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Complementary Tree Domination Number of Circular-Arc Graphs

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***Abstract*— In this present paper, we concentrated on the theory of complimentary tree domination in graphs and focused on resolving the complementary tree domination number of circular-arc graphs. Some categorized circular-arc graphs are chosen in this course of study.**

***Index Terms*—Circular-Arc Graph, Dominating Set, Complementary Tree Dominating Set, Complementary Tree Domination Number.**

I. INTRODUCTION

Complementary tree dominating set problem is the problem of finding whether the given graph has a complementary tree dominating set of a specified size. A subset S of the vertex set V of the graph G = (V, E) is a dominating set of the graph G if every vertex in the vertex set V is either a member of S or is adjacent to some vertex of S. The domination number of the graph G is denoted by γ (G) and is the minimum cardinality of a dominating set of G. Dominating sets play an imperative role in algorithmics and in combonatorics. Allan and Laskar [1] derived many results pertaining to dominating sets. Hedetniemi and Laskar in [2] listed over 300 papers related to domination in graphs. In complexity theory, dominating set was one of the first problems recognized as NP-complete. In [3] - [4] Haynes et al. discussed dominating sets in detail. A dominating set S $\subseteq$ V of a graph G with vertex set V (G) and edge set E (G) is a complementary tree dominating set if the induced sub graph < V – S > is a tree. Complementary tree domination number is the minimum cardinality of a complementary tree dominating set of G. It is denoted by γ_{ctd} (G). The notion of complementary tree dominating set is due to S. Muttamai et al. [5]. Some results pertaining to the bounds of Complementary tree domination number is obtained by them. Circular-arc graphs are a new class of intersection graphs, defined for a set of arcs on a circle. A graph is a circular - arc graph, if it is the intersection graph of a finite set of arcs on a circle. That is, there exists one arc for each vertex of G and two vertices in G are adjacent in G, if and only if the corresponding arcs intersect. A vertex is said to dominate another vertex if there is an edge between the two vertices. Let

A = {A_1, A_2, A_3,…, A_n} be a circular – arc family on a circle, where all the arcs together cover the entire circle. An arc A_i that begins at endpoint p_i and ends at end point q_i considered in the clockwise direction is denoted by (p_i, q_i). Two arcs A_i and A_j are said to intersect each other if they have non-empty intersection. A representation of a graph with arcs helps in the solving of combinatorial problems on the graph. Neighborhood of an arc A_i is defined as the set of all arcs belonging to A that intersect the arc A_i. Graphs considered in this paper are all undirected, connected and simple graphs. Throughout this paper, for the graph G = (V, E) and for S $\subseteq$ V, the sub graph of G induced by the vertices in S is denoted by ‹ S ›. A vertex of degree one is called a support. In section 2, preliminary information regarding the bounds of a complementary tree domination number and the stand point information regarding the relation between the domination number and complementary tree domination number are outlined.

A. Our Results:

In our recent paper [6], we put forth the findings related to complementary tree domination number of interval graphs. The exact value of complementary tree domination number and minimal complementary tree domination sets of some particular classes of interval graphs are obtained. In this paper, we extended our study on complementary tree domination number and we put forward some results regarding the complementary tree domination number of circular-arc graphs.

II. SOME RESULTS

The following results are obtained by S. Muttamai et al. [5] and these results characterize ctd-sets.

Result 2.1: Bounds of complementary tree domination number Let G be a connected graph of order k $\geq$ 2. Then γ_{ctd} (G) $\leq$ k-1.

Result 2.2: Relation between domination number and complementary tree domination number

Let G be a connected interval graph of order n $\geq$ 2. Then γ (G) $\leq$ γ_{ctd}(G) . For any graph G, every complementary tree dominating set is a dominating set. But every dominating set need not be a complementary tree dominating set. Hence the result follows.

Result 2. 3: Every pendant vertex is a member of all ctd-sets.

III. COMPLEMENTARY TREE DOMINATION NUMBER OF SOME CIRCULAR-ARC GRAPHS

Theorem 3.1: Let A = {A_1, A_2, A_3… A_n}, n $\geq$ 3 be the circular-arc family corresponding to a circular-arc graph G. For any three consecutive arcs A_i, A_j and A_k, if A_j does not dominate any arc other than A_i and A_k and furthermore, domination of A_1 by A_n is the only other domination that occurs in the family, then

$$\gamma_{ctd} (G) = n-2$$

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Proof: Let G be the circular-arc graph, whose circular-arc family A= $\{A_1, A_2, A_3, \ldots, A_n\}$ satisfies the conditions mentioned in the theorem. By the hypothesis,
the arc A_1 dominates the arcs A_2 and A_n;
the arc A_2 dominates the arcs A_1 and A_3;
the arc A_3 dominates the arcs A_2 and A_4;
the arc A_4 dominates the arcs A_3 and A_5;
………………………………………;
………………………………………;
the arc A_{n-2} dominates the arcs A_{n-3} and A_{n-1};
the arc A_{n-1} dominates the arcs A_{n-2} and A_n;
and the arc A_n dominates the arcs A_{n-1} and A_1.
Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices corresponding to the arcs $A_1, A_2, A_3, \ldots, A_n$ respectively. Let $S_i = \{ v_1, v_2, \ldots, v_{i-1}, v_{i+2}, v_{i+3}, \ldots, v_n\}$ for $i = 1, 2, 3, \ldots, n-1$ and $S_n = \{ v_2, v_3, \ldots, v_{n-1}\}$.
Then, $V - S_i = \{v_i, v_{i+1}\}$ for $i = 1, 2, 3, \ldots, n-1$. Since the set of arcs A_{i-1}, A_i and A_{i+1} and the set of arcs A_i, A_{i+1} and A_{i+2} are two sets of three consecutive arcs, in the set $V - S_i$, the vertex v_i is adjacent to the vertex v_{i-1} and the vertex v_{i+1} is adjacent to the vertex v_{i+2} for $i = 1, 2, \ldots, n - 1$. So the set S_i is a dominating set and also the induced subgraph $< V - S_i >$ consists of only two vertices v_i and v_{i+1} with an edge between them and it is a tree. Therefore, the set S_i is a ctd-set for each $i = 1, 2, 3, \ldots, n-1$. Also, $V - S_n = \{v_1, v_n\}$. By hypothesis, v_1 is adjacent to v_2 and v_n is adjacent to v_{n-1} and it follows that the induced subgraph $< V - S_n >$ is a tree. Therefore, the set S_n is a ctd-set. Hence, the set S_i is a ctd-set for each $i = 1, 2, 3, \ldots, n$. Cardinality of S_i is n-2 for each $i =1, 2, \ldots, n$. Let $S^1= \{S_i/i=1, 2, \ldots, n\}$. It can be observed that, any set of cardinality n-2 other than in S^1 is not a ctd-set. Therefore, any ctd-set with cardinality less than n-2 is a subset of S_i for some i. Cardinality of S_i is n-2. It follows that,

$$\gamma_{ctd}(G) \leq n - 2.$$

Now we shall show that any subset of S_i is not a ctd-set for any i, where i=1, 2, …, n.
First, let $S_i' = S_i - v_j$ for i=1, 2, ……., n-1 and $j \neq i, i+1$ and $1 \leq j \leq n-1$.
Here three cases will arise.
Case (i): The vertex v_j may be predecessor of v_i. Then v_i is not adjacent to any vertex of S_i'. Implies S_i' is not a dominating set. It follows that S_i' is not a ctd-set.
Case (ii): The vertex v_j may be successor of the vertex v_{i+1}. Then v_{i+1} is not adjacent to any vertex of S_i'. Implies S_i' is not a dominating set. It follows that S_i' is not a ctd-set.
Case (iii): The vertex v_j may be neither predecessor of v_i nor successor of the vertex v_{i+1}. Then S_i' is a dominating set. But the induced sub graph $< V - S_i' >$ is not a connected graph in this case. The graph $< V - S_i' >$ is not a tree. Implies the set S_i' is not a ctd-set. It follows that; S_i' is not a ctd-set in all the possible three cases. Secondly, let

$$S_n' = S_n - v_j, \text{ where } 2 \leq j \leq n-1.$$

Again three cases will arise.
Case (i): The vertex v_j may be v_2. Then v_1 is not adjacent to any vertex of S_n'. Implies S_n' is not a dominating set. It follows that S_n' is not a ctd-set.
Case (ii): The vertex v_j may be v_{n-1}. Then v_n is not adjacent to any vertex of S_n'. Implies S_n' is not a dominating set. It follows that S_n' is not a ctd-set.
Case (iii): The vertex v_j may be different from v_2 and v_{n-1}. Then S_n' is a dominating set. But the induced sub graph $< V - S_n' >$ is not a connected graph in this case. The graph $< V - S_n' >$ is not a tree. Implies, the set S_n' is not a ctd-set. It follows that; S_n' is not a ctd-set in all the possible three cases. Hence, any subset of S_i for i=1, 2, …., n is not a ctd-set.

$$\gamma_{ctd}(G) \geq n - 2$$

From (1) and (2), it follows that

$$\gamma_{ctd}(G) = n - 2$$

Illustration: Let the circular-arc family A= $\{A_1, A_2, A_3, \ldots, A_n\}$, $n \geq 3$ corresponding to a circular-arc graph G be as in Fig. 1

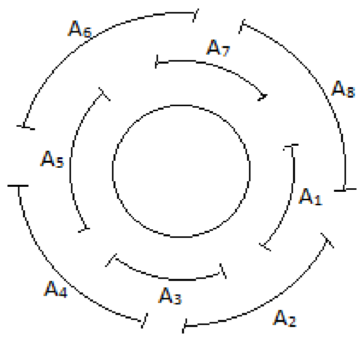

Fig.1. Circular-arc family

The circular-arc family satisfies all the conditions mentioned in the theorem for n = 8.The complementary tree domination number of the graph, $\gamma_{ctd}(G) = 6$. The minimal ctd-sets of the graph are
$\{A_3, A_4, A_5, \ldots, A_8\}$;
$A_1, A_4, A_5, \ldots, A_8\}$;
$\{A_1, A_2, A_5, A_6 \ldots, A_8 \}$;
………………………….;
$\{A_1, A_2, A_3, \ldots, A_{8-2}\}$

Theorem3. 2: Let A = $\{A_1, A_2, \ldots, A_n\}$, $n \geq 2$ be a circular-arc family analogous to a circular-arc graph G. In a condition, wherein the arcs A_i, A_j are intersecting arcs and any arc of A other than A_i and A_j does not dominate any other arc except A_i or A_j, but not both, then

$$\gamma_{ctd}(G) = n-1$$

iff all the arcs in A- $\{ A_i, A_j \}$ dominate A_i or all the arcs in A-$\{A_i, A_j\}$ dominate A_j, where $i \neq j$, $1 \leq i \leq n$ and $1 \leq j \leq n$.

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Proof: Let A= {A_1, A_2, A_3, ….., A_n}, $n \geq 2$ be the circular-arc family corresponding to a circular-arc graph G. Let v_1, v_2, v_3, …, v_n be the vertices corresponding to the arcs A_1, A_2, A_3, ….., A_n respectively.

First, let the circular-arc family A analogous to a circular-arc graph G satisfy the conditions mentioned in the theorem and let all the arcs in A- { A_i, A_j } dominate A_i . Then G is a $k_{1, n-1}$ graph. Then the partites of the graph are U={v_i} and V={ v_1, v_2, …., v_{i-1}, v_{i+1} ……, v_n}for some i, where $1 \leq i \leq n$. Vertices v_1, v_2, ……, v_{i-1}, v_{i+1} …, v_n are pendant vertices. Every ctd-set consists of all pendant vertices. Therefore the vertices v_1, v_2, ….., v_{i-1}, $v_{i+1,}$ ……, v_n are members of every ctd-set. As a result,

$$\gamma_{ctd}(G) \geq n-1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Similarly, it can be proved that result (i) holds even if all the arcs in A- { A_i, A_j } dominate the arc A_j. But for any connected graph G with $n \geq 2$, $\gamma_{ctd}(G) \leq n-1$……..(ii)

From (i) and (ii)

$$\gamma_{ctd}(G) = n-1$$

Conversely, let $\gamma_{ctd}(G) = n-1$ ……………..(iii)

Let D ={ v_1, v_2, ………..…, v_{i-1}, v_{i+1} …………, v_n}for some i=1, 2, ……., n be the minimal ctd-set of G. Then V (G) – D ={v_i} for some i = 1, 2, ….., n. D is a ctd-set of G. Then D is a dominating set of G. Every vertex of the vertex set V not in D is adjacent to some vertex in D. So, the vertex v_i is adjacent to some vertex of D, say v_j for some j, where $1 \leq j \leq n$ and $j \neq i$. The vertex v_j is not adjacent to any vertex in D. If not, let v_j be adjacent to some vertex of D, say v_k for some k, where $1 \leq k \leq n$ and $k \neq j \neq i$. Also, the vertex v_i is adjacent to another vertex of D other than v_j. If v_i is adjacent only to the vertex v_j, then v_i is a pendant vertex and must be a member of ctd-set, which is not the case here. So, other than the vertex v_j, vertex v_i is adjacent to another vertex in D. Implies the set D-{v_j} is also a dominating set of the graph G. Also the induced sub graph < (V – D) ∪ {v_j} > is a tree. The set D-{v_j} is a ctd-set. As a result, $\gamma_{ctd}(G) \leq n-2$. This contradicts our supposition that $\gamma_{ctd}(G) = n-1$. Hence v_j is not adjacent to any of the vertex in D and is a pendant vertex. This is true for every vertex in D that is adjacent to v_i. Now suppose that there exists a vertex v_k in D that is not adjacent to v_i. Graph G is a connected graph. So vertex v_k must be adjacent some vertex v_p ϵ D. Vertex v_p may be or may not be adjacent to v_i. Case(i):let v_p be adjacent to v_i. Then V – D – {v_p} = {$v_{i,}$ v_p} and D - v_p is a dominating set as vertex v_i is adjacent to vertex v_j and vertex v_p is adjacent to the vertex v_k. Moreover, the induced sub graph < (V – D) ∪ {v_p} > is a tree. It follows that, the set D – {v_p} is a ctd-set which contradicts (iii). So vertex v_p is not adjacent to v_i. Case (ii): let v_p be not adjacent to v_i. G is a connected graph. There exists a path between the pair of vertices v_p and v_i. In the path from v_p to $v_{i,}$ let the vertex adjacent to v_i be v_r. Since v_i is not a pendant vertex, the set D – {v_r} is a dominating set and the induced sub graph < (V – D) ∪ {v_r} > is a tree. It follows that, the set D – {v_r} is a ctd-set which contradicts (iii). Therefore, there does not exist a vertex in D that is not adjacent to v_i and no vertex in D is adjacent to none of its vertices. Hence, the graph G is a $k_{1, n-1}$ graph. Hence the circular-arc family must satisfy the conditions mentioned in the theorem.

Illustration: Let the circular-arc family A= {A_1, A_2, A_3, ….., A_n}, $n \geq 3$ corresponding to a circular-arc graph G be as in Fig. 2

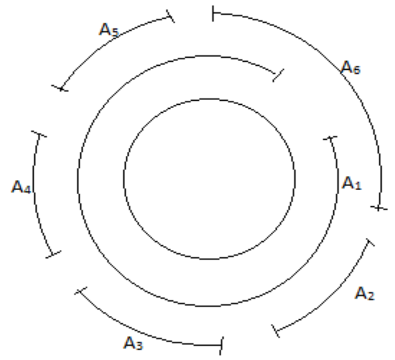

Fig.2. Circular-arc family

The circular-arc family satisfies all the conditions mentioned in the theorem for n = 6, i = 1and j = 2, 3, 4, 5, 6.The complementary tree domination number of the graph, $\gamma_{ctd}(G) = 5$. The minimal ctd-set of the graph is {A_2, A_3, A_4, A_5, A_6}.

Theorem 3. 3: Let A = {A_1, A_2, ………, A_n}, $n \geq 4$ be a circular-arc family analogous to a circular-arc graph G. In a condition, wherein the arcs A_i, A_j are intersecting arcs and any arc of A other than A_i and A_j doesn't dominate any other arc except A_i or A_j, but not both, then

$$\gamma_{ctd}(G) = n-2$$

if some of the arcs in A-{ A_i, A_j } dominate A_i and some of the arcs in A-{A_i, A_j} dominate A_j .

Proof: Let the circular-arc family A = {A_1, A_2, ….., A_n}, $n \geq 4$ satisfy the conditions mentioned in the hypothesis. Some of the arcs in A-{$A_{i,}$ A_j} may dominate A_i and the remaining arcs in A-{$A_{i,}$ A_j} may dominate A_j. Let $v_{1,}$ $v_{2,}$ $v_{3,}$ ….., v_n be the vertices corresponding to the arcs A_1, A_2, ………………., A_n respectively. By the conditions for domination between the arcs of the circular-arc family A, it is clear that except the vertices v_i and v_j, the remaining n-2 vertices are pendant vertices. Every pendant vertex is a member of complementary tree dominating set. Therefore

$$\gamma_{ctd}(G) \geq n-2 \qquad \ldots\ldots\ldots\ldots (1)$$

Let S = V - {$v_{i,}$ v_j}. Then V – S = {$v_{i,}$ v_j}. Since, some of the vertices in V are adjacent to v_i and some of the vertices are adjacent to v_j, vertices v_i and v_j are adjacent to at least one vertex in the set S. The set S is a dominating set. As the arcs A_i and A_j are intersecting arcs, the induced sub graph < V – S >

consists of two vertices v_i and v_j with an edge between them. Implies sub graph < V – S > is a tree. It follows that S is a ctd-set, wherein the cardinality of S is n -2. Hence

$$\gamma_{ctd}(G) \leq n\text{-}2 \ \ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2), it is clear that $\gamma_{ctd}(G) = n\text{-}2$ with minimal ctd-set as A - { A_i, A_j }.

Illustration: Let the circular-arc family A= {A_1, A_2, A_3,……….., A_7}, $n \geq 4$ corresponding to a circular-arc graph G be as in Fig. 3

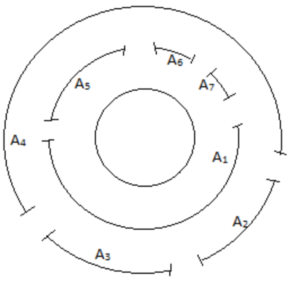

Fig.3. Circular-arc family

The circular-arc family satisfies all the conditions mentioned in the theorem for n = 7, i = 1and j = 4.The complementary tree domination number of the graph, $\gamma_{ctd}(G) = 5$. The minimal ctd-set of the graph is {A_2, A_3, A_5, A_6, A_7}.

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