ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 12, June 2014

# Weierstrass Approximation Theorem Using Newton Interpolating Polynomials

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***Abstract*—The celebrated and famous Weierstrass approximation theorem characterizes the set of continuous functions on a compact interval via uniform approximation by algebraic polynomials. This theorem is the first significant result in approximation Theory of one real variable and plays a key role in the development of General Approximation Theory. The famous Weierstrass approximation theorem states that any continuous function $f: [a, b] \to \mathbb{R}$ can be approximated by a polynomial with a maximum error as small as one likes. There are several approaches to proving this theorem, in this paper, we present a proof using Newton Interpolating Polynomials. We also showed that this approximation is uniform on closed and bounded interval of $\mathbb{R}$.***

***Index Terms*—Weierstrass Approximation Theorem, Newton Interpolating Polynomials**

## I. INTRODUCTION

Weierstrass Approximation theorem is widely used in Mathematical Analysis. Numerical techniques like Interpolation and Numerical Integration can be derived from Weierstrass Approximation Theorem, stated as "If $f: [a, b] -$ is continuous function then for each $\epsilon$ : there exist a polynomial $p_n$ of degree such that $|f(x) - p_n(x)|$ for all $x \in [$" i.e. any continuous function on closed and bounded interval can be approximated by some polynomial on the other way we can say that set of all polynomials are dense in $C[a$ (set of all continuous function on closed interval $[a$).

In this paper we are going to prove Weierstrass approximation theorem using Newton Interpolating polynomials i.e. we are going to prove that if $f: [a, b] -$ is continuous function then for each $\epsilon$ : there exist a Newton Interpolating polynomial $p_n$ of degree such that $|f(x) - p_n(x)|$ for all $x \in [a$.

In this paper we construct a sequence of Newton Interpolating polynomial $p_n($ then we show that $p_n($ converges uniformly to . We also discuss some application to this theorem.

## II. PRELIMINARIES

Definition 2.1 Continuous function:[1]A function $A.\ f: [a, b] -$ is said to be continuous at $A.\ x_0 \in [a$ if for every $A.\ \epsilon$ there exist $A.\ \delta$ such that $A.\ |f(x) - f(x_0)|$ whenever $A.\ |x - x_0|$ and function is continuous over $A.\ [a$ if it is continuous for all value $A.\ x_0 \in [a$.

Definition2.2 Sequence of functions: [1]Let $\{A.\}$ be the sequence of functions defined on [a, b] then $\{A.\}$ ⟶f point wise if $A.\ \lim_{n \to \infty} f_n(x) = f$, and we say that a sequence of functions $\{A.\}$ converges uniformly on [a, b] if for every $A. > 0$ there is an integer N such that $A.\ |f_n(x) - f(x)|$ whenever $A.\ n \geq$ and for all $A.\ x \in [a$.

Theorem 2.3 Weierstrass M-test:[1]Suppose $A.\ \lim_{n \to \infty} f_n(x) = f$ for all $A.\ x \in [a$, put $A.\ M_n = \sup|f_n(x) - f|$. Then $\{A.\}$ ⟶f uniformly on [a, b] if and only if $A.\ M_n -$as $A.\ n -$.

Definition 2.4 Partition of an interval [a, b]: [1] Let [a, b] be given interval. By partition P of [a, b] we mean a finite set of points $x_0, x_1, \dots$ where $a = x_0 < x_1 < \cdots < x_n$ .

Definition 2.5 Outer Measure: [2] Let be any subset of then outer measure on is denoted by $m^*$ and defined as

$$m^*(A) = \inf \left\{ \sum l(I_n) \,\middle|\, \{I_n\} \text{be the sequence of intervals} \ni A \subseteq \bigcup \right.$$

Definition 2.6 Lebesgue Measurable: [2] Let be any subset of then is measurable if for any set ,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap$$

Definition 2.7 Measurable function: [2]A function $f: [a, b] -$ is measurable if inverse image of open set is measurable.

Preposition 2.8 Littlewoods' second principle: [2]A function $f: [a, b] -$ is measurable then for every $\epsilon$ : there exist a continuous function $g: [a, b] -$ such that $|f(x) - g(x)|$ almost everywhere.

Definition 2.9 Forward Difference: [3] Let $f: [a, b] -$ be any function then first order forward difference is denoted by and defined as, $\Delta f(x) = f(x + h) - f$. Second order

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 12, June 2014

forward difference is defined as $\Delta^2 f(x) = \Delta f(x+h) - \Delta$ and in general n order forward difference is defined as $\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1}$.
If $(x_0, y_0), (x_1, y_1), \dots, (x_n,$ be $(n +$ equidistance data points with step size "h" then first forward difference is $\Delta y_i = y_{i+1}$ - for $i = 0,1,2,\dots,n$ . Second difference is $\Delta^2 y_i = \Delta y_{i+1} -$ for $i = 0,1,2,\dots,n$ . On same way "k" difference is $\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-}$ for $i = 0,1,2,\dots,n$ .

Definition 2.10 Newton forward interpolating polynomial: [3]
If $(x_0, y_0), (x_1, y_1), \dots, (x_n,$ be $(n +$ equidistance data points with step size "h" then Newton interpolating polynomial of "n" degree is given by,

$$p_n(x) = y_0 + \frac{(x-x_0)}{h}\Delta y_0 + \frac{(x-x_0)(x-x_1)}{h^2 2!}\Delta^2 y_0 + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{}\Delta^n y_0$$

## III. WEIERSTRASS APPROXIMATION THEOREM

Theorem 3.1 If $f: [a,b] \to \mathbb{R}$ is continuous function then for each $\epsilon > 0$ there is polynomial $p_n(x)$ of degree n such that $|f(x) - p_n(x)| < \epsilon$ for all $x \in [a,b]$.

*Proof:* Without loss of generality we assume a=0 and b=1, then we have to show that if $f: [0,1] \to \mathbb{R}$ is continuous function then for each $\epsilon > 0$ there is polynomial $p_n(x)$ of degree $n$ such that $|f(x) - p_n(x)| < \epsilon$ for all $x \in [0,1]$. For any positive integer n consider a partition of [0, 1], $0 < {}^1/_n < {}^2/_n < {}^3/_n \dots\dots < 1$ and define,

$$p_n(x) = f(0) + \frac{x}{(1/n)}\Delta f(0) + \frac{x(x-1/n)}{(1/n)^2}\frac{\Delta^2 f(0)}{2!} + \dots + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\dots\dots(x-1)}{(1/n)^n}\frac{\Delta^n f(0)}{n!}$$

$$= f(0) + nx\,\Delta f(0) + n^2 x\left(x - {}^1/_n\right)\frac{\Delta^2 f(0)}{2!} + \dots + n^n x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)\dots(x-1)\frac{\Delta^n f(0)}{n!}$$

Then for each $n \in \mathbb{N}$, $f\left(\frac{i}{n}\right) = P_n\left(\frac{i}{n}\right)$ For every integer i=1, 2, 3,…, n.

Since f(x) is continuous at x ∊ [0, 1] therefore for ${}^\epsilon/_2 > 0$ there exist positive integer "N" such that
$|f(x) - f(y)| < {}^\epsilon/_2$ Whenever $|x - y| < \frac{1}{N}$.

And $P_n(x)$ is continuous at x ∊ [0, 1] therefore for ${}^\epsilon/_2 > 0$ there exist positive integer "M" such that
$|P_n(x) - P_n(y)| < {}^\epsilon/_2$ Whenever $|x - y| < \frac{1}{M}$.

Taken $= \max\{M, N\}$.
(*This is the "n" used in the partition of [0, 1] and hence degree of* $p_n(x)$)
If x ∊ [0, 1] then $x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$ and take ∊>0

Consider

$$|f(x) - p_n(x)| = \left|f(x) - f\left(\frac{i}{n}\right) + f\left(\frac{i}{n}\right) - P_n\left(\frac{i}{n}\right) + P_n\left(\frac{i}{n}\right) - p_n(x)\right|$$

$$\le$$

$$\left|f(x) - f\left(\frac{i}{n}\right)\right| + \left|f\left(\frac{i}{n}\right) - P_n\left(\frac{i}{n}\right)\right| + \left|P_n\left(\frac{i}{n}\right) - P_n(x)\right|$$

$$< {}^\epsilon/_2 + 0 + {}^\epsilon/_2 = \epsilon$$

Since ∊ > 0 is arbitrarily chosen.
Therefore $f: [0,1] \to \mathbb{R}$ continuous function is then for each $\epsilon > 0$ there is polynomial $p_n(x)$ of degree $n$ such that $|f(x) - p_n(x)| < \epsilon$ for all $x \in [0,1]$.

Corollary 3.1 If $f: [a,b]$ - is continuous function then for each $\epsilon$ : there is sequence of Newton Interpolating polynomials { $p_n$( such that $\{p_n($ converges uniformly on [a, b].

*Proof:* We proved that if $f: [a,b] \to \mathbb{R}$ is continuous function then for each $\epsilon > 0$ there is polynomial $p_n(x)$ of degree $n$ such that $|f(x) - p_n(x)| < \epsilon$ for all $x \in [a,b]$. Therefore we can say that the sequence $\{p_n(x)\}$ converges to $f(x)$ for all $x \in [a,b]$.
Define $Q_n(x) = f(x) - p_n(x)$ then $Q_n(x)$ is sequence of continuous function on $[a,b]$ and $\{Q_n(x)\}$ converges to 0.
Let, $M_n = \sup\{|Q_n(x)| / x \in [a,b]\}$ then $\{M_n\}$ is sequence of real numbers converges to 0. Therefore by Weierstrass M test
$\{Q_n(x)\}$ Converges to 0 uniformly, hence $\{P_n(x)\}$ converges to $f(x)$ uniformly.

Now we generalize above statement as, any measurable function can be approximated by a Newton interpolating polynomial almost everywhere. This means that set of points where function cannot be approximated is of measure zero.

Corollary 3.2 If $f: [a,b] \to \mathbb{R}$ is measurable function then for each $\epsilon > 0$ there is polynomial $P_n(x)$ of degree $n$ such that $|f(x) - P_n(x)| < \epsilon$ for almost all $x \in [a,b]$.

*Proof:* Let $\epsilon > 0$ and $f: [a,b] \to \mathbb{R}$ is measurable therefore by Littlewoods second principle there exist a continuous function $g: [a,b] \to \mathbb{R}$ such that $|f(x) - g(x)| < \frac{\epsilon}{2}$ almost everywhere and since $g: [a,b] \to \mathbb{R}$ is continuous therefore there exist Newton interpolating polynomial $P_n(x)$ such that $|g(x) - P_n(x)| < \frac{\epsilon}{2}$ for all value of $x \in [a,b]$. Combining these two statements we get If $f: [a,b] \to \mathbb{R}$ is measurable function then for each $\epsilon > 0$ there is polynomial $P_n(x)$ of degree $n$ such that $|f(x) - P_n(x)| < \epsilon$ for almost all $x \in [a,b]$.

## IV. CONCLUSION

From above two results we can conclude that any continuous function can be approximated by Newton interpolating polynomial and this approximation is uniform

over closed and bounded interval of . Third result shows that any measurable function over the closed interval can be approximated by a polynomial almost at all points of the

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 12, June 2014

interval (i.e. points at which approximation fails is of measure zero).

## ACKNOWLEDGMENT

We are thankful to Dr. H. R. Kataria and Dr. R. M. Patel for their motivation. We are also thankful to Mr. MrudulJani for his valuable suggestions.

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