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Inversion Theorem of Two Dimensional Fractional Fourier Transform

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***Abstract*— Fourier and fractional-Fourier transformations are widely used in theoretical physics, in data processing, quantum mechanics, phase retrieval, optics, signal processing, filter designing, water marking and many other branches. In this paper inversion theorem for two dimensional fractional Fourier transform is proved.**

***Key words*—Fourier transform, Fractional Fourier transform (FRFT), Testing Function space.**

I. INTRODUCTION

Some fractional transforms arise under consideration of different problem: description of paraxial diffraction in free space and in a quadrant refractive index medium, resolution of nonstationary Schrodinger equation in quantum mechanics, phase retrieval and so forth. Other fractional transform can be constructed for their own sake, even if their direct application may not be obvious yet [1].

Fourier analysis is frequently used tool in signal processing and analysis. Fourier transform provides the signal's spectral content, but it fails when we need the time location of the spectral components. Time-frequency patterns are important in analysis of signals like non-stationary or time-varying signals. In order to analyze such signals, time-frequency representations are used [11]. Since the Fourier Transform plays an important role in data processing, its generalization the fractional Fourier transform was probably the most intensively studied among all fractional transforms. Although the Fourier transform can be divided into fractions in different ways, the canonical fractional Fourier transform certainly has advantages for application in optical information processing. First because this fractional Fourier transform can easily be realized experimentally by using simple optical set ups [2], and secondly, because it produces mere rotation of two phase space distributions.

The concept of the fractional Fourier transform was originally described by Condon and was later introduced for signal processing in 1980 by Namias as a Fourier transform of fractional order. Sumiyoshi et al also made an interesting generalization on fractional Fourier transform in 1994[10]. The fractional Fourier transform is generalization of classical Fourier transform. The canonical fractional Fourier transform was introduced more than 60 years ago in the mathematical literature [3], after that, it was reinvented for applications in quantum mechanics [4] [9], optics [5] [6] [7], and signal processing [6]. After the main properties of the fractional Fourier transform were established, the perspectives for its implementations in filter design, signal analysis phase retrieval, water marking and so forth became clear. Moreover, the use of refractive optics for analog realizations of the fractional Fourier transform opened a way for the fractional Fourier transform optical information processing [1]. Recently the fractional Fourier transform has been reintroduced twice with optical applications in mind. Hence Fourier transform and fractional Fourier transform has important properties and applications as describe in [12,13,14,15.16,17,18].

A. Definition of The conventional two-dimensional the fractional Fourier transform

The conventional two-dimensional fractional Fourier transform is defined as

$$FRFT\{f(t,x)\} = F_\alpha(s,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x)K_\alpha(s,u,t,x)\,dtdx$$

where

$$K_\alpha(s,u,t,x) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{i}{2\sin\alpha}[(t^2+s^2+x^2+u^2)\cos\alpha-2(ts-xu)]}$$

where $0<\alpha<\frac{\pi}{2}$.

$K_\alpha(s,u,t,x)$ belongs to the testing function space and $f(t,x)$ lies in its dual space.

B. Generalized Distributional Two-Dimensional Fractional Fourier Transform

The two dimensional fractional Fourier transform developed as a generalization of two dimensional Fourier transform through an angle α is defined as

$$FRFT\{f(t,x)\} = F_\alpha(s,u) = \langle f(t,x), K_\alpha(t,s,x,u)\rangle$$

where the kernel

$$K_\alpha(s,u,t,x) = \sqrt{\frac{1-i\cot\phi}{2\pi}}\, e^{\frac{i}{2\sin\alpha}[(t^2+s^2+x^2+u^2)\cos\alpha-2(ts-xu)]}$$

where $C_{1\alpha} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$ $\quad C_{2\alpha} = \frac{1}{2\sin\alpha}$

$$K_\alpha(s,u,t,x) = C_{1\alpha} e^{iC_{2\alpha}[(t^2+s^2+x^2+u^2)\cos\alpha-2(ts-xu)]}$$

C. The Test Function

An infinitely differentiable complex valued function $\phi(x,y)$ on R belongs to $E(R)$ if for each compact set $K\subset S_\alpha$, $L\subset S_\alpha$ where

$$S_{a,b} = \{x,y : x,y\in R, |x|\le a, |y|\le b, a>0, b>0\}$$

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$$\gamma_{E,K,L}[\phi(x,y)] = \sup_{\substack{x\in k\\ y\in l}} \left|D_x^k D_y^l \phi(x,y)\right| < \infty$$

Thus $E(R)$ will denote the space of all $\phi \in E(R)$ with compact support contained in S_α. Note that the space E is complete and therefore a Frechet space. Moreover, we say that f is a fractional Fourier transformable if it is a member of E.

In the present work, Inversion Theorem for the generalized two-dimensional Fractional Fourier transform is proved.

II. INVERSION THEOREM

Let, $f(x,y) \in E'(R^n)$, $0 < \alpha \le \dfrac{\pi}{2}$ and $\sup f \subset S_{a,b}$,

$S_{a,b} = \{x,y : x,y \in R, |x| \le a, |y| \le b, a > 0, b > 0\}$ and let, $F_\alpha(\xi,\eta)$ be the distributional fractional Fourier transform of f as defined by

$$FRFT\{f(x,y)\}(\xi,\eta) = F_\alpha(\xi,\eta)$$
$$= \langle f(x,y), K_\alpha(x,y,\xi,\eta) \rangle .$$

Then for each $\phi \in D(I)$ we have

$$\lim_{\substack{N\to\infty\\ P\to\infty}} \left\langle \frac{1}{4\pi^2} \int_{-N}^{N}\int_{-P}^{P} \overline{K_\alpha}(x,y,\xi,\eta) F_\alpha(\xi,\eta)\, d\xi\, d\eta, \phi(x,y) \right\rangle$$
$$= \langle f, \phi \rangle$$

To prove this inversion theorem, we prove the following two lemmas.

A. Lemma 1

Let $FRFT\{f(x,y)\} = K_\alpha(\xi,\eta)$ for $0 < \alpha \le \dfrac{\pi}{2}$

$\sup f \subset S_{a,b} = \{x,y : x,y \in R, |x| \le a, |y| \le b, a > 0, b > 0\}$

for $\phi(x,y) \in D(I)$ set .

$$\varphi(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{K_\alpha}(x,y,\xi,\eta)\phi(x,y)\,dxdy$$

Then, for any fixed number $r, s, -\infty < r < \infty$, $-\infty < s < \infty$

$$\int_{-r}^{r}\int_{-s}^{s} \varphi(\xi,\eta) \langle f(u,v), K_\alpha(u,v,\xi,\eta) \rangle\, d\tau d\mu$$
$$= \left\langle f(u,v), \int_{-r}^{r}\int_{-s}^{s} \varphi(\xi,\eta) K_\alpha(u,v,\xi,\eta)\, d\tau d\mu \right\rangle$$

$\xi = \sigma + i\tau$, $\eta = \lambda + i\mu$

where $\xi, \eta \in C^n$ and u & v are restricted a compact subset of R^n.

Proof- The case $\phi(x,y) = 0$ is trivial, so that we consider $\phi(x,y) \ne 0$.

It can be easily seen that

$$\int_{-r}^{r}\int_{-s}^{s} \varphi(\xi,\eta) K_\alpha(u,v,\xi,\eta)\, d\tau d\mu,$$

$\xi = \sigma + i\tau$, $\eta = \lambda + i\mu$ is a C^∞ -function of u and v and it belongs to E. Hence the RHS of (1) is meaningful. To prove the equality, we construct the Riemann sums for this integral and write

$$\int_{-r}^{r}\int_{-s}^{s} \langle f(u,v), K_\alpha(u,v,\xi,\eta) \rangle \varphi(\xi,\eta)\, d\tau d\mu$$
$$= \lim_{\substack{m\to\infty\\ l\to\infty}} \sum_{n=-m}^{m-1} \sum_{k=-l}^{l-1} \langle f(u,v), K_\alpha(u,v,\sigma + i\tau_{n,m}, \lambda + i\mu_{k,l}) \rangle$$
$$\varphi(\sigma + i\tau_{n,m}, \lambda + i\mu_{k,l}) \Delta\tau_{n,m} \Delta\mu_{k,l}$$
$$= \lim_{\substack{m\to\infty\\ l\to\infty}} \Big\langle f(u,v), \sum_{n=-m}^{m-1} \sum_{k=-l}^{l-1} K_\alpha(u,v,\sigma + i\tau_{n,m}, \lambda + i\mu_{k,l})$$
$$\varphi(\sigma + i\tau_{n,m}, \lambda + i\mu_{k,l}) \Delta\tau_{n,m} \Delta\mu_{k,l} \Big\rangle$$

We show that last summation converges in E to the integral on RHS of (1).

Carrying the operator $D_{u,v}^{t,i}$ within the integral and summation signs, which is easily justified, we get

$$\gamma_{K,k,j}\Big\{ \sum_{n=-m}^{m-1} \sum_{k=-l}^{l-1} K_\alpha(u,v,\sigma + i\tau_{n,m}, \lambda + i\mu_{k,l})$$
$$\varphi(\sigma + i\tau_{n,m}, \lambda + i\mu_{k,l})\ \Delta\tau_{n,m} \Delta\mu_{k,l} - \int_{-r}^{r}\int_{-s}^{s} \varphi(\xi,\eta)$$
$$K_\alpha(u,v,\xi,\eta)\, d\tau d\mu \Big\} =$$
$$\sup_{\substack{u\in K\\ v\in K}} \Big| \sum_{n=-m}^{m-1} \sum_{k=-l}^{l-1} D_u^t D_v^j K_\alpha(u,v,\sigma + i\tau_{n,m}, \lambda + i\mu_{k,l}) - \int_{-r}^{r}\int_{-s}^{s} \varphi(\xi,\eta)$$
$$D_u^t D_v^j K_\alpha(u,v,\xi,\eta)\, d\tau d\mu \Big|$$

As $\displaystyle\lim_{\substack{m\to\infty\\ l\to\infty}} \sum_{n=-m}^{m-1} \sum_{k=-l}^{l-1} D_u^t D_v^j K_\alpha(u,v,\sigma + i\tau_{n,m}, \lambda + i\mu_{k,l})$

$$\phi(\sigma + i\tau_{n,m}, \lambda + i\mu_{k,l}) \Delta\tau_{n,m} \Delta\mu_{k,l}$$
$$= \int_{-r}^{r}\int_{-s}^{s} \varphi(\xi,\eta) D_{u,v}^{t,j} K_\alpha(u,v,\xi,\eta)\, d\tau d\mu \text{ for all } u,v \in K .$$

It thus follows that for every $m \,\&\, l$, the summation is a member of E and it converges in E to the integral on the RHS of (1).

B. Lemma 2

For $\phi(x,y) \in D(I)$, set $\varphi(\xi,\eta)$ as in lemma 1 above for $\xi, \eta \in C^n$, $u \,\&\, v$ restricted to compact subset of R^n then

$$M_{r,l}(u,v) = \frac{1}{4\pi^2} \int_{-r}^{r}\int_{-l}^{l} \varphi(\xi,\eta) K_\alpha(u,v,\xi,\eta)\, d\tau d\sigma$$

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$$=\frac{1}{4\pi^2}\int_{-r}^{r}\int_{-l}^{l}K_\alpha(u,v,\xi,\eta)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\overline{K}_\alpha(x,y,\xi,\eta)dxdyd\tau d\sigma\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\overline{K}_\alpha(x,y,\xi,\eta)d\tau d\sigma dxdy$$

(1)

converges in E to $\phi(u,v)$ as $r\to\infty, l\to\infty$.

Proof- We shall show that $M_{r,l}(u,v)\to\phi(u,v)$ in E as $r\to\infty, l\to\infty$.

That is to below:

$$\gamma_{K,k,t}[M_{r,l}(u,v)-\phi(u,v)=\sup_{\substack{u\in K\\ v\in K}}\left|D_u^k D_v^t[M_{r,l}(u,v)-\phi(u,v)\right|\to 0$$

as $r\to\infty, l\to\infty$

We note that for $k=0, t=0$

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_\alpha(u,v,\xi,\eta)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\overline{K}_\alpha(x,y,\xi,\eta)dxdyd\tau d\sigma$$
$$=\phi(u,v)$$

This is to say that $\lim_{\substack{m\to\infty\\ l\to\infty}}M_{r,l}(u,v)=\phi(u,v)$

Since the integrand is a C^∞-function of u, v and $\phi\in D(I)$, we can repeatedly differentiate under the integral sign in (1) and integrals are uniformly convergent, we have

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}D_u^k D_v^t(u,v,\xi,\eta)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\overline{K}_\alpha(x,y,\xi,\eta)dxdyd\tau d\sigma$$
$$=\phi(u,v)\ ,\quad \forall u,v\in K$$

Hence proved.

C. *Proof of Inversion theorem*

Now let $\phi(x,y)\in D(I)$ we shall show that

$$\left\langle\frac{1}{4\pi^2}\int_{-r}^{r}\int_{-l}^{l}\overline{K}_\alpha(x,y,\xi,\eta)F_\alpha(\xi,\eta)d\tau d\sigma,\phi(x,y)\right\rangle$$ tends to

$\langle f(x,y),\phi(x,y)\rangle$ as $r\to\infty, l\to\infty$ (2)

From the analyticity of $F_\alpha(\xi,\eta)$ on C^n and the fact that $\phi(x,y)$ has compact support in I, it follows that the left side expression in (2) is merely a repeated integral with r to x, y and ξ,η and the integral in (2) is a continuous function of x,y as the closed bounded domain of the integration. Therefore, we write (2) as

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\int_{-r}^{r}\int_{-l}^{l}\overline{K}_\alpha(x,y,\xi,\eta)F_\alpha(\xi,\eta)d\tau d\sigma dxdy$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\int_{-r}^{r}\int_{-l}^{l}\overline{K}_\alpha(x,y,\xi,\eta)$$
$$\langle f(u,v),K_\alpha(u,v,\xi,\eta)\rangle d\tau d\sigma dxdy$$

$$=\frac{1}{4\pi^2}\int_{-r}^{r}\int_{-l}^{l}\langle f(u,v),K_\alpha(u,v,\xi,\eta)\rangle$$

Since $\phi(x,y)$ is of compact support, and the integrand is a continuous function of (x,y,ξ,η), the order of integration may be changed. The change in the order of integration is justified by appeal to lemma (1). This yields

$$=\frac{1}{4\pi^2}\int_{-r}^{r}\int_{-l}^{l}\langle f(u,v),\overline{K}_\alpha(u,v,\xi,\eta)\rangle\varphi(\xi,\eta)d\tau d\sigma$$

where $\varphi(\xi,\eta)$ is as in lemma (1). Hence by lemma (1)

$$=\left\langle f(u,v),\frac{1}{4\pi^2}\int_{-r}^{r}\int_{-l}^{l}\overline{K}_\alpha(u,v,\xi,\eta)\right\rangle\varphi(\xi,\eta)d\tau d\sigma$$

(3)

$=\langle f(u,v),\varphi(u,v)\rangle$ as $r\to\infty, l\to\infty$ ($\because$ by lemma(2))

This completes the proof.

III. CONCLUSION

In this paper Inversion theorem for two-dimensional fractional Fourier transform is proved.

REFERENCES

[1] Tatiana Alieva, Martin J. Bastiaans, Maria Luisa Calvo, "Fractional Transform in Optical Information Processing", EURASIP Journal on Applied Signal Processing 2005,10,1-22, Hindawi Publishing Corporation.

[2] A. W. Lohmann, "Image rotation, Wigner rotation and fractional Fourier Transform", Journal of Optical Society of America A, vol. 10, no.10, pp. 2181-2186, 1993.

[3] H. Kober, "Wurzeln aus der Hankel-Fourier- und aus andren stetigen Transformational", Quart J. Math. Oxford Ser., Vol. 10, pp 45-59, 1939.

[4] A.C. Mc. Bridge & F.H.Kerr, "On Namias fractional Fourier Transform"s,IMA J of Applied Mathematics, vol. 39, no.2, pp. 159-175, 1987.

[5] B.N. Bhosale & Chaudhary M.S.," Fractional Fourier Transform of distribution of compact support" Bull. Cal. Math Soc., 94 (5), pp 349-358, (2008).

[6] W. Lohmann, "Image rotation, Wigner rotation and fractional Fourier Transform, Journal of Optical Society of America A, vol. 10, no.10, pp. 2181-2186, 1993.

[7] T. Alieva, V. Lopez, F. Agullo-Lopez & L.B.Almeida, "The Fractional Fourier Transform in optical propagation problems", J. of Modern Optics, Vol.41, No.5, pp 1037-1044 (1994).

[8] H.M.Ozaktas & D.Mendolvic, "Fractional Fourier Transforms & their optical implementation", Journal of Optical Society of America A, vol. 10, no.9, pp. 1875-1881, 1993.

[9] V.Namias, "The Fractional order Fourier Transform and its application to quantum mechanics", J. Inst. Appl. Math 25, 241-265,(1980).

[10] Jun-hua Chen, Hong-yi Fan, "Quantum mechanical perspectives and generalization of the fractional Fourier transformation*", Department of Material Science and

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Engineering, University of Science and Technology of China, 2 CAS Key Laboratory of Materials for Energy Conversion Hefei, Anhui, China, July 30, 2013.

[11] Muhammad Irfan, Liying Zheng and Haroon Shahzad, "Review of Computing Algorithms for Discrete Fractional Fourier Transform", Research Journal of Applied Sciences, Engineering and Technology 6(11): 1911-1919, 2013 ISSN: 2040-7459; e-ISSN: 2040-7467 © Maxwell Scientific Organization, 2013.

[12] R. S. Pathak, Akhilesh Prasad, Manish Kumar, "Fractional Fourier transform of tempered distributions and generalized pseudo-differential operator" Springer Link, June 2012, Volume 3, Issue 2, pp 239-254 .

[13] A. A. Kilbas[a], Yu. F. Luchko[b*], H. Martínez[c] & J. J. Trujillo[d] , " Fractional Fourier transform in the framework of fractional calculus operators", J. Integral Transforms and Special Functions ,Volume 21, Issue 10, 2010 .

[14] V.D.Sharma, "Generalized Two-Dimensional Fractional Fourier Transform and Inversion Formula" Vol. 4(2) July 2013, pp. 191-195.

[15] Feng Zhang ; Ran Tao ; Yue Wang , "Discrete fractional Fourier transform computation by adaptive method", Fourier Optics and Optical Signal Processing, Opt. Eng. 52(6), 068202 (Jun 03, 2013).

[16] M.C. Anumaka, "Analysis and applications of Laplace/Fourier transformations in electric", circuit, IJRRAS, 12(2) (2012), 333-339.

[17] A.H. Zemanian, Generalized Integral Transformation, Interscience Publisher, New York, 1968.

[18] K. K. Sharma and Abdul Samad, "Application Specific Integrated Circuit Implementation of Discrete Fractional Fourier Transform", IJIEE 2013 Vol.3 (5).