ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 2, August 2013

Operational Calculus on Generalized Two-Dimensional Fractional Fourier Transform

V.D.Sharma
Mathematics Dept., Arts, Commerce and Science College, Amravati (MS) India.

***Abstract*— The Fractional Fourier transform is a useful mathematical operation that generalizes the well- known continuous Fourier transforms. It has been found wide applications in information processing, optical image encryption, beam analysis and shaping. In this paper operational calculus on the generalized two-dimensional fractional Fourier transform (2DFrFT) is presented.**

***Keywords*- Fractional Fourier Transforms, Signal Processing, Generalized Function, Testing Function Space.**

I. INTRODUCTION

Namias introduced fractional Fourier transform in the field of quantum mechanics for solving some classes of differential equations efficiently. Later, Ozaktas et.al came up with the discrete implementation of fractional Fourier transform. Since then, a number of applications of fractional Fourier transform have been developed, mostly in the field of optics. However, it remains relatively unknown in acoustics.

Little needs to be said of the importance and ubiquity of the ordinary Fourier transform and frequency-domain concepts in many diverse areas of science and engineering. As a generalization of the ordinary Fourier transform, the fractional Fourier transform is only richer in theory and more flexible in applications, but not more costly in applications. Therefore, the transform is likely to have something to offer in every area in which Fourier transforms and related concepts are used. The FrFT is basically a time-frequency distribution. It provides us with an additional degree of freedom (order of the transform), which in most cases results in significant gain over the classical Fourier transform. With the development of FrFT and related concepts, we see that the ordinary frequency domain is merely a special case of a continuum of fractional Fourier domains. Every property and application of the ordinary Fourier transform becomes a special case of the FrFT. So in every area in which Fourier transforms and frequency domain concepts are used, there exists the potential for improvement by using the FrFT [1], [2].

As the one-dimensional (1-D) Fourier transform can be extended into the 1-D fractional Fourier transform (FrFT), we can also generalize the Two-dimensional (2-D) fractional Fourier transform. In our previous work in [3], [10].

A. *Conventional Two- dimensional Fractional Fourier transforms*

The two dimensional fractional Fourier transform with parameter $\propto$ of $f(x,y)$ denoted by $FRFT\{f(x,y)\}$ performs a linear operation given by the integral transform,

$$FRFT\{f(x,y)\} = F_\propto\{f(x,y)\}(u,v)$$

$$= F_\propto(u,v)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, K_\propto(x,y,u,v)\,dx\,dy, \quad \ldots\ldots (2.1)$$

Where the kernel,

$$K_\propto(x,y,u,v) = \sqrt{\frac{1-icot\propto}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+u^2+y^2+v^2)cos\alpha-2(xu+yv)]} \quad \ldots\ldots (2.2)$$

B. *The testing function space E*

An infinitely differentiable complex values smooth function $\emptyset$ on R^n belongs to $E(R^n)$ if for each compact set $I < S_{a,b}$

where,

$S_{a,b} = \{x,y : x,y \in R^n, |x| \le a, |y| \le b, a > 0, b > 0\}, I \in R^n$,

$\gamma_{E,p,q}(\emptyset) = \sup_{x,y\in I} |D_{x,y}^{p,q}\, \emptyset(x,y)|$

$< \infty$, where $p,q = 1,2,3, \ldots$.

Thus $E(R^n)$ will denote the space of all $\emptyset \in E(R^n)$ with support contained in $S_{a,b}$.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that f is a fractional Fourier transformable if it is a member of E^*, the dual space of E.

FRFT has great potential in sonar signal processing as it takes advantage of the knowledge of transmitted waveform. It has many applications in solution of differential equations, optical beam propagation and spherical mirror resonators optical diffraction theory, quantum mechanics, statistical optics, signal detectors, pattern [5], [6], [9] recognition, space image recovery etc.

In the present work, Generalization of 2DFrFT is given. Some properties of the kernel and operation transform formulae on 2-D Fractional Fourier Transform are proved.

II. DISTRIBUTIONAL TWO-DIMENSIONAL FRACTIONAL FOURIER TRANSFORM

The two distributional two-dimensional fractional Fourier transform of $f(x,y) \in E(R^n)$ can be defined by,

$$FRFT\{F(X,Y)\} = F_\propto(u,v) = \langle f(x,y), K_\propto(x,y,u,v)\rangle \quad \ldots\ldots (3.1)$$

When

$$K_\propto(x,y,u,v) = C_{1\propto} e^{iC_{2\propto}[(x^2+u^2+y^2+v^2)cos\alpha-2(xu+yv)]},$$

$$C_{1\propto} = \sqrt{\frac{1-icot\propto}{2\pi}},\quad C_{2\propto} = \frac{1}{2sin\alpha} \quad \ldots\ldots (3.2)$$

The right hand side of (3.1) has a meaning as the application of $f \in E^*$ to $K_\propto(x,y,u,v) \in E$.

It extended to the complex space as an entire function given by

$$FRFT\{F(X,Y)\} = F_\propto(g,h) = \langle f(x,y), K_\propto(x,y,g,h)\rangle \quad \ldots\ldots (3.3)$$

Where

$$K_\alpha(x,y,g,h) = C_{1\alpha} e^{iC_{2\alpha}[(x^2+g^2+y^2+h^2)\cos\alpha - 2(xg+yh)]}$$

The right hand side of (3.3) is meaningful because for each $g, h \in c^n$, $K_\alpha(x,y,u,v) \in E$, as a function of x, y.

III. PROPERTIES OF KERNEL OF 2DFRFT

A. Result-1

To prove $k_{(-\alpha)}(x,u,y,v) = k_\alpha^*(x,u,y,v)$ where $*$ denotes the conjunction.

$$k_\alpha(x,u,y,v) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{i}{2\sin\alpha}[(x^2+u^2+y^2+v^2)\cos\alpha - 2(xu+yv)]}$$

$$k_{-\alpha}(x,u,y,v) = \sqrt{\frac{1+i\cot\alpha}{2\pi}}\, e^{-\frac{i}{2\sin\alpha}[(x^2+y^2+u^2+v^2)\cos\alpha - 2(xu+yv)]}$$

$$= \sqrt{\frac{1-(-i)\cot\alpha}{2\pi}}\, e^{-\frac{i}{2\sin\alpha}[(x^2+y^2+u^2+v^2)\cos\alpha - 2(xu)]}$$

$$= k_\alpha^*(x,u,y,v).$$

where $*$ denotes the conjunction.

B. Result-2

To prove $k_{(\alpha)}(-x,-y,u,v) = \frac{1}{e^2} k_\alpha(x,y,u,v)$

Consider

$$k_\alpha(x,y,u,v) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{i}{2\sin\alpha}[(x^2+y^2+u^2+v^2)\cos\alpha - 2(xu+yv)]}$$

$\therefore k_\alpha(-x,-y,u,v)$

$$= \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{i}{2\sin\alpha}[(x^2+y^2+u^2+v^2)\cos\alpha - 2((-x)u+(-y)v)]}$$

$$= \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{i}{2\sin\alpha}(x^2+y^2+u^2+v^2)\cos\alpha)}\, e^{-\frac{i}{2\sin\alpha}2(-x)u}\, e^{-\frac{i}{2\sin\alpha}2(-y)v}$$

$$= \frac{1}{e^2} k_\alpha(x,y,u,v).$$

C. Result-3

$$k_\alpha(x,y,0,0) = e^{-\frac{i}{2\sin\alpha}[(u^2+v^2)\cos\alpha - 2(xu+yv)]} k_\alpha(x,y,u,v)$$

$$k_\alpha(x,y,u,v) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{i}{2\sin\alpha}[(x^2+y^2+u^2+v^2)\cos\alpha - 2(xu+yv)]}$$

$$k_\alpha(x,y,0,0) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{i}{2\sin\alpha}[(x^2+y^2)\cos\alpha]}$$

$$= \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{i}{2\sin\alpha}[(x^2+y^2+u^2+v^2)\cos\alpha - 2(xu+yv)]}\, e^{-\frac{i}{2\sin\alpha}(u^2+v^2)\cos\alpha}\, e^{\frac{2i}{2\sin\alpha}(xu+yv)}$$

$$= e^{-\frac{i}{2\sin\alpha}[(u^2+v^2)\cos\alpha - 2(xu+yv)]} k_\alpha(x,y,u,v)$$

IV. PROPOSITION

Generalization of 2-D fractional Fourier transform reduces to conventional 2D Fourier transform if $\theta = \frac{\pi}{2}$.

$$2DFRFT\{f(x,y)\} = F_\alpha(u,v) = \langle f(x,y), k_\alpha(x,y,u,v)\rangle$$

$$F_\alpha(u,v) = \int_0^\infty \int_0^\infty f(x,y)\, C_{1\alpha}\, e^{iC_{2\alpha}[(x^2+y^2+u^2+v^2)\cos\alpha - 2(xu+yv)]} dx\, dy$$

$$F_{\frac{\pi}{2}}(u,v) = \int_0^\infty \int_0^\infty f(x,y) \sqrt{\frac{1-i\cot\frac{\pi}{2}}{2\pi}}\, e^{\frac{i}{2\sin\frac{\pi}{2}}[(x^2+y^2+u^2+v^2)\cos\frac{\pi}{2} - 2(xu+yv)]} dx\, dy$$

$$= \sqrt{\frac{1}{2\pi}} \int_0^\infty \int_0^\infty f(x,y)\, e^{-i(xu+yv)}\, dx\, dy$$

$$= 2DF\{f(x,y)\}$$

V. OPERATION TRANSFORMS FORMULAE FOR 2DFRFT

A. Linearly property

If $2DFRFT\{f(x,y)\}$ & $2DFRFT\{g(x,y)\}$ is generalized fractional Fourier transform of $f(x,y)$ and $g(x,y)$ then

$2DFRFT\{C_1 f(x,y) + C_2 g(x,y)\}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [C_1 f(x,y) + C_2 g(x,y)]\, k_\alpha(x,y,u,v)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_1 f(x,y)\, k_\alpha(x,y,u,v)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_2 g(x,y)\, k_\alpha(x,y,u,v)\, dx\, dy$$

$$= C_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, k_\alpha(x,y,u,v)\, dx\, dy + C_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, k_\alpha(x,y,u,v)\, dx\, dy$$

$$= C_1[2DFrFT(x,y)](u,v) + C_2[2DFrFT(x,y)](u,v)$$

B. Differential property

Result:

$2DFrFT\{f'(x,y)\}(u,v) = (-i\cot\alpha)\, 2DFRFT\{xf(x,y)\}(u,v) + (iu\, \mathrm{cosec}\alpha)\, 2DFRFT\{f(x,y)\}(u,v)$

Consider

$2DFrFT\{f'(x,y)\}(u,v) =$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{1\alpha} e^{\frac{i}{2\sin\alpha}[(x^2+y^2+u^2+v^2)\cos\alpha - 2(xu+yv)]} f'(x,y)\, dx\, dy$$

$$= C_{1\alpha} e^{\frac{i}{2}[(u^2+v^2)\cot\alpha]} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[(x^2+y^2)\cot\alpha - i(xu+yv)]\,\mathrm{cosec}\alpha} f'(x,y)\, dx\, dy$$

$$= C_{1\alpha} e^{\frac{i}{2}[(u^2+v^2)\cot\alpha]} \int_{-\infty}^{\infty} e^{\frac{i}{2}y^2\cot\alpha - iyv\,\mathrm{cosec}\alpha} \int_{-\infty}^{\infty} e^{\frac{i}{2}x^2\cot\alpha - ixu\,\mathrm{cosec}\alpha} f'(x,y)\, dx\, dy$$

$$= C_{1\alpha} e^{\frac{i}{2}[(u^2+v^2)\cot\alpha]} \int_{-\infty}^{\infty} e^{\frac{i}{2}y^2\cot\alpha - iyv\,\mathrm{cosec}\alpha} \left\{\left[e^{\frac{i}{2}x^2\cot\alpha - ixu\,\mathrm{cosec}\alpha} f(x,y)\right]_{-\infty}^{\infty} - \right.$$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 2, August 2013

$$\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2 \cot\alpha - ixu\,cosec\alpha}\left(\frac{i}{2}2x\cot\alpha - iu\,cosec\alpha\right) f(x,y)\,dx\Big\}dy$$

$$= C_{1\alpha} e^{\frac{i}{2}[(u^2+v^2)\cot\alpha]} \int_{-\infty}^{\infty} e^{\frac{i}{2}y^2\cot\alpha - iyv\,cosec\alpha}$$

$$\Big\{-\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2\cot\alpha - ixu\,cosec\alpha}(ix\cot\alpha - iu\,cosec\alpha)\, f(x,y)\,dx\Big\}\,dy$$

$$= (-i\cot\alpha)C_{1\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[(u^2+v^2)\cot\alpha]} e^{\frac{i}{2}[(x^2+y^2)\cot\alpha - i(xu+yv)]\,cosec\alpha} f(x,y)\,dx\,dy$$

$$+iu\,cosec\,\alpha\, C_{1\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[(u^2+v^2)\cot\alpha]} e^{\frac{i}{2}[(x^2+y^2)\cot\alpha - i(xu+yv)]\,cosec\alpha} f(x,y)\,dx\,dy$$

$$= (-i\cot\alpha)2DFrFT\{xf(x,y)\}(u,v) + (iu\,cosec\alpha)2DFrFT\{f(x,y)\}(u,v)$$

Similarly we can find

$$(-i\cot\alpha)2DFrFT\{yf(x,y)\}(u,v) + (iv\,cosec\alpha)2DFrFT\{f(x,y)\}(u,v)$$

C. *First shifting property*

$$2DFrFT\{e^{i(ax+by)} f(x,y)\}(u,v) = e^{\left(\frac{a^2+b^2}{2}\right)\sin 2\alpha}\, e^{i[a(u-a\sin\alpha)+b(v-b\sin\alpha)]\cos\alpha}\,[2FRFT(f(x,y)](u-a\sin\alpha)\,(v-b\sin\alpha)$$

Proof: Consider

$$2DFrFT\{e^{i(ax+by)} f(x,y)\}(u,v)$$

$$= C_{1\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2\sin\alpha}[(x^2+y^2+u^2+v^2)\cos\alpha - 2(xu+yv)]}\, e^{i(ax+by)} f(x,y)\,dx\,dy$$

$$= C_{1\alpha}e^{\frac{i}{2}[(u^2+v^2)\cot\alpha]}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[(x^2+y^2)\cot\alpha - i(xu+yv)]\,cosec\alpha}\, e^{i(ax+by)} f(x,y)\,dx\,dy$$

$$= C_{1\alpha}e^{\frac{i}{2}[(u^2+v^2)\cot\alpha]}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}. e^{\frac{i}{2}[(x^2+y^2)\cot\alpha - ix[u\,cosec\alpha - a\sin\alpha\,cosec\alpha] - iy[v\,cosec\alpha - b\sin\alpha\,cosec\alpha]]} f(x,y)\,dx\,dy$$

$$= C_{1\alpha}e^{\frac{i}{2}[(u^2+v^2)\cot\alpha]}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}. e^{\frac{i}{2}[(x^2+y^2)\cot\alpha - ix\,cosec\alpha[u-a\sin\alpha] - iy\,cosec\alpha[v-b\sin\alpha]]} f(x,y)\,dx\,dy$$

$$= C_{1\alpha}e^{\frac{i}{2}[((u+a\sin\alpha-a\sin\alpha)^2+(v+b\sin\alpha-b\sin\alpha)^2)\cot\alpha]}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[(x^2+y^2)\cot\alpha - ix\,cosec\alpha[u-a\sin\alpha] - iy\,cosec\alpha[v-b\sin\alpha]]} f(x,y)\,dx\,dy$$

$$= C_{1\alpha}e^{\frac{i}{2}\left[\left(\begin{matrix}[(u-a\sin\alpha)^2-2a\sin\alpha(-u+a\sin\alpha)]+\\ [(v-b\sin\alpha)^2-2b\sin\alpha(-v+b\sin\alpha)]\end{matrix}\right)\cot\alpha\right]}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[(x^2+y^2)\cot\alpha - ix\,cosec\alpha[u-a\sin\alpha] - iy\,cosec\alpha[v-b\sin\alpha]]}\, e^{(a^2+b^2)\sin^2\alpha\cos\alpha} f(x,y)\,dx\,dy$$

$$= e^{\frac{i}{2}[2a(u-a\sin\alpha)\cos\alpha]}\, e^{\frac{i}{2}[2b(v-b\sin\alpha)\cos\alpha]}\, e^{(a^2+b^2)\sin\alpha\cos\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{1\alpha}e^{\frac{i}{2}[(u-a\sin\alpha)^2+(v-b\sin\alpha)^2]\cot\alpha}\, e^{\frac{i}{2}[(x^2+y^2)\cot\alpha - ix\,cosec\alpha[u-a\sin\alpha] - iy\,cosec\alpha[v-b\sin\alpha]]} f(x,y)\,dx\,dy$$

$$= e^{\left(\frac{a^2+b^2}{2}\right)\sin 2\alpha}\, e^{i[a(u-a\sin\alpha)+b(v-b\sin\alpha)]\cos\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{1\alpha}e^{\frac{i}{2\sin\alpha}\left\{\begin{matrix}[x^2+y^2+(u-a\sin\alpha)^2+(v-b\sin\alpha)^2]\cos\alpha\\ -2[x(u-a\sin\alpha)+y(v-b\sin\alpha)]\end{matrix}\right\}} f(x,y)\,dx\,dy$$

$$= e^{\left(\frac{a^2+b^2}{2}\right)\sin 2\alpha}\, e^{i[a(u-a\sin\alpha)+b(v-b\sin\alpha)]\cos\alpha}\,[2FRFT(f(x,y)](u-a\sin\alpha)\,(v-b\sin\alpha)$$

D. *Scaling property*

$$2DFrFT\{f(ax,by)\}(u,v) = \frac{1}{ab}\{2DFrFT\ g(x,y)\}(u,v),$$

where

$$g(x,y) = e^{\frac{i}{2}}\left\{\left(\frac{1-a}{a}\right)\left[\left(\frac{1+a}{a}\right)(ax)^2\cot\alpha - 2u(ax)cosec\alpha\right] + \left(\frac{1-b}{b}\right)\left[\left(\frac{1+b}{b}\right)(by)^2\cot\alpha - 2v(by)cosec\alpha\right]\right\}$$

Consider

$$2DFrFT\{f(ax,by)\}(u,v)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{1\alpha}e^{\frac{i}{2\sin\alpha}[(x^2+y^2+u^2+v^2)\cos\alpha - 2(xu+yv)]} f(ax,by)\,dx\,dy$$

$$= C_{1\alpha}e^{\frac{i}{2}(u^2+v^2)\cot\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2)\cot\alpha - i(xu+yv)cosec\,\alpha} f(ax,by)\,dx\,dy$$

putting $ax = P$, $by = Q$

$a\,dx = dP$, $b\,dy = dQ$

$dx = \frac{dP}{a}$, $dy = \frac{dQ}{b}$

$$2DFrFT\{f(ax,by)\}(u,v)$$

$$= C_{1\alpha}e^{\frac{i}{2}(u^2+v^2)\cot\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{P^2}{a^2}+\frac{Q^2}{b^2}\right)\cot\alpha - i\left(\frac{P}{a}u+\frac{Q}{b}v\right)cosec\,\alpha} f(P,Q)\frac{dP}{a}\frac{dQ}{b}$$

$$= C_{1\alpha}e^{\frac{i}{2}(u^2+v^2)\cot\alpha}\frac{1}{ab}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(1+\frac{1-a^2}{a^2}\right)P^2\cot\alpha}\, e^{\frac{i}{2}\left(1+\frac{1-b^2}{b^2}\right)Q^2\cot\alpha}\, e^{-i\left(\frac{P}{a}u+\frac{Q}{b}v\right)cosec\,\alpha} f(P,Q)\,dP\,dQ$$

$$= \frac{C_{1\alpha}}{ab}e^{\frac{i}{2}(u^2+v^2)\cot\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[P^2+Q^2]\cot\alpha}\, e^{-i[uP+vQ]cosec\alpha}\, e^{\frac{i}{2}\left[\left(\frac{1-a^2}{a^2}\right)P^2+\left(\frac{1-b^2}{b^2}\right)Q^2\right]\cot\alpha}\, e^{-i\left[uP\left(\frac{1-a}{a}\right)+vQ\left(\frac{1-b}{b}\right)\right]cosec\alpha} f(P,Q)\,dP\,dQ$$

$$= \frac{C_{1\alpha}}{ab}e^{\frac{i}{2}(u^2+v^2)\cot\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[P^2+Q^2]\cot\alpha - i[uP+vQ]cosec\alpha}\, e^{\frac{i}{2}\left[\left(\frac{1-a^2}{a^2}\right)P^2+\left(\frac{1-b^2}{b^2}\right)Q^2\right]\cot\alpha - i\left[uP\left(\frac{1-a}{a}\right)+vQ\left(\frac{1-b}{b}\right)\right]cosec\alpha} f(P,Q)\,dP\,dQ$$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 2, August 2013

$$= \frac{1}{ab}\{[2DFrFT\; e^{\frac{i}{2}\left[\left(\frac{1-a^2}{a^2}\right)P^2+\left(\frac{1-b^2}{b^2}\right)Q^2\right]\cot\alpha - i\left[uP\left(\frac{1-a}{a}\right)+vQ\left(\frac{1-b}{b}\right)\right]cosec\alpha} f(P,Q)](u,v)\}$$

$$= \frac{1}{ab}\{[2DFrFT\; g(x,y)](u,v)\}$$

$$where\; g(x,y)$$

$$= e^{\frac{i}{2}\left[\left(\frac{1-a^2}{a^2}\right)P^2+\left(\frac{1-b^2}{b^2}\right)Q^2\right]\cot\alpha - i\left[uP\left(\frac{1-a}{a}\right)+vQ\left(\frac{1-b}{b}\right)\right]cosec\alpha} f(P,Q)$$

$$= e^{\frac{i}{2}\left[\left(\frac{(1-a)(1+a)}{a^2}\right)P^2+\left(\frac{(1-b)(1+b)}{b^2}\right)Q^2\right]\cot\alpha - i\left[uP\left(\frac{1-a}{a}\right)+vQ\left(\frac{1-b}{b}\right)\right]cosec\alpha} f(P,Q)$$

$$= e^{\frac{i}{2}\left\{\left(\frac{1-a}{a}\right)\left[\frac{1+a}{a}P^2\cot\alpha - 2uP\; cosec\alpha\right]\right\}} e^{\frac{i}{2}\left\{\left(\frac{1-b}{b}\right)\left[\frac{1+b}{b}Q^2\;\cot\alpha - 2vQ\; cosec\alpha\right]\right\}} f(P,Q)$$

$$= e^{\frac{i}{2}\left\{\begin{array}{c}\left(\frac{1-a}{a}\right)\left[\frac{1+a}{a}(ax)^2\cot\alpha - 2u(ax)cosec\alpha\right] \\ +\left(\frac{1-b}{b}\right)\left[\frac{1+b}{b}(by)^2\cot\alpha - 2v(by)\,cosec\alpha\right]\end{array}\right\}} f(P,Q)$$

REFERENCES

[1] R. N. Bracewll, “The Fourier Transform and its Application”, 2nd ed., McGraw-Hill, New York, 1986.

[2] H.M. Ozaktes, A. Zalevsky, M.A. Katay, “The FRFT with Application in optics and signal processing”, John Wiky, Chichester New York, USA, 2001.

[3] V.D.Sharma, “Generalization of Two-dimensional Fractional Fourier transforms”, Asian Journal of current Engineering and Maths, 46-48, 2012.

[4] V.D.Sharma, “Operation transform Formulae on Generalized FrFT”, Processing of Int. conf. in Recent trends in Inf. Tech. and Comp. Science published in Int. J. of computer applications 15-18, 2012.

[5] Bultheel A. et.al., “Recent developments in the theory of the fractional Fourier & linear canonical transform”, Preprint (2004), 1-33.

[6] Felipe Salazon, “An Introduction to the Fractional Fourier transform and its applications”, Dec. 7, (2011) 1-8

[7] Zemanian A.H., “Generalized Integral transform”, Interscience Publication, New York, 1968.

[8] B.N.Bhosale, “Integral transformation of Generalized functions”, Discovery publishing hour, 2005.

[9] Ervin S., Djurovic I. et.al., “Fractional Fourier transform as a signal processing tool; an review of recent developments”, signal processing (2011), 1351-1369.

[10] V.D. Sharma, “Generalized Two-Dimensional Fractional Fourier Transform and Inversion Formula”, Journal of Fractional Calculus and Applications, Vol. 4(2) July 2013, pp. 191- 195.