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Two Problems of Contact Deformation under Real Scheme of Contact Bodies are considered

X.I.Musayev[1], R.S.Guliyev[2], A. O. Yuzbashiyeva[3]
[1,2,3] National Academy of Sciences of Azerbaijan Institute of Mathematics and Mechanics
Baku, AZ 1143, str. B. Vahabzadeh 9

***Abstract-** Existing solutions of flat contact deformation problems are based on Flaman's problem about deformation elastic half space was affected by concentrated force, and force is submitted as distribution of superficial pressure on conditionally cut out cylindrical surface of small radius. Such statement does not correspond to a real picture of interaction of bodies at contact.*

Two problems of flat contact deformation are considered under real contact conditions. The solution of a problem on introduction of the elastic cylinder into absolutely rigid flat body is received through the biharmonic equation. The solution of a problem on introduction of absolutely rigid cylinder into an elastic flat body is received through a general view of stress function for two-dimensional problems in polar coordinates. Owing to absolute rigidity of one body contact surfaces in both problems are determined on the basis of geometrical reasons. The received strict solutions yield the results close to Hertz's solution.

Index Terms: **contact, deformation, elastic, cylinder, stress, platform, surface.**

I. INTRODUCTION

In the theory of elastic strength the decision of the flat contact task is based on Hertz's theory, in which flat deformation is considered as an extreme case of contact of bodies on elliptic area, when one half-axle of an ellipse aspires to infinity [1]. Other approach consists in consideration of a task of cave-in of a stamp in elastic half-space [1].

In a basis of the received decisions of Flaman's task about deformation elastic half-space from the concentrated force, in which force lays in which the force is represented by distribution of superficial pressure on the conditionally cut out cylindrical surface of small radius, that strictly does not correspond to a real picture of interaction of the bodies at contact.

II. THE PROBLEM FORMULATION

Let's consider contact tasks, when the surface of contact is a flat platform that is observed at contact of cylinders of identical radius, at contact of the cylinder to an absolutely rigid flat surface. In particular, we shall consider the decision of a contact task for the elastic cylinder of radius R and absolutely rigid body (Fig.1). We shall present the pressure on a platform of contact of the elastic cylinder as $p = p_0 \cdot f(x)$, and at $x=0$, $f(x)=1$, and $f'(x)=0$, at $x=b$, $f(x)=0$, where $b >> R$, half breadth of the contact zone. Then force of cave-in of the cylinder

$$F = 2p_0 \cdot \int_0^b f(x)dx. \quad (1)$$

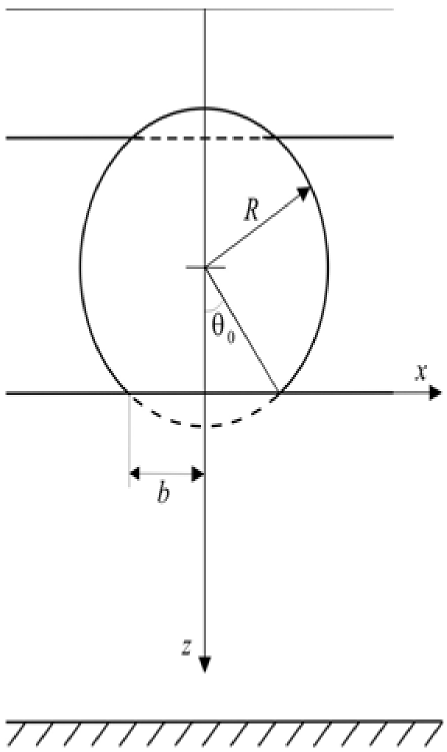

Fig.1. Scheme elastic cylinder of radius R and the absolutely rigid body

We shall present stresses σ_z as

$$\sigma_z = \sigma_0(\varphi(z)-1) - p_0 \cdot \varphi(z) \cdot f(x),$$

where $\sigma_0 = \sigma_0(x,z)$ - is a stress in a cylinder (disk) from compressing force F_1 [1]. Function $\varphi(z)$: on a surface at $z=0$, $\varphi_{(z)}=1$, stresses $\sigma_z = p$. At $z \to L$, $\varphi(z) \to 0$, then pressure $\sigma_z \to \sigma_0$.

From the equation of balance on an axis z

$$\tau_{xz} = p_0 \cdot \varphi'_{(z)} \left[\int f(x)dx - \int \frac{\partial(\varphi_{(z)}\sigma_0)}{\partial z} dx \right].$$

The expression for σ_z and τ_{xz} consists of two members: one depends on the distribution of external pressure and works within the limits of a zone of contact $0 \le x \le b$, another depends on the general decision, in which stresses work on all section.

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Further we shall consider the decision in the field of contact, then

$$\sigma_z = -p_0 \cdot \varphi(z) \cdot f(x),$$
$$\tau = \tau_{zx} = \tau_{xz} = p_0 \cdot \varphi'(z) \int f(x)dx,$$
$$\sigma_x = -p_0 \cdot \varphi''(z) \iint f(x)dxdx.$$

That is we receive the known decision bidimensional of a flat task with function of stresses [2], [3]

$$\Phi(x,z) = -p_0 \varphi(z) F(x),$$

where $F(x) = \iint f(x)dxdx$.

Functions $\varphi(z)$ and $F(x)$ should also satisfy to the differential equation

$$\varphi(z) f''(x) + 2\varphi''(z) f(x) + \varphi^{IV}(z) \iint f(x)dxdx = 0 \quad (2)$$

If we accept that $f_1(x) = p_i \cos k_i x$, then decision of the (2) will be function $\varphi_i(z)$ as $\varphi_i(z) = (1 + k_i z)e^{-k_i z}$, and $\varphi_i'(z) = -k_i^2 z e^{-k_i z}$; $\varphi_i''(z) = -k_i^2 (1 - k_i z) e^{-k_i z}$. Whence follows satisfaction of boundary conditions on surface at $z = 0$, $\tau_{xz} = \tau_{zx} = 0$.

The decision as well will be the function

$$\Phi(x.z) = \sum_i p_i \left(\frac{(1 + k_i z)}{k_i^2} \right) e^{-k_i z} \cos k_i x.$$

When we shall receive expression for the stresses

$$\sigma_z = -\sum_i p_i (1 + k_i z) e^{-k_i z} \cos k_i x,$$
$$\sigma_x = -\sum_i p_i (1 - k_i z) e^{-k_i z} \cos k_i x,$$
$$\tau_{xz} = -\sum_i p_i k_i z \cdot e^{-k_i z} \sin k_i x.$$

On a surface of contact $p = \sum_i p_i \cos k_i x$, at $x = 0$, $p_0 = \sum_i p_i$.

Relative deformations

$$\varepsilon_z = -\frac{1}{E} \sum_i p_i [(1 - \nu) + (1 + \nu) k_i z] \cdot e^{-k_i z} \cdot \cos k_i x,$$

$$\varepsilon_x = -\frac{1}{E} \sum_i p_i [(1 - \nu) - (1 + \nu) k_i z] \cdot e^{-k_i z} \cdot \cos k_i x.$$

Movements

$$W = \int \varepsilon_z dz = \frac{1}{E} \sum_i p_i \left[\frac{2}{k_i} + (1 + \nu) \cdot z \right] \cdot e^{-k_i z} \cdot \cos k_i x, \quad (3)$$

$$U = \int \varepsilon_x dx = -\frac{1}{E} \sum_i p_i \left[\left(\frac{1 - \nu}{k_i} \right) - (1 + \nu) \cdot z \right] \cdot e^{-k_i z} \cdot \sin k_i x.$$

at cave-in of the cylinder from geometrical parities of moving W of a contact surface $(z = 0)$.

$$W(x) = R\left(\sqrt{1 - \frac{x^2}{R^2}} - \sqrt{1 - \frac{b^2}{R^2}} \right) = \frac{b^2}{2R}\left(1 - \frac{x^2}{b^2} \right).$$

The received expression can be submitted as [4]

$$W(x) = \frac{32}{\pi^3} \cdot \frac{b^2}{2R} \cdot \sum_{i=0}^{\infty} \frac{(-1)^i \cos\left[\frac{(2i+1)\pi x}{2b} \right]}{(2i+1)^3}.$$

Having equated this expression to the decision (3) at $z = 0$, we shall receive that

$$k_i = \frac{(2i+1)\pi}{2b}, \text{ but } p_i = \frac{4bE}{\pi^2 R} \cdot \frac{(-1)^i}{(2i+1)^2}.$$

Whence follows that

$$p = \sum_0^{\infty} p_i \cos k_i x = -\frac{4bE}{\pi^2 R} \left(\frac{1}{2} \cdot \int_0^{\frac{\pi}{2}(1 - x/b)} \ln\left(tg \frac{z}{2} \right) dz \right), \quad (4)$$

$$p_0 = \sum_0^{\infty} p_i = \frac{4bE}{\pi^2 R} \cdot \sum_0^{\infty} \frac{(-1)^i}{(2i+1)^2} = \frac{3.66bE}{\pi^2 R}.$$

From (1) follows:

$$F = \frac{16b^2 F}{\pi^3 R} \cdot \sum_0^{\infty} \frac{1}{(2i+1)^3} = \frac{16.827b^2 E}{\pi^3 R},$$

whence half breadth of the zone of contact $b = 1.357\sqrt{\frac{FR}{E}}$, pressure in the center $p_0 = 0.5034\sqrt{\frac{FE}{R}}$, where for a flat task $E = \frac{E_0}{1 - \nu^2}$, in the center of the contact area

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$$\left.\begin{aligned} \overline{\sigma_z} = \frac{\sigma_z}{p_0} = \frac{\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} \cdot \left(1 + \frac{(2i+1)\pi z}{2b}\right) \cdot e^{-\frac{(2i+1)\pi z}{2b}}}{\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2}} \\ \overline{\sigma_x} = \frac{\sigma_x}{p_0} = \frac{\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} \cdot \left(1 - \frac{(2i+1)\pi z}{2b}\right) \cdot e^{-\frac{(2i+1)\pi z}{2b}}}{\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2}} \end{aligned}\right\}$$

(5)

as $\frac{\partial \tau}{\partial x} \equiv 0$ at $x = b$, that shearing stresses have their maximum on border of a zone of contact

$$\overline{\tau_{x=b}} = \overline{\tau_i} = \frac{\tau_i}{p_0} = \frac{\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)} \cdot \left(\frac{\pi z}{2b}\right) \cdot e^{-\frac{(2i+1)\pi z}{2b}}}{\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2}}.$$

(6)

The greatest shearing stresses $\max[\tau_1] = 0.392$ and $\frac{z}{b} = 0.75$.

The results of accounts under the (4, 5, 6) are submitted in a Fig. 2. It is necessary to note that use of the received decision in a return task-about cave-in of the absolutely rigid cylinder in an elastic body is not lawful. Since the contact surface will be cylindrical, and the pressure will be perpendicular of this surface [5], [6]. We shall consider in polar coordinates cave-in in an elastic body by the sizes $A \times H$ of the absolutely rigid cylinder of radius R, and $R << A$ (Fig.1).

Basic decision [7]

$$\sigma_r^0 = -\sigma_0 \cos^2 \theta, \; \sigma_\theta^0 = -\sigma_0 \sin^2 \theta,$$
$$\tau_{r\theta}^0 = \sigma_0 \sin\theta \cos\theta,$$

where $\sigma_0 = \frac{F}{A}$.

Contact pressure we shall receive, proceeding from known functions of a pressure [1], [8]:

$$\varphi = \sum_n \left(a_n \rho^{-n} + b_n \rho^{-n-2}\right) \cos n\theta.$$

We shall enter variable $r = \rho / R$.

From a condition: at $r = 1$, $\tau_{r\theta} = 0$, we shall receive

$$\varphi = \sum_k p_k \cos\beta_k\theta \cdot \left(\frac{r^{-\beta_k - 2}}{\beta_k - 1} - \frac{r^{-\beta_k}}{\beta_k + 1}\right).$$

Then contact pressure [9]

$$\sigma_r = \sum_k \frac{1}{2} p_k \left[\beta_k r^{-\beta_k - 2} - (\beta_k + 2) r^{-\beta_k}\right] \cos\beta_k\theta,$$
$$\sigma_\theta = -\sum_k \frac{1}{2} p_k \left[\beta_k r^{-\beta_k - 2} - (\beta_k - 2) r^{-\beta_k}\right] \cos\beta_k\theta,$$
$$\tau_{r\theta} = \sum \frac{1}{2} p_k \left[r^{-\beta_n - 2} - r^{-\beta_n}\right] \beta_k \sin\beta_k\theta,$$

at $\bar{r} = 1$; $\sigma_r = \sigma_\theta = -\sum p_k \cos\beta_k\theta$, $\tau_{r\theta} = 0$.

Totally radial pressure in an elastic body

$$\sigma_r = \sum_k \frac{1}{2} p_k \cos\beta_k\theta \cdot \left[\beta_k r^{-\beta_k - 2} - (\beta_k + 2) r^{-\beta_k}\right] - \sigma_r^0 \left[1 + \frac{1}{2}(\right.$$

,

are similarly represented σ_θ and $\tau_{r\theta}$.

At $\bar{r} = 1$ the second composed, determined by the basic decision addresses in 0. Further, since $A >> R$, then $\sigma_0 \approx 0$, and second composed it is possible to neglect, considering the decision only in a zone of contact, i.e. within the limits of $0 < \theta \le \theta_0$. From the condition $F = r \int_0^{\theta_0} \sigma_r R \cos\theta d\theta$, we shall receive:

$$F = 2R\cos\theta_0 \sum \frac{p_k (-1)^k \beta_k}{\beta_k^2 - 1}.$$

(7)

Relative deformation

$$\varepsilon_r = \frac{\sigma_r - \nu\sigma_0}{E} = \frac{1}{E} \sum \frac{1}{2} p_k \left[\beta_k r^{-\beta_k - 2}(1+\nu) - \left[(\beta_k + 2) - \nu(\beta_k - 2)\right] \cdot r^{-\beta_k}\right] \cos\beta_k\theta$$

Radial moving

$$U = \int \varepsilon_r R dr = \frac{R}{E} \sum p_k \left(\frac{2\beta_k + (1-\nu)}{\beta_k^2 - 1}\right) \cos\beta_k\theta.$$

Radial moving at cave-in of the absolutely rigid cylinder from geometry

$$U = R - R \cdot \frac{\cos\theta_0}{\cos\theta} = R \cdot \left(\frac{\cos\theta - \cos\theta_0}{\cos\theta}\right) \cong \frac{R\theta_0^2}{2\cos\theta} \left(1 - \frac{\theta^2}{\theta_0^2}\right)$$

The expression $1 - \frac{\theta^2}{\theta_0^2}$ can be presented as:

$$1 - \frac{\theta^2}{\theta_0^2} = \frac{32}{\pi^3} \sum \frac{(-1)^k \cos\left[(2k+1) \cdot \frac{\pi\theta}{2\theta_0}\right]}{(2k+1)^3}.$$

The accounts show, that in limits θ_0 from 0^0 to 20^0 within 0.5%

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$$\frac{1}{2}\cdot\frac{\theta_0^2}{\cos\theta}\left(1-\frac{\theta^2}{\theta_0^2}\right)=0.405\theta_0^2\left(1-\frac{\theta^2}{\theta_0^2}\right).$$

Equating expressions for U, we shall receive:

$$\beta_k=(2k+1)\frac{\pi}{2\theta_0},$$

$$p_k=12.96\cdot\frac{E\theta_0^2}{\pi^3}\cdot\frac{(-1)^k}{(2k+1)^3}\cdot\frac{\beta_k^2-1}{2\beta_k+(1-\nu)}.$$

Substituting the received expressions in (7), we shall receive

$$F=R\cos\theta_0\cdot\frac{12.96E\theta_0^2}{\pi^3}\sum\frac{1}{(2k+1)^3\left(1+\frac{(1-\nu)\theta_0}{(2k+1)\pi}\right)}$$

where contact angle

$$\theta_0=\sqrt{\frac{F}{RE}\cdot\frac{\pi^3}{12.96F_{1(\theta_0)}}}=1.547\sqrt{\frac{F}{REF_{1(\theta_0)}}},$$

where

$$F_{1(\theta_0)}=\cos\theta_0\sum\frac{1}{(2k+1)^3\left(1+\frac{(1-\nu)\theta_0}{(2k+1)\pi}\right)}.$$

Function $F_{1(\theta_0)}$ we shall present as

$$F_{1(\theta_0)}=k_1(\theta_0,\nu)\sum\frac{1}{(2k+1)^3}=k_1(\theta_0,\nu)\cdot 1.05172\ .$$

Accounts of factor $k_1(\theta_0,\nu)$ at $\nu=0;\ 0.3;\ 0.5$ when $\overline{\theta_0}=\frac{\theta_0}{\pi}=0.02;\ 0.04;\ 0.08;\ 0.10;\ 0.15$ are submitted in the Table I.

Table I. Meaning of factors $k_1(\theta_0,\nu)/k_2(\theta_0,\nu)$

ν	$\overline{\theta_0}$				
	0.02	0.04	0.08	0.10	0.15
0.5	0.989/0.993	0.975/0.985	0.938/0.963	0.916/0.949	0.850/0.866
0.3	0.985/0.991	0.967/0.980	0.924/0.954	0.900/0.939	0.828/0.892
0	0.980/0.988	0.957/0.974	0.905/0.965	0.876/0.924	0.800/0.872

In limits θ_0 from 0^0 till 20^0 influences of Puasson's factor does not exceed 5%. Factor $k_1(\theta_0,\nu)$ is possible to approcsimate by simple dependence

$$k_1(\theta_0,\nu)=1-\frac{\theta_0}{\pi},\ \text{then}\ \sqrt{k_1(\theta_0,\nu)}\approx 1-\frac{\theta_0}{2\pi}.$$

In a result

$$\theta_0=\frac{1.508}{\left(1-\frac{\theta_0}{2\pi}\right)}\cdot\sqrt{\frac{F}{RE}}.$$

Pressure at the centre of contact $p_0=\sum_{k=0}^{\infty}p_k$.

Expression for p_k after substitution β_k and transformations shall present as

$$p_k=\frac{3.24E\theta_0}{\pi^2}\left\{\frac{(-1)^k}{(2k+1)^2\left(1+\frac{(1-\nu)\theta_0}{(2k+1)\pi}\right)}-4\left(\frac{\theta_0}{\pi}\right)^2\cdot\frac{(-1)^k}{(2k+1)^4\left(1+\frac{(1-\nu)\theta_0}{(2k+1)\pi}\right)}\right\},$$

then $p_0=0.508\sqrt{\frac{FE}{R}}\cdot F_{2(\theta_0,\nu)}$,

where

$$F_{2(\theta_0,\nu)}=\frac{1}{\sqrt{F_{1(\theta_0,\nu)}}}\cdot\sum\left[\frac{(-1)^k}{(2k+1)^2\left(1+\frac{(1-\nu)\theta_0}{(2k+1)\pi}\right)}-4\left(\frac{\theta_0}{\pi}\right)^2\cdot\frac{(-1)^k}{(2k+1)^4\left(1+\frac{(1-\nu)\theta_0}{(2k+1)\pi}\right)}\right].$$

Function $F_{2(\theta_0,\nu)}$ we shall present as

$$F_{2(\theta_0,\nu)}=k_{2(\theta_0,\nu)}\cdot\frac{1}{\sqrt{\sum\frac{1}{(2k+1)^3}}}\cdot\sum\frac{(-1)^k}{(2k+1)^2}=0.8922k_2(\theta_0,\nu)$$

.Accounts of factor $k_2(\theta_0,\nu)$ at $\nu=0;\ 0.3;\ 0.5$ and $\overline{\theta_0}=\frac{\theta_0}{\pi}=0.02;\ 0.04;\ 0.08;\ 0.10;\ 0.15$ are submitted in Table 1. As show accounts influence of Puasson's factor on meaning of factor does not exceed 3%. For $\theta_0\le 20^0$ it is possible to accept, that

$$k_2(\theta_0,\nu)\approx 1-\frac{\theta_0}{2\pi}.$$

As a result

$$p_0=\left(1-\frac{\theta_0}{2\pi}\right)\cdot 0.453\sqrt{\frac{FE}{R}},$$

at $E=\frac{E_0}{(1-\nu^2)}$ - for the flat contact deformation.

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Computing of $\overline{\sigma_r} = \frac{\sigma_r}{p_0}$, $\overline{\sigma_\theta} = \frac{\sigma_\theta}{p_0}$ for $\theta = 0$ and $\overline{\tau_1} = \frac{\tau_1}{p_0}, (\tau_1 = \max \tau_{r\theta})$ versus of $\Delta r = \frac{\rho - R}{S_b}$, where $S_b = R \cdot \theta_0$ for $\theta_0 = 0.02;\ 0.04;\ 0.08$ are represented in Fig. 3.

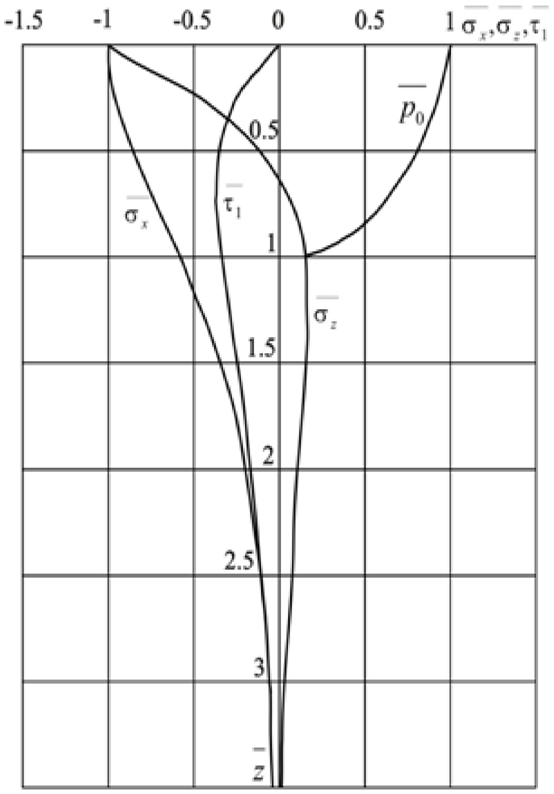

Fig.2. Introduction of the elastic cylinder into absolutely rigid flat body $(b >> R)$

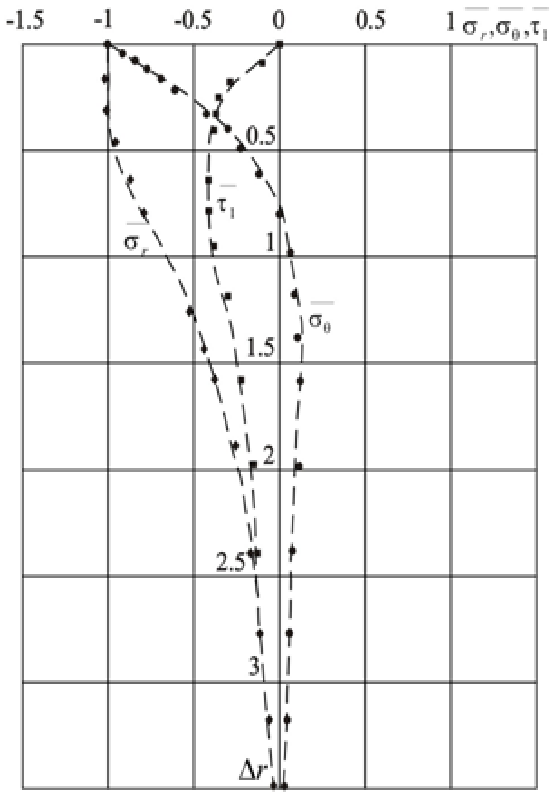

Fig.3. Introduction of absolutely rigid cylinder into an elastic flat body $(R << A)$

III. CONCLUSIONS

Computing results are showing that for small angles of contact the computed points are lying practically on one curve. And these three curves are nearly juxtaposed with corresponding curves shown in Fig. 2.

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AUTHOR BIOGRAPHY

Musayev Xasay Ibish oqli
Candidate of Physics and Mathematics Sciences, docent
National Academy of Sciences of Azerbaijan
Institute of Mathematics and Mechanics
Baku, AZ 1143, str. B. Vahabzadeh 9.
Email: afa7803@rambler.ru

Guliyev Rahim Sameddin oqli
Candidate of Physical and Mathematical Sciences
National Academy of Sciences of Azerbaijan
Institute of Mathematics and Mechanics
Baku, AZ 1143, str. B. Vahabzadeh 9.
Email: guliyev-rahim@rambler.ru
(+99450) 3515448

Yuzbashiyeva Afat Oqtay kizi
Candidate of Physical and Mathematical Sciences
National Academy of Sciences of Azerbaijan
Institute of Mathematics and Mechanics
Baku AZ1143, B.Vaxabzade St., 9
Email: afa7803@rambler.ru
(+99455) 7861078