working paper, Department of Economics, University of Texas at Austin.169

[8] Tang F.F: Anticipatory Learning in Two-person Games: Some Experimental Results," Journal of Economic Behavior and Organization, 44 (2001), 221-232.

[9] Von Neumann and Morgenstern: The Theory of Games and Economic Behavior, Princeton: Princeton University Press, 1944.

[10] Weibull J: Evolutionary Game Theory, Cambridge: MIT Press, 1995.

AUTHOR BIOGRAPHY

Dr. N.W. Khobragade for being M.Sc in statistics and Maths he attained Ph.D. He has been teaching since 1986 for 27 years at PGTD of Maths, RTM Nagpur University, Nagpur and successfully handled different capacities. At present he is working as Professor. Achieved excellent experiences in Research for 15 years in the area of Boundary value problems and its application. Published more than180 research papers in reputed journals. Fourteen students awarded Ph.D Degree and four students submitted their thesis in University for award of Ph.D Degree under their guidance.