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Prey Predator Model with Fuzzy Initial Conditions

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Abstract - In this paper we consider a fuzzy differential equation describing a prey predator model with fuzzy initial condition. Prey predator model in the fuzzy setup is more realistic depiction of the phenomena, here the initial conditions are considered as fuzzy because the initial population estimates may not be precisely known in the real life situation. The results for the existence of the solution are discussed in paper.

***Keywords* - Fuzzy differential equation, Prey predator model, Fuzzy initial condition.**

I. INTRODUCTION

In the population model, consider two species of animals which are part of food chain, predator eats prey and prey depends on other food, the prey are assumed to have unlimited food supply and to reproduce exponentially unless subject to predation, this exponential growth is represented by the term ax. The rate of predation upon the prey is assumed to be proportional to the rate at which prey and predator interact, represented as *bxy*. Secondly, the growth of predator is proportional to *xy* with the proportionality constant *d* and *cy* represents the loss rate of the predator due to natural death or absence of prey.

Thus, the two species population model can be represented as a system of two first order nonlinear differential equations which is also known as Lotka-Volterra equation.

$$\dot{x}(t) = ax - bxy;$$
$$\dot{y}(t) = -cy + dxy; \qquad (1)$$

with initial condition $x(0) = x_0$ and $y(0) = y_0$, where a, b, c and d are positive constants as described above, $x(t)$ denotes the population of prey species and $y(t)$ denotes the population of predator species, x_0 and y_0 is the initial estimates of the species.

For the system as given by (1) it may not be possible to have the exact estimates of initial population, then such a scenario fits into fuzzy setup where the initial estimates are represented by fuzzy numbers, the concept of fuzzy sets was proposed by L. A. Zadeh [8].

System (1) with fuzzy initial condition is given by:

$$\dot{x}(t) = ax - bxy;$$
$$\dot{y}(t) = -cy + dxy; \qquad (2)$$

with $x(0) = \tilde{x}_0$ and $y(0) = \tilde{y}_0$.

For such system as given by (2), Muhammad Zaini Ahmad, Bernard De Baest [9] proposed the numerical solution by generalized numerical method and Omer Akin, Omer Oruc [10] proposed the solution by strongly generalized derivative concept.

System (2) can be written in compact form as

$$\dot{X} = AX + f(X(t))$$

Where, $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $A = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$ and $f = \begin{pmatrix} -bxy \\ dxy \end{pmatrix}$

In our paper, we adopt analytical approach to solve the fuzzy prey predator model, which gives the estimate of number of prey and number of predator at time *t*. To get the approximate solution first we linearize the equation about equilibrium point by Taylor's expansion then for this linearized system the solution is obtained which satisfies the fuzzy initial condition. The paper is organized in the following manner in the next section the preliminaries are listed, in the section 3 we describe the technique to obtain crisp solution followed by the fuzzy solution. Illustrative example is given at the end.

II. PRELIMINARIES

A. α-cut

An α-cut or α-level set of a fuzzy set $A \subseteq X$ is an ordinary set $A^\alpha \subseteq X$, such that $A^\alpha = \mu_A(X) \geq \alpha, \forall x \in X$.

B. Fuzzy Number

A fuzzy set is said to be fuzzy number if it qualifies following condition:

- A is a convex fuzzy set, i.e. $A(r \lambda + (1-\lambda)s) \geq \min [A(r), A(s)]$, $\lambda \in [0,1]$ and $r, s \in X$;
- A is normal, i.e. $\exists x_0 \in X$ with $(A(x_0) = 1$;
- A is upper semi-continuous i.e. $A(x_0) \geq \lim_{x \to x_0^{\pm}} A(x)$;
- $[A]^0 = \overline{\sup p(A)} = \overline{\{x \in R | A(x) \geq 0\}}$ is compact, where $\bar{A}$ denotes the closure of A.

C. Fuzzy Operation

For $u, v \in R_f$ and $\lambda \in R$ the sum $u + v$ and the product $\lambda . u$ is defined as

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha = [\overline{u}\ \underline{u}] + [\overline{v}, \underline{v}] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}]$$

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& $[\lambda . u]^{\alpha} = \lambda\, [\overline{u}\ \underline{u}] = [\lambda\, \overline{u}, \lambda\, \underline{u}]$, for all $\alpha \in [0,1]$.

D. H-Difference

Let $u, v \in \mathcal{F}$. If there exist $w \in \mathcal{F}$ such that $u = v \oplus w$ then w is called the *H-difference of* u and v is denoted by $u \ominus v$.

E. Hukuhara Derivative

Consider a fuzzy mapping $F: (a,b) \to \mathcal{F}$ and $t_0 \in (a,b)$. We say that F is differentiable at $t_0 \in (a,b)$ if there exist an element $F'(t_0) \in \mathcal{F}$ such that for all $h > 0$ sufficiently small $\exists F(t_0+h) \ominus F(t_0), F(t_0) \ominus F(t_0-h)$ and the limits

$$\lim_{h\to 0+} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0-} \frac{F(t_0) \ominus F(t_0-h)}{h}$$

exist and are equal to $F'(t_0)$.

III. CRISP PREY-PREDATOR MODEL

We first obtain the equilibrium point by solving

$$AX + f(X(t)) = 0$$

That is $\begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -bxy \\ dxy \end{pmatrix} = 0$

Giving $\begin{pmatrix} x_e \\ y_e \end{pmatrix} = \begin{pmatrix} c/d \\ a/b \end{pmatrix}$

as the equilibrium point of the system.

To obtain the solution of such system we linearize the system (1) about the equilibrium point (x_e, y_e) with the help of Taylor's expansion considering the first order term and neglecting the higher order terms, we get,

$$\dot{X}(t) = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{ac}{d} \\ \frac{-ac}{b} \end{bmatrix} \qquad (3)$$

That is, $\dot{X}(t) = CX + B$

where C is 2x2 matrix and given as $\begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}$ and B is 2x1 vector and given as $\begin{bmatrix} \frac{ac}{d} \\ \frac{-ac}{b} \end{bmatrix}$.

The eigen values of C are $\lambda_1 = i\sqrt{ac}$ and $\lambda_2 = -i\sqrt{ac}$. We construct fundamental matrix $\Psi(t)$ with the columns as the linearly independent eigen vectors corresponding to these complex eigen values. The solution of system (3) is then given by:

$$X(t) = \Psi(t)\, \Psi^{-1}(0)\, X(0) + \Psi(t) \int_0^t \Psi^{-1}(s)\, B\, ds$$

IV. FUZZY PREY-PREDATOR MODEL

The Prey-Predator model with fuzzy initial condition in the linearized form is given as

$$\dot{X}(t) = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{ac}{d} \\ \frac{-ac}{b} \end{bmatrix} \qquad (4)$$

with $X(0) = \tilde{X}_0$ and $Y(0) = \tilde{Y}_0$

The solution of system (4) is given by,

$$\tilde{X}(t) = \Psi(t)\, \Psi^{-1}(0) \tilde{X}(0) + \Psi(t) \int_0^t \Psi^{-1}(s)\, B\, ds$$

Taking $\alpha - cut$ on both sides we get,

$$[\overline{X}, \underline{X}] = \Psi(t)\, \Psi^{-1}(0) [\overline{X}_0, \underline{X}_0] + \Psi(t) \int_0^t \Psi^{-1}(s)\, B\, ds$$

Comparing the elements of interval we get

$$\overline{X} = \Psi(t)\, \Psi^{-1}(0)\, \overline{X}_0 + \Psi(t) \int_0^t \Psi^{-1}(s)\, B\, ds \qquad (5)$$

$$\underline{X} = \Psi(t)\, \Psi^{-1}(0)\, \underline{X}_0 + \Psi(t) \int_0^t \Psi^{-1}(s)\, B\, ds \qquad (6)$$

The state vector $\tilde{X}(t)$ can be constructed from (5) and (6) using First decomposition theorem, Klir [11].

The solution $\tilde{X}(t)$ obtained using (5) and (6) will be fuzzy solution if for $t > 0$.

(i) $\forall \alpha \in [0,1],\ {}^{\alpha}\underline{x}_i(t) \le {}^{\alpha}\overline{x}_i(t)$

(ii) $\forall \alpha, \beta \in [0,1], \alpha \le \beta$

$${}^{\alpha}\underline{x}_i(t) \le {}^{\beta}\underline{x}_i(t) \le {}^{\beta}\overline{x}_i(t) \le {}^{\alpha}\overline{x}_i(t)$$

Hence, $\forall \alpha \in [0,1],\ \tilde{x}_i(t) = \left[{}^{\alpha}\underline{x}_i(t), {}^{\alpha}\overline{x}_i(t)\right]$, for $i = 1,2$ where,

- ${}^{\alpha}\underline{x}_i$ is a bounded left continuous non-decreasing function over [0,1].
- ${}^{\alpha}\overline{x}_i$ is a bounded left continuous non-increasing function over [0,1].

V. NUMERICAL ILLUSTRATIVE

Consider the following example of Prey-Predator Model in crisp set up.

$$(x(\dot{t})) = 0.1x - 0.005xy;$$

$$(y(\dot{t})) = -0.4y + 0.008xy;$$

$$X(0) = 130;\ \ Y(0) = 40$$

The system has two critical points, the trivial one is origin and the other is (50, 20). First, we linearize this problem at (50, 20) by Taylor's expansion and get,

$$\dot{X}(t) = -0.25y + 5$$

$$\dot{Y}(t) = 0.16x - 8 \qquad (7)$$

The linearized system has 2x2 coefficient matrix C which is given as $\begin{bmatrix} 0 & -0.25 \\ 0.16 & 0 \end{bmatrix}$ and B is 2x1 vector which is given as $\begin{bmatrix} 5 \\ -8 \end{bmatrix}$.

Eigen values of this matrix are $-0.2i$ and $0.2i$ and corresponding eigenvectors are $\begin{bmatrix} 1 \\ -0.8i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0.8i \end{bmatrix}$ respectively and the fundamental matrix is given by

$$\Psi(t) = \begin{bmatrix} \cos(0.2t) & \sin(0.2t) \\ 0.8\sin(0.2t) & -0.8\cos(0.2t) \end{bmatrix}$$

Thus, the solution of system (7) is given by:

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$$X(t) = \Psi(t)\Psi^{-1}(0)X(0) + \Psi(t)\int_0^t \Psi^{-1}(s)B\,ds$$

That is,

$$X(t) = \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{-5}{4} \end{bmatrix} \begin{bmatrix} 130 \\ 40 \end{bmatrix}$$
$$+ \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix}$$
$$\int_0^t \left(\begin{bmatrix} cos(0.2s) & -\frac{sin(0.2s)}{8i} \\ sin(0.2s) & \frac{cos(0.2s)}{8i} \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right) ds$$

And we get,

$$x(t) = 50 + 80\cos(0.2t) - 25\sin(0.2t)$$
$$y(t) = 20 + 20\,cos(0.2t) + 64\,sin(0.2t)$$

The evolution of system in small time interval is as shown in Fig. (1).

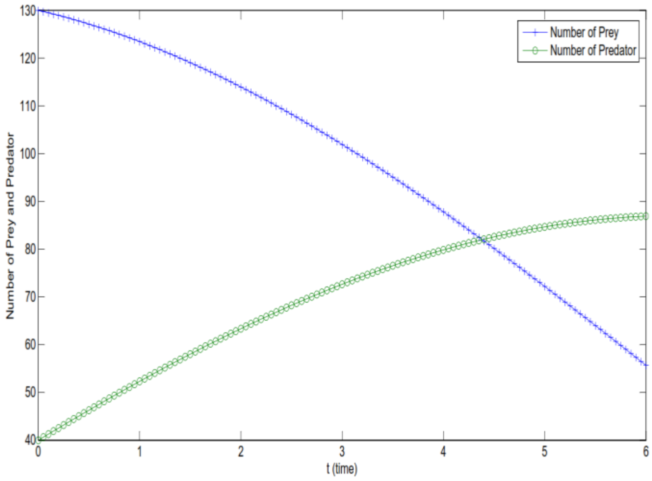

Fig. 1: Evolution of Crisp Prey Predator Population

In system (8), when we consider fuzzy initial condition as below,

$$\widetilde{X_0} = \begin{cases} \frac{x-120}{10} & 120 < x \le 130 \\ \frac{150-x}{20} & 130 < x \le 150 \end{cases}$$

$$\widetilde{Y_0} = \begin{cases} \frac{y-20}{10} & 20 < y \le 40 \\ \frac{50-y}{10} & 40 < y \le 50 \end{cases} \qquad (8)$$

Then, $(\widetilde{X_0})^{\alpha} = [10\alpha + 120, 150 - 20\alpha]$

$(\widetilde{Y_0})^{\alpha} = [20\alpha + 20, 50 - 10\alpha]$

where $\alpha \in [0,1]$.

The solution is now obtained as,

$$\tilde{X}(t) = \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{-5}{4} \end{bmatrix}$$
$$\begin{bmatrix} 10\alpha + 120, 150 - 20\alpha \\ 20\alpha + 20, 50 - 10\alpha \end{bmatrix}$$
$$+ \begin{bmatrix} cos(0.2t) & sin(0.2t) \\ 0.8\,sin(0.2t) & -0.8\,cos(0.2t) \end{bmatrix}$$
$$\int_0^t \left(\begin{bmatrix} cos(0.2s) & -\frac{sin(0.2s)}{8i} \\ sin(0.2s) & \frac{cos(0.2s)}{8i} \end{bmatrix} \begin{bmatrix} 5 \\ -8 \end{bmatrix} \right) ds$$

Now, comparing the components, we get

$\underline{X} = 70\,cos(0.2t) + 50$ and $\underline{Y} = 20 + 56\sin(0.2t)$

$$\overline{X} = 50 + 100\,cos(0.2t) - 37.5\sin(0.2t) \text{ and}$$
$$\overline{Y} = 20 + 80\,sin(0.2t) + 30\,cos(0.2t).$$

The evolution of Prey and Predator population in system (8), in small time interval are as shown in Fig. (2) and Fig. (3) respectively.

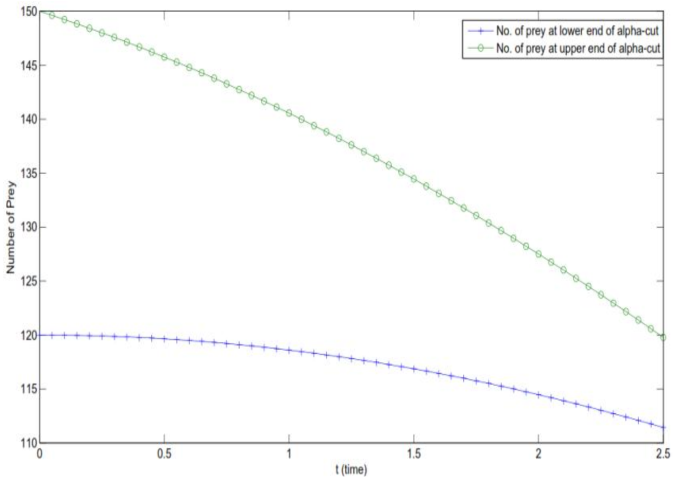

Fig. 2: Evolution of Fuzzy Prey Population

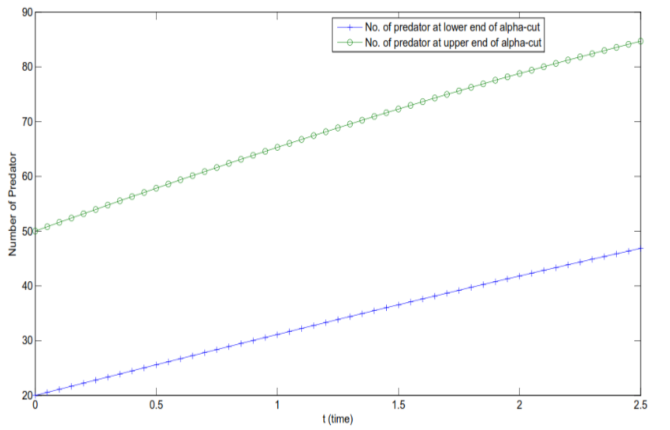

Fig. 3: Evolution of Fuzzy Predator Population

VI. CONCLUSION

Here, we discuss approximate solution of Prey Predator model with fuzzy initial condition over small time interval. In future, for the same we can study the case with the entries of A represented as fuzzy numbers.

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