ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

Complexity and Chaos in Volterra Like 3-Dimensional Food Chain System and Control of Chaos

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Abstract*—a discrete-time food chain model of three species, comprising of a set of three nonlinear difference equations, has been considered. Bifurcation diagrams have been drawn by varying a system parameter along coordinate axes and the motion has been analyzed. Time series plots and phase plane attractor are drawn showing chaos in the system. In order to control the chaotic motion, certain periodic changes in the system parameter λ have been proposed for each equation. The system becomes regular with such a change in the model. Numerical calculations have been carried out for each case to obtain plots for Lyapunov exponents and topological entropy and also, correlation dimension for chaotic system has been obtained. The investigation is further extended to obtain DLI for regular and chaotic evolution of food chain model.

Index Terms—**chaotic attractor, correlation dimension, Food chain model, Lyapunov exponents and topological entropy.**

I. INTRODUCTION

Chaotic phenomena are reality while studying nonlinear systems. Chaos is exhibited in nonlinear systems and can be viewed by observing bifurcations by varying a parameter of the system. As natural systems are mostly nonlinear, existence of chaos in nature is also quite natural. We say a system evolve chaotically if it shows divergence in behavior of two trajectories initiated at slightly different initial conditions, Such sensitivity to initial condition was first noticed by Poincaré [20], and later termed as chaos, Lorenz [17]. Lyapunov characteristic exponents, (LCE), are considered as very effective tools to distinguish regular and chaotic motions and provide a clear measure of chaos. If the divergence is exponential in time with a constant factor, say λ, in the exponent, then λ is a LCE of the system and if $\lambda > 0$, then the system becomes chaotic. The system is regular as long as $\lambda \leq 0$,[8,12,18,21,22]. Some recent articles suggest that models of food chain written in the form of difference equations are more appropriate than continuous equation when the interactions of species are non-overlapping generations,[9, 12, 13].

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II. DESCRIPTION OF LYAPUNOV, EXPONENTS ATTRACTOR, TOPOLOGICAL ENTROPY, CORRELATION

A. Lyapunov exponents (Lyapunovnumbers)

Lyapunov exponents are dynamical measure capable to characterize deterministic chaos in the system which features to the highly sensitive dependence on initial conditions. Actually it means the exponential divergence of orbits originated closely with very small difference in initial conditions. It is an important and effective element to identify regularity and chaos in the system and can be explained in the following ways: Chaos in a dynamical system is characterized by the exponential divergence of orbits originated closely. Such complexity of behavior in solution can be measured by a quantity called Lyapunov number. Lyapunov exponents are the measure of divergence of two orbits originated with slightly different initial conditions. Let us consider a one dimensional map defined in some interval (a, b),

$$x_{n+1} = f(x_n) \qquad (1)$$

And two of its orbits starting at x_0 and $x_0 \pm \delta_0$, where δ_0 is very small. Then, expanding $f(x_0 + \delta_0)$ by tailors series, we obtain after one iteration the distance between the orbits as

$$\delta_1 = \left|f'(x_0)\right| \Big/ \delta_0 = M_0\delta_0 \qquad (2)$$

M_0 is known as first step magnification factor. Similarly, at the second iteration, the distance between the orbits can be written as

$$\delta_2 = \left|f'(x_1)\right| \Big/ \delta_1 = M_1\delta_1 = M_1 M_0 \delta_0 \qquad (3)$$

Continuing in this manner, separation between the orbits at nth iteration is $\delta_n = |f'(x_{n-1})|/\delta_{n-1}$

$= M_{n-1}\delta_{n-1} = M_{n-1}M_{n-2} \dots M_0\delta_0 \qquad (4)$

The product $M_{n-1}M_{n-2} \dots M_0\delta_0$ is the accumulation of magnification factors, so it is meaningful to consider an average of it. The most convenient is the geometric average

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ISO 9001:2008 Certified
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$$(M_{n-1}M_{n-2}\dots M_0\,\delta_0)^{\frac{1}{n}}$$

Taking log, one obtains the arithmetic average

$$\lambda = ln\,(M_{n-1}M_{n-2}\dots M_0\,\delta_0)^{\frac{1}{n}} = \frac{1}{n}(\ln M_{n-1} + \ln M_{n-2} + \dots =$$

$$\frac{1}{n}[\ln(|f'(x_0)| + \ln(|f'(x_1)| + \ln(|f'(x_2)| + \cdots\cdots + \ln(|f'(x_{n-1})|\,] \quad (5)$$

Then, the condition of stability implies that, if average magnification is less than 1, the orbit is stable and if it is greater than 1 the orbit is unstable, i.e. $\lambda < 0$ implies stable orbit and $\lambda > 0$ implies unstable orbit. For more accurate result, one should take iterations nas large as possible. This leads to the following definition of Lyapunov exponents:

Def. 1: **Lyapunov exponents** of a smooth map f on R with x_0 as initial point be defined as

$$\boldsymbol{\lambda(x_0)} = \lim_{n\to\infty}\frac{1}{n}[\ln(|f'(x_0)| + \ln(|f'(x_1)| + \ln(|f'(x_2)| + \cdots\cdots + \ln(|f'(x_{n-1})|\,]$$

provided the limit exists. Lyapunov number is the exponent of Lyapunov exponent and is given by

$$L(x_0) = e^{\lambda(x_0)} \qquad (6)$$

Def. 2: A bounded orbit $\{x_0, x_1, x_2, x_4 \dots\dots x_n\}$ of the map f on R is called chaotic if following conditions are satisfied:

(a) $\{x_0, x_1, x_2, x_4 \dots\dots x_n\}$ is not asymptotically periodic.

(b) $\lambda(x_0)$ is exactly equal to zero , and

(c) $\lambda(x_0) > 0$ or equivalently, $L(x_0) > 1$.

From above definition, a clear interpretation for Lyapunov exponent is given as: it is the measure of loss of information during the process of iteration.

For higher dimensional system, we can generalize the above one dimensional case to higher dimension and obtain

$$\lambda(X_0, U_0) = \lim_{n\to\infty}\frac{1}{n}\log\left\|\prod_{t=0}^{n-1} J(X_t)U_0\right\| \qquad (7)$$

and

$$\|X_n - Y_n\| \approx e^{\lambda(X_0, U_0)n},$$

Where $X \in R^n, F: R^n \to R^n, U_0 = X_0 - Y_0$ and J is the Jacobian matrix of map F.

Quantitatively, two trajectories in phase space with initial separation δx_0 diverge (provided that the divergence can be treated within the linearized approximation)

$$|\delta x(t)| \approx e^{\lambda t}|\delta x(0)| \qquad (8)$$

where $\lambda > 0$ is the Lyapunov exponent.

The system described by the map f be *regular* as long as $\lambda \le 0$ and *chaotic* when $\lambda > 0$.

B. Attractor

Attractor in a complex dynamical system's phase space that shows sensitivity to initial conditions. Because of this property, once the system is on the attractor nearby states diverge from each other exponentially fast. Such types of attractors have fixed geometric structures, despite the fact that the trajectories moving within them appear unpredictable. Such type of attractors has non integer dimensions and they are also self similar objects and behave like fractals. In our discussion we got special line attractor for the following three dimensional food chain model given below.

C. Topological Entropy

The usefulness of Lyapunov exponents are limited because of the following important features:

- Lyapunov exponents are local in nature and are not necessarily constant throughout the evolution and so ergodicity is also required to characterize chaos.
- As per their definitions, Lyapunov exponents are time dependent and this leads to a serious drawback for systems arising from relativistic considerations.

A chaotic attractor is composed of a complex pattern. To investigate chaotic behavior in a wide variety of systems evolving time, an alternate replacement of Lyapunov exponents which could be more reliable and acceptable as indicator is the topological entropy [1, 2, 3, 4, 11].Topological entropy describes the *rate of mixing* of a dynamical system. It has a relationship to both Lyapunov exponents, through the dependence of rate, and to the ergodicity, because of the association of mixing. For a system having non-zero topological entropy, the rate of mixing must be exponential which reminiscent of of Lyapunov exponents. But such exponentially of mixing is not relative to time, but rather to the number of discrete steps through which the system has evolved. Positivity of Lyapunov exponent and topological entropy are characteristic of chaos. A mathematical definition of topological entropy can be obtained from the book by Nagashima and Baba [19] Topological entropy h(f) for a map f defined in a close interval **I** = [a, b], is closely related to Li and Yorke chaos, Nagashima and Baba [19] and measures the complexity of the map f. If f be a continuous map from **I** to **I** and if α be an open initial cover of **I**, then the topological entropy h(f) can be described by the supremum, sup h(α,f), for all the covers of interval **I** such that

$$h(\alpha, f) = \lim_{n\to\infty}\frac{1}{n}\log N\left(\bigvee_{i=0}^{n-1} f^{-1}\alpha\right) \qquad (9)$$

then

$$h(f) = \sup h(\alpha, f). \qquad (10)$$

When the map f is piecewise-monotonic over I, the topological entropy can be determined by the lap number, $lap(f^n)$ of the iterated map f^n as follows:

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ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
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$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{lap}(f^n). \qquad (11)$$

The lap number of f grows with n in general. If the growth obeys the power law, $lap(f^n) \sim kn^{\alpha}$,
then by (10),

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(kn^{\alpha}) = \lim_{n\to\infty} \frac{\alpha}{n} \log n = 0. \qquad (12)$$

However, if it grows exponentially, $lap(f^n) \sim kn^{\alpha}, (\alpha > 1)$, then

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(k\alpha^n) = \log\alpha. \qquad (13)$$

This shows that h(f) is determined by the way $lap(f^n)$ increases. In case of super stable periodic orbits, the method of structure matrix **M** can be employed. [19]. If λ_{max} be the largest eigenvalue of **M**, then the topological entropy can be obtained as

$$h = \log(\lambda_{max}) \qquad (14)$$

D. Correlation Dimension

As stated, chaos may exist in nonlinear systems during evolution and that can be seen easily by observing the bifurcation diagrams. A chaotic set, an strange attractor, has fractal structure. Correlation dimension gives a *measure of dimensionality* of the chaotic set. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. To determine correlation dimension we use statistical method. It is a very practical and efficient method than other methods, like box counting etc. The procedure to obtain correlation dimension follows the following steps, [18]: Consider an orbit $O(\mathbf{x}_1) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \ldots\}$, of a map f: U → U, where U is an open bounded set in R^n. To compute correlation dimension of $O(x_1)$, for a given positive real number r, we form the correlation integral, [8],

$$C(r) = \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i\neq j}^{n} H\left(r - \left\| x_i - x_j \right\|\right), \qquad (15)$$

Where

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases},$$

is the unit-step function, (Heaviside function). The summation indicates the number of pairs of vectors closer to r when $1 \leq i, j \leq n$ and $i \neq j$. C(r) measures the density of pair of distinct vectors $\mathbf{x}_i$ and $\mathbf{x}_j$ that are closer to r.
The correlation dimension D_c of $O(\mathbf{x}_1)$ is defined as

$$D_c = \lim_{r\to 0} \frac{\log C(r)}{\log r} \qquad (16)$$

To obtain D_c, log C(r) is plotted against log r and then we find a straight line fitted to this curve. The y- intercept of this straight line provides the value of the correlation dimension D_c.

III. THREE DIMENSIONAL FOOD CHAIN MODEL

Many papers have been devoted to the study of two dimensional coupled logistic maps [6, 7, 14, 16] and Some of them can be considered as biological models, corresponding to interactions between species. In this paper, we have considered a model of same kind in three dimensional, which corresponds to symbiotic interaction between correlative pairs of species as 3-species food chain model:

$$\begin{aligned} x_{n+1} &= \lambda(3y_n + 1)x_n(1 - x_n) \\ y_{n+1} &= \lambda(3z_n + 1)y_n(1 - y_n) \\ z_{n+1} &= \lambda(3x_n + 1)z_n(1 - z_n) \end{aligned} \qquad (3.1)$$

Where λ is real positive parameter and (x,y, z) represents the species population. The model contains only one parameter i.e. λ which has important significance for the formulation of the model. A route of chaos has already been studied for this map of logistic type by Daniele Founier-prunaret.The parameter while changing show regular and chaotic behavior which one has to analyzed sincerely. Bifurcation diagrams while varying its parameter λ produce some interesting features.
Bifurcation diagram for map (3.1) by varying λ in range $1 \leq \lambda \leq 1.5$

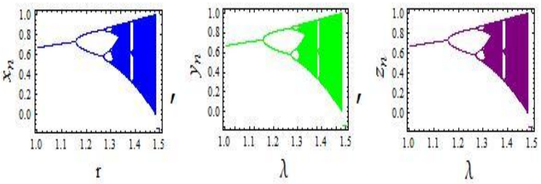

Fig. 1: Bifurcation diagrams of food chain model; above when λ is varied with initial conditions x0 = 0.5, y_0 = 0.5, z_0 = 0.5.

Time series plots are drawn for λ = 1.48, Fig. 2, showing chaos in the system. The chaotic attractor for this case is shown in Fig. 3 below

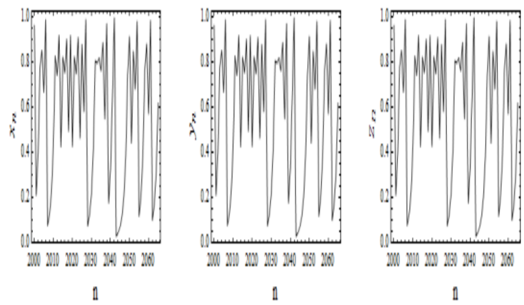

Fig. 2: Time series plots for chaotic evolutions when λ = 1.48 and initial values x_0 = 0.5, y_0 = 0.5, z_0 = 0.5.

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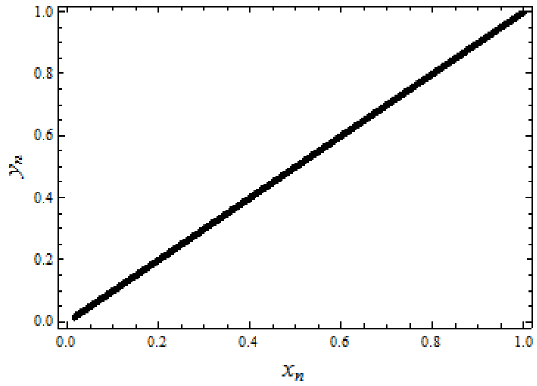


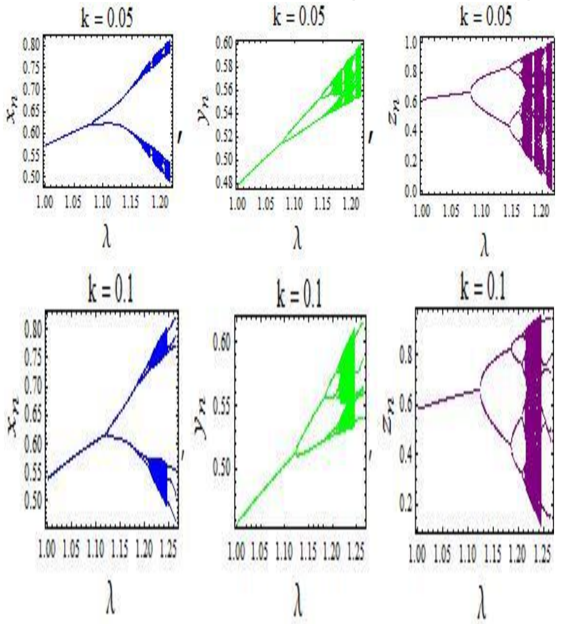
Fig. 3: Time chaotic attractor in the x – y plane of the map (3.1) when λ = 1.48 and initial values x_0 = 0.5, y_0 = 0.5, z_0 = 0.5.

IV. CONTROLLED SYSTEM

In natural food chain system always there is a possibility of co-existence of species. A chaotic or randomness evolution acompanied by uncertainity in their existence which may lead to extinctingtion of one or more species. This does not hold every time and so, one may assume certain type of chaos controlling agent already existing in the model and chaotic evolutions may be controlled. In this regard, we propose a certain changes in the model (3.1). We replace the parameter λ by λ (1 – k cos x_n), λ (1 – k cos y_n), λ (1 – k cos z_n), respectively, in first, second and third equations. Finally the model looks like

$$x_{n+1} = \lambda(1 - k\cos x_n)(3y_n + 1)x_n(1 - x_n)$$
$$y_{n+1} = \lambda(1 - k\cos y_n)(3z_n + 1)y_n(1 - y_n)$$
$$z_{n+1} = \lambda(1 - k\cos z_n)(3x_n + 1)z_n(1 - z_n) \qquad (4.1)$$

This type of assumption can be justified as the parameter λ stand for growth rate of the species and as certain periodic changes may be possible. With such changes, we have drawn the bifurcation diagrams of equation (4.1) for k = 0.05, 0.1, 0.2. 0.3, 0.35, 0.4 and 0.5 below in Fig. 4(A) and in Fig. 4(B).



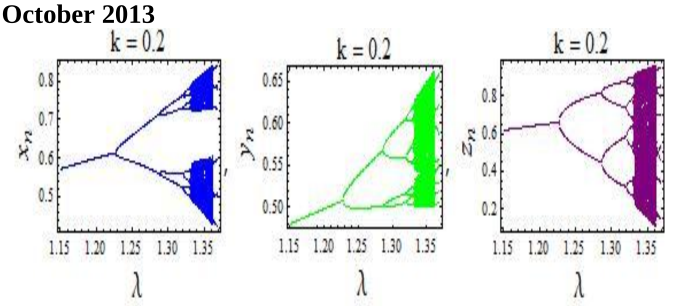


Fig. 4 (A): Bifurcation diagrams for different values of k and varying theλ.

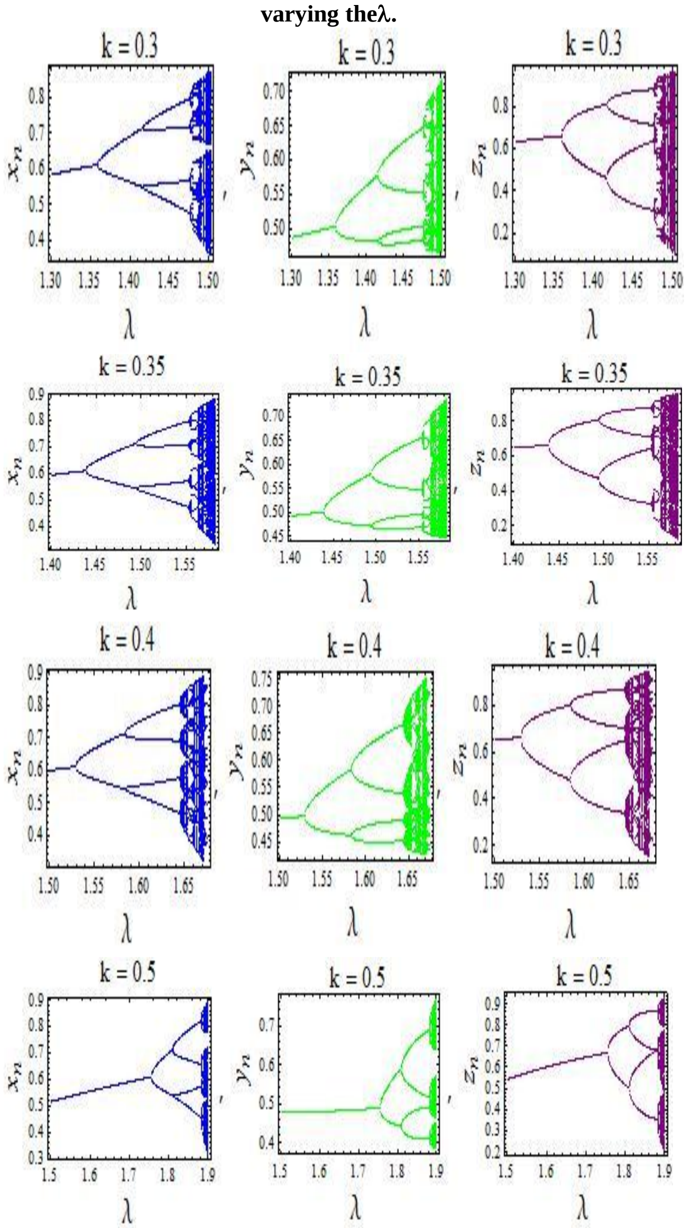


Fig. 4 (B): Bifurcation diagrams for different values of k and varying theλ.

The above figures indicate that as we change the values of k from k = 0 to k = 0.5 chaotic evolution observed earlier getting controlled and the motion turns into regularity. For example, initially when k = 0 and λ = 1.48, system evolution was chaotic but the system becomes regular when k = 0.2, 0.3, 0.4, 0.5 for this same value of λ. In case of k = 0.5,

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
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system shows regularity even at λ = 1.85. At these values of k, the system shows either show periodic of finite period or quasi-periodic.

V. CALCULATIONS OF LYAPUNOV EXPONENTS, TOPOLOGICAL ENTROPY, CORRELATION DIMENSION AND DLI

A. Lyapunov Exponents

Lyapunov exponents have been computed numerically for parameter value λ =1.48 and initial conditions (x, y, z) = (0.5, 0.5,0.5) for the uncontrolled system, k = 0 and plotted and shown in Fig. 5. Also, similar calculations have been repeated for same value of λ, i.e. λ =1.48, and same initial conditions for the controlled system when k = 0.47, 0.48, 0.49, 0.5. These are plotted and shown in Fig. 6.

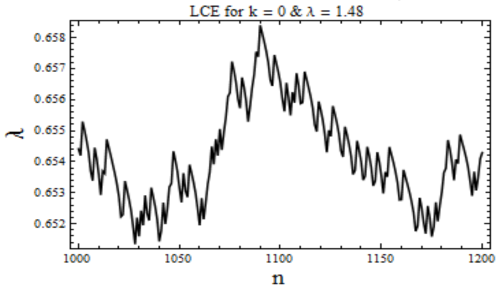
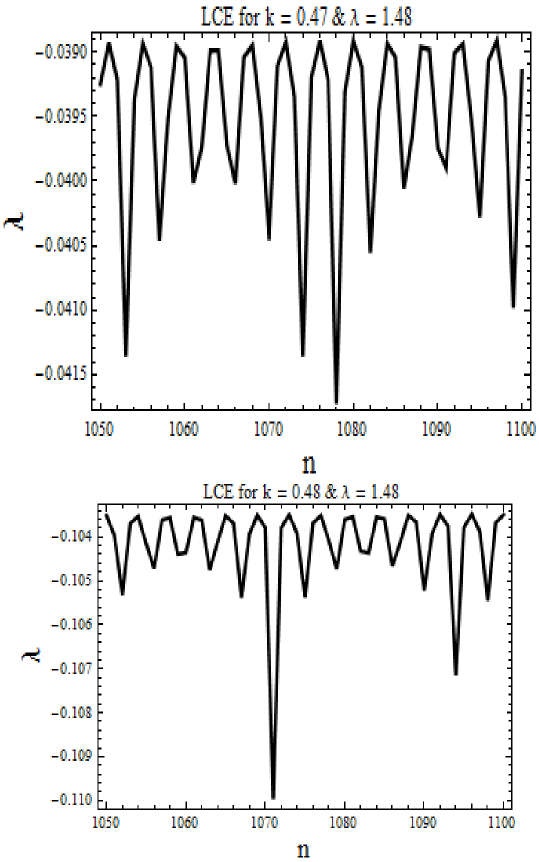


Fig. 5: Plot of Lyapunov exponents shown for uncontrolled system when k = 0 and λ =1.48.



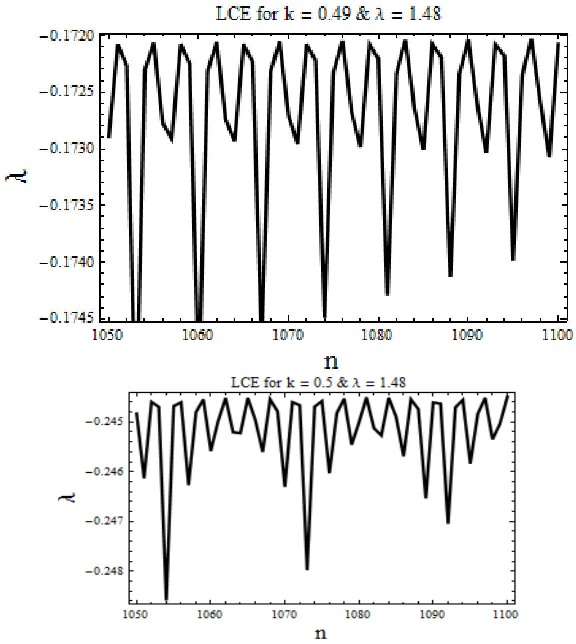
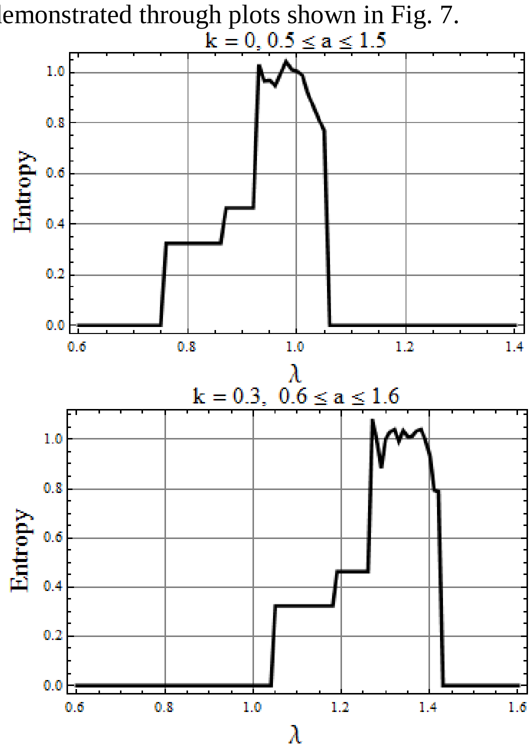


Fig. 6: Plots of Lyapunov exponents for controlled system when k = 0.47, 0.48, 0.49 & 0.5.

Plots of LCE in Fig. 6 show all Lyapunov exponents are negative and the system (4.1) evolve in regular manner. This implies co-existence of all the species of the food chain.

B. Topological Entropy

The topological entropy provides the measure of complexity of the chaotic system. Numerically we have computed the topological entropies for k = 0, 0.3, 0.4 and 0.5 and demonstrated through plots shown in Fig. 7.



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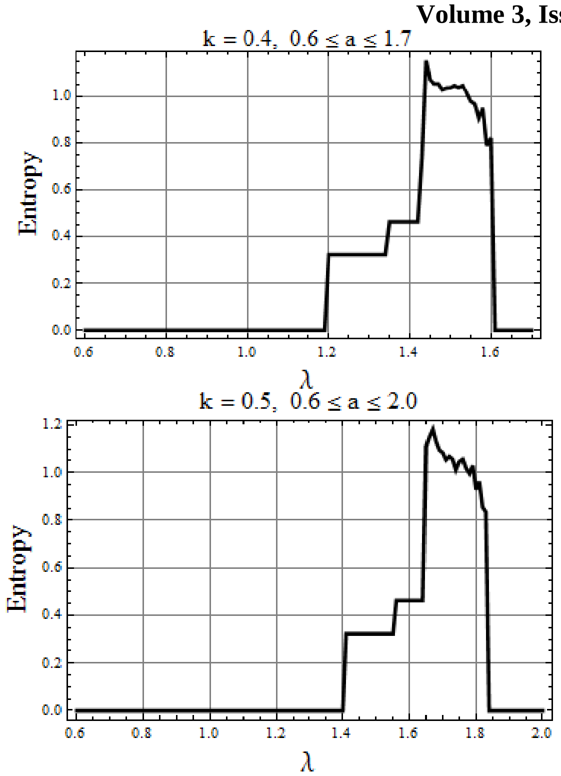
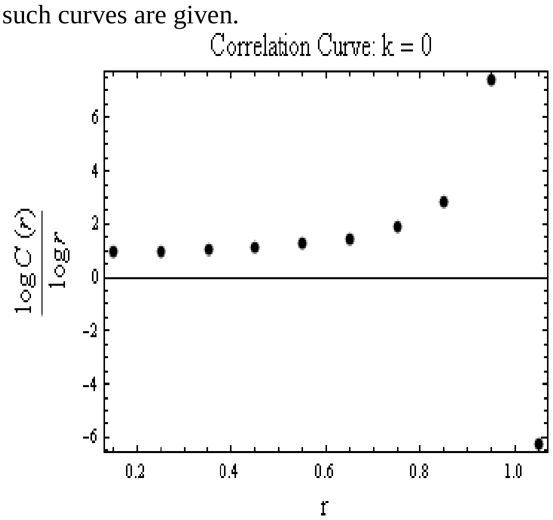


Fig. 7: Plots of topological entropies for four different values of k and for different ranges of λ.

One observes from plots of topological entropies in Fig. 7, that for k = 0 the complexity region confined to approximately between 0.75 ≤λ≤ 1.05. But for k = 0.3, k= 0.4 and k = 0.5, the complexity regions are different and shifting towards right on λ axis. The approximate complexity regions for above cases are, respectively, be given as 1.05 ≤λ≤ 1.45, 1.2≤λ≤ 1.6, 1.4≤λ≤ 1.85.

C. Correlation Dimension

Correlation dimension measures the dimensionality of the system at any stage. Such a dimension is referred as a fractal dimension. In case of our model (4.1), we follow the method of Martelli [18] and for fixed parameter value λ = 1.48, first we obtain the plots for the correlation curves for k = 0 and k = 0.5. At these values, respectively, we have already observed chaos and regularity in the system. Below in Fig. 8 plots of two such curves are given.



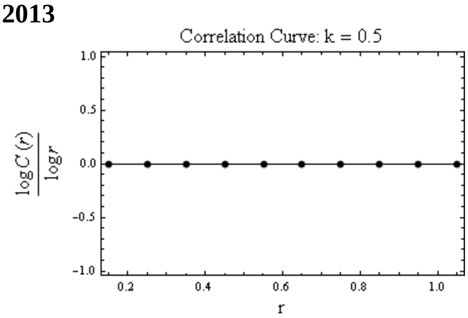


Fig. 8: Plots of correlation curves with fixed λ = 1.48 and k = 0 and 0.5.

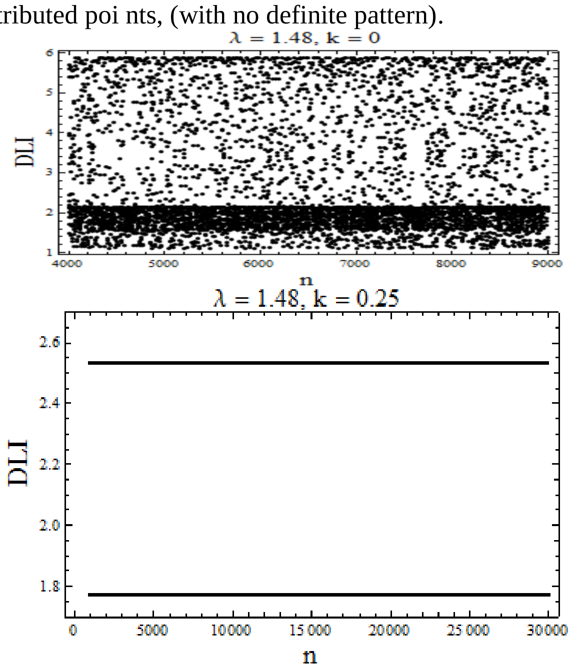
Then, by using the least square linear fit to the correlation data, we obtain the equation of the straight line exactly fitting the correlation data. The y-intercept of this straight line represents the correlation dimension of the system. In case of k = 0, equation of such straight line is

$$y = 1.5762 - 0.511576\, x \qquad (5.1)$$

Therefore the correlation dimension for this case is 1.5762 ≈ 1.58. In the second case, when k = 0.5, x –axis is itself is the straight line fitting the correlation data which is clearly visible from the right hand graph. So the correlation dimension is zero. It has been observed that this is also true for any value of k in 0.25 < k ≤ 0.53. Here, the system becomes no more chaotic.

D. DLI Plots

As an indicator of chaos, dynamic Lyapunov exponents (DLI) has been introduced recently by Saha and Budhraja [23] and then it's working ability is tested for various discrete systems by applied in articles by Yuasa and Saha [25], Saha and Tehri [24], Deleanu [11]. These studies confirm that DLI could be reliable indicator for distinguishing regularity and chaos. When plotted after numerical simulations, it has been observed that DLI's, form a definite pattern for the motion is regular motion and for chaotic motion it shows randomly distributed poi nts, (with no definite pattern).



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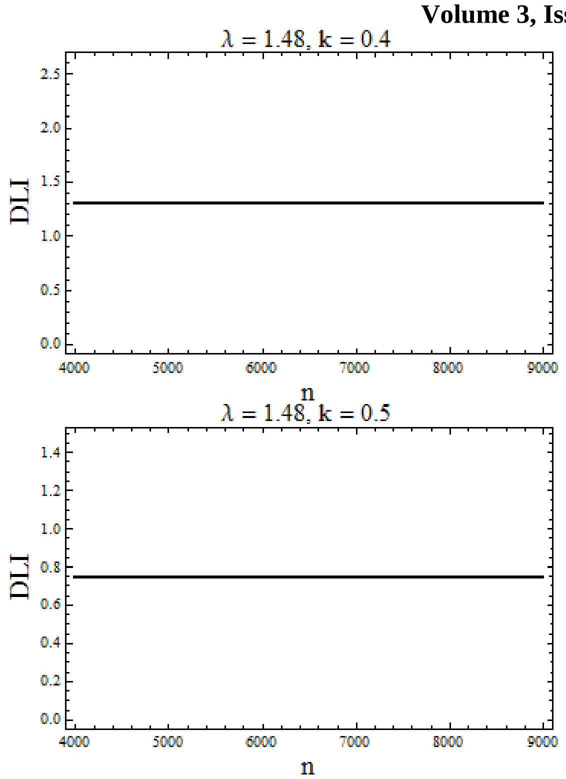

Fig. 9: DLI plots for the chaotic case and regular cases.

VI. DISCUSSION

We have studied the nonlinear behavior of a food chain system (3.1) together with certain measure for chaotic evolution. Bifurcation diagrams of this food chain model have been drawn by varying the parameter λ in Fig. 1. These figures provide information regarding evolution with stable solutions as well as chaotic nature of nonlinear properties and limitation for parameter space. Chaotic time series graphs and attractors in x-y plan have been drawn in Fig. 2 –Fig. 3.In order to control chaotic evolution in this particular food chain model, the parameter λ has been changed for each equation of the original model by assuming this rate of change subject to some periodic change in populations. With this the original model (3.1) has been replaced by equation(4.1). Bifurcation diagrams have been drawn by varying λ for different fixed values of K in Fig 4. Plots of Lyapunov exponents for controlled system when k =0, 0.47, 0.48, 0.49, 0.5 and λ= 1.48 are shown in Fig. 5 – Fig. 6.we have also computed the topological entropies for k = 0, 0.3, 0.4 and 0.5 and demonstrated through plots shown in Fig. 7. To measure the dimensionality of the chaotic attractor, numerical simulations have been extended to evaluate an correlation integral and collect appropriate correlation data. Plot of correlation data resulting in correlation curve is shown in Fig.8. Then, by using the method of least square linear fit, we have obtained the equation of the straight line approximately fitting the data given by equation (5.1) and its y-intercept provides the correlation dimension shown there. Finally, we have applied the recently introduced indicator DLI to the food chain model for distinguishing regular and chaotic motion in Fig.9.

VII. ACKNOWLEDGMENT

We are thankful to the department of mathematics, Chitkara University, HPand ChitkaraEducation trust, for all kind of support for this work.

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ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
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