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Perfect Domination and Packing of Graph

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Abstract: In this paper we consider two concepts namely Perfect Domination and Packing in Graphs. We define Big Perfect Domination Number of a Graph and prove necessary and sufficient conditions under which this number decreases or remains same when a vertex is removed. We also prove a necessary and sufficient condition under which a Perfect Dominating set is a maximal Perfect Dominating set. We also consider the maximum Packing number of the Graph G and prove two necessary and sufficient conditions under which ρ decreases when vertex is removed from the Graph.

Key Words: **Perfect domination, Perfect domination number, Minimal perfect dominating set, Γ_{pr} set, γ_{pr} set, Big perfect domination number, Maximal perfect dominating set, Packing and Maximum packing.**

I. INTRODUCTION

In this paper we consider so called perfect dominating sets. Perfect dominating sets are closely related to perfect codes which have applications in coding theory. In this paper we consider minimal perfect dominating sets with maximum cardinality. The cardinality of any such set is called the big perfect domination number of the graph. We prove necessary and sufficient conditions under which this number decreases or remains same. Let G be a graph and , u and v be two vertices of G. then the distance between u and v, denoted as d(u,v), is the length of the shortest path in G joining u and v. If there is no path joining u and v. We write $d(u,v) = \infty$ and we accept that $d(u,v) > k$, for all positive integer k.

II. PRELIMINARIES AND NOTATIONS

For a graph G, V(G) will denote the vertex set of the graph G. G-v will denote the sub graph obtained by removing the vertex v from the graph G. We assume that all graphs are simple and have no isolated vertices.

Definition: 2.1 [5]

A subset S of V(G) is said to be a perfect dominating set if for every vertex v not in S, v is adjacent to exactly one vertex of S. Note that every perfect dominating set is a dominating set.

Definition: 2.2

A perfect dominating set S is said to be a minimal perfect dominating set if for every vertex v in S, S – v is not a perfect dominating set.

Definition:2.3 [5]

A perfect dominating set with smallest cardinality is called a minimum perfect dominating set. It is also called γpr-set of G. The cardinality of a γpr-set is called a perfect domination number of the graph G and is denoted as γpr(G).

Definition: 2.4

Let G be a graph, S be a subset of V(G) and v $\in$ S, then the perfect private neighbourhood of v with respect to S is pprn(v,s) = { w $\in$ V(G) / w does not belongs to S and $n[w] \cap S = \{v\}$ } $\cup$ {v}. If v is adjacent to no vertex of S or at least two vertices of S.

Definition: 2.5

A minimal perfect dominating set with highest cardinality is called Γpr – set. The number of elements of such a set is called the big perfect domination number of G, and is denoted as Γpr(G).

Definition: 2.6

Let G be a graph and S be a proper subset of V(G) then S is said to be a maximal perfect dominating set if for every vertex v not in S,

$S \cup \{v\}$ is not a perfect dominating set.

Definition: 2.7[5]

A subset S of V(G) is said to be packing of G if $d(u,v) > 2$, for all distinct vertices u and v of S.

Definition:2.8

A packing with largest cardinality is called a maximum packing of G. A cardinality of such a set is denoted as ρ(G). It may be noted that a subset S of V(G) is a packing if and only if for any two distinct vertices u and v of S, $N[u] \cap N[v] = \emptyset$.

Notations:

d(u,v)	Distance between u and v in a graph.
N[v]	$N(v) \cup \{v\}$.
$N_k(v)$	$\{ w \in V(G) \ni 1 \leq d(u,w) \leq k \}$.
pprn(v,S)	{ w $\in$ V(G) / w does not belongs to S and $n[w] \cap S = \{v\}$ } $\cup$ {v}.

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V_{pr}^{+} { v ∈ V(G) / γ_{pr}(G - v) > γ_{pr}(G) }.

V_{pr}^{-} { v ∈ V(G) / γ_{pr}(G - v) < γ_{pr}(G) }.

V_{pr}^{0} { v ∈ V(G) / γ_{pr}(G - v) = γ_{pr}(G) }.

We note that the above three sets are mutually disjoint and their union is V(G).

W_{pr}^{-} { v ∈ V(G) / Γ_{pr}(G – v) < Γ_{pr}(G) }.

W_{pr}^{0} { v ∈ V(G) / Γ_{pr}(G – v) = Γ_{pr}(G) }.

Note that the above two sets are disjoint and their union is the vertex set V(G).

γ_{pr}-set	Minimum perfect dominating set of a graph G.
γ_{pr}(G)	Perfect domination number of a graph G.
Γ_{pr}-set	Minimum perfect dominating set with highest cardinality.
Γ_{pr}(G)	Big perfect domination number of a graph G.
ρ(G)	The size of a maximum packing.

III. MAIN RESULTS

Theorem: 3.1 A perfect dominating set S is a minimal perfect dominating set if and only if for every v ∈ S pprn(v,S) is non empty.

Proof: Suppose S is a minimal perfect dominating set and v ∈ S. Now S – v is not a perfect dominating set. There fore there is a vertex w not in S – v which is either adjacent to at least two vertices of S – v or is adjacent to no vertex of S – v.

If w ≠ v and w is adjacent to at least two vertices of S – v then N[w] ∩ S contains at least two vertices. Thus w does not belongs to S and w is adjacent to at least two vertices of S. Which is a contradiction.

Thus w is adjacent to no vertices of S – v . since w is not in S , and S is a perfect dominating set, w must be adjacent to v only in S. That is N[w] ∩ S = {v}. Hence w ∈ pprn(v,S).

If w = v and if w is adjacent to at least two vertices of S – v, then w = v ∈ pprn(v,S).

If w = v and w is non adjacent to any vertex of S – v then it means that v is not adjacent to any vertex of S. Thus w = v ∈ pprn(v,S).

Thus in all cases pprn(v,S) is non empty.

IV. CONVERSELY

Suppose pprn(v,S) is non empty for every v in S. Let w be a vertex in pprn(v,S). If w = v then w is not adjacent to any vertex of S. Thus w does not belongs to S – v and w is not adjacent to S – v . if w = v and w is adjacent to at least two vertices of S then w does not belongs to S – v and w is adjacent to at least two vertices of S – v. If w ≠ v then w does not belongs to S. Since w ∈ pprn(v,S), N[w] ∩ S = {v}. Thus w is not adjacent to any vertex of S – v . Thus in all cases either w is adjacent to no vertex of S – v or adjacent to at least two vertices of S – v . Hence S – v is not a perfect dominating set. Hence S is a minimal perfect dominating set.■

Lemma: 3.2 Let G be a graph and v ∈ V(G), then Γ_{pr}(G – v) ≤ Γ_{pr}(G).

Proof:

Let S be a Γ_{pr} – set of G – v . if v is adjacent to exactly one vertex w of S then S is a minimal perfect dominating set of G. There fore Γpr (G) ≥ |S| = Γ_{pr}(G – v). If v is adjacent to no vertex of S or is adjacent to at least two vertices of S then S_1 = S ∪ {v} is a minimal perfect dominating set of G. There fore Γpr (G) > |S_1| > |S| = Γ_{pr}(G – v).

Thus Γpr (G – v) < Γ_{pr}(G).■

Theorem: 3.3 Let G be a graph and v ∈ V(G) then v ∈ W_{pr}^{0} if and only if there is a Γ_{pr} – set S of G not containing v and a vertex w in S such that pprn(w,S) contains at least two vertices and one of them is v.

Proof:

Suppose v ∈ W_{pr}^{0}. Let S be a Γ_{pr} – set of G – v .

Claim : v is adjacent to exactly one vertex of S.

Proof of the claim: If v is adjacent to no vertex of S or at least two vertices of S then S_1 = S ∪ {v} is a minimal perfect dominating set of G and hence Γ_{pr}(G – v) < Γ_{pr}(G). Which contradicts our assumption then v ∈ W_{pr}^{0} . Thus v is adjacent to exactly one vertex of S.

Let w be the only vertex of S to which v is adjacent. There fore v ∈ pprn (w,S). Also S is a minimal perfect dominating set of G – v. There fore pprn(w,S) contains a vertex v' of G – v . Thus pprn(w,S) contains at least two vertices and one of them is v.

IV. CONVERSE

Let S be a Γ_{pr} – set of G not containing v such that for some vertex w in S, pprn(w,S) contains at least two vertices and one of them is v. Thus pprn(w,S) contains a vertex of G – v , also for any other vertex p of S, pprn(p,S) contains a vertex v' of G. This vertex v' cannot be equal to v , because otherwise v would be adjacent to two distinct

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vertices w and p of S. Which contradicts that S is a perfect dominating set in G. Thus v' is different from v. Thus for every point z of S pprn(z,S) is non empty in G – v . Hence S is a minimal perfect dominating set of G – v . There fore $\Gamma_{pr}(G - v) \geq |S| = \Gamma_{pr}(G)$. But it is impossible that $\Gamma_{pr}(G - v) > \Gamma_{pr}(G)$, because of Lemma 4.9. There fore $\Gamma_{pr}(G - v) = \Gamma_{pr}(G)$. Hence $v \in W_{pr}^{0}$. ■

Corollary: 3.4 Let G be a graph and v ∈ V(G) then $v \in W_{pr}^{-}$ if and only if for every Γ_{pr} – set S of G either v ∈ S or there is a unique vertex w in S such that pprn(w,S) is equal to v.

Proof:

Suppose $v \in W_{pr}^{-}$ then v does not belongs to W_{pr}^{0}.

Let S be a Γ_{pr} – set of G. If v ∈ S then the corollary is proved.

Suppose v does not belongs to S. Let w be the unique vertex of S which is adjacent to v(S is a perfect dominating set in G). Then v ∈ pprn(w,S). If there is another vertex w ≠ v such that v' ∈ pprn(w,S) then it means that pprn(w,S) contains at least two vertices and one of them is v. This implies that $v \in W_{pr}^{0}$ by above theorem and we have a contradiction. Thus pprn(w,S) = {v}.

Converse:

Suppose $v \in W_{pr}^{0}$ then by above theorem there is a Γ_{pr} – set S of G not containing v and a vertex w of S such that pprn(w,S) contains at least two vertices and one of them is v. This contradicts our assumption, and hence $v \in W_{pr}^{-}$. ■

Example: 3.5

Consider the cycle C_5 with vertices 1,2,3,4,5. Note that the vertex set $V(C_5)$ it self is a perfect dominating set. Also if we remove any vertex i from the graph the remaining set is not a perfect dominating set. There fore $V(C_5)$ is a minimal perfect dominating set of the graph C_5. In fact $\Gamma_{pr}(C_5) = 5$.

If we remove any vertex i from C_5. The remaining graph is a path graph with four vertices and its big perfect dominating number is 2. There fore $\Gamma_{pr}(C_5 - i) = 2$. Thus every vertex belongs to W_{pr}^{-}.

Remarks: 3.6

It may be noted that a set S is a minimal dominating set if and only if S – v is not a dominating set for every vertex v in V(G) if and only if no proper subset of S is a dominating set. However for perfect domination the situation is not exactly similar. That is we cannot say that if a set S is a minimal perfect dominating set then no proper subset of S is a perfect dominating set.

For example consider the cycle C_5 with vertex set { 1,2,3,4,5 } = V(G). Then V(G) is a minimal perfect dominating set. However the set S_1 = { 1,2,3 } which is a proper subset of S is also a minimal perfect dominating set of C_5.

Theorem: 3.7 Let G be a graph and S be a proper subset of V(G) and S is a perfect dominating set, then S is a maximal perfect dominating set if and only if it contains all pendent vertices of the graph G.

Proof:

Suppose S is a maximal perfect dominating set and suppose that there is some pendent vertex v of G such that v does not belongs to S. Then it is easily verified that S ∪ {v} is a perfect dominating set. Which is a contradiction. Thus v ∈ S.

Converse

Suppose S is not a maximal perfect dominating set. Then there is some vertex v does not belongs to S such that S ∪ {v} is a perfect dominating set.

<u>Claim</u> v is a pendent vertex of G.

<u>Proof of the claim:</u> If v is not a pendent vertex of G, then let w_1 and w_2 be two neighbours of v. If w_1 and w_2 belongs to S then we have a contradiction because S is a perfect dominating set.

When either w_1 ∈ S or w_2 ∈ S. Suppose w_1 ∈ S and w_2 does not belongs to S. Now S ∪ {w_2} is a perfect dominating set(by assumption). However v is adjacent to two distinct vertices w_1 and w_2 of S ∪ {w_2}. This is a contradiction. Thus v must be a pendent vertex of G.

Thus we have proved that if S is not maximal perfect dominating set then there is a pendent vertex out side of S. ■

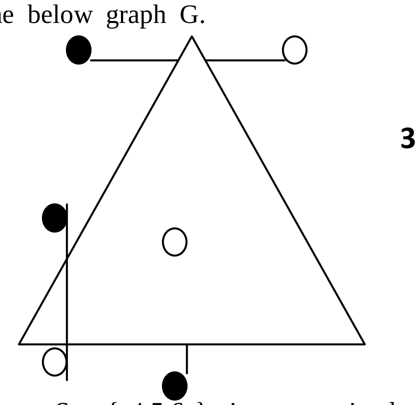
Remark: 3.8

It is usual to expect that a minimal set is not a maximal set and vice versa.

However this does not happen in the case of perfect domination.

Example: 3.9

Consider the below graph G.



Then the set S = { 4,5,6 } is a maximal perfect dominating set, because it contains all pendent vertices, and also it is a minimal perfect dominating set.

Theorem: 3.10 Let G be a graph which has no pendent vertices then every minimal perfect dominating set of G is a maximal perfect dominating set.

Proof:

A minimal perfect dominating set contains the set of all pendent vertices (because it is empty) and there fore by above theorem it is a maximal perfect dominating set. ■

IV : Packing and Maximum Packing

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Let G be a graph and , u and v be two vertices of G. then the distance between u and v, denoted as d(u,v), is the length of the shortest path in G joining u and v. If there is no path joining u and v. We write $d(u,v) = \infty$ and we accept that $d(u,v) > k$, for all positive integer k.

Remak: 4.1

It may be easily verified that a subset S of V(G) is a packing if and only if for every vertex $v \in V(G)$, $N[v] \cap S$ is either empty or a singleton set.

Theorem: 4.2

Let G be a graph and $v \in V(G)$ then the following statements are equivalent.

(I) $\rho(G\text{-}v) < \rho(G)$.

(II) There is a maximum packing S in G – v such that $N_2(v) \cap S = \emptyset$.

(III) Every maximum packing T contains v of G and T – v is a maximum packing in G – v .

Proof:

(I) Implies (II).

Let S_1 be a maximum packing of G. If v does not belongs to S_1 then s_1 is a packing of G – v also and hence $\rho(G) \leq \rho(G\text{-}v)$. Which is a contradiction to our assumption. Thus v must belongs to S_1. Since S_1 is a packing in G, $d(v,x) > 2$ for all x in S_1 with $x \neq v$. Now let $S = S_1 - \{v\}$ then S is a maximum packing in G – v and since $d(v,x) > 2$ for all x in S, and $N_2(v) \cap S = \emptyset$.

(II) Implies (I).

Let S be a maximum packing in G – v such that $N_2(v) \cap S = \emptyset$. Let $S_1 = S \cup \{v\}$ then S_1 is a packing in G with $|S_1| > |S|$. There fore $\rho(G) > \rho(G\text{-}v)$.

(I) Implies (III)

Suppose there is a maximum packing T in G such that v does not belongs to T then T is a packing in G – v . there fore $\rho(G\text{-}v) \geq \rho(G)$. which contradicts (I). There fore every maximum packing T of G contains v. Since T is a packing in G, T – v is also a packing in G, and hence a packing in G – v .Since $\rho(G\text{-}v) < \rho(G)$, T – v must be a maximum packing in G – v.

(III) Implies (I)

Let T be a maximum packing in G then $v \in T$ and T – v is a maximum packing of G – v. There fore $\rho(G\text{-}v) < \rho(G)$.

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