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Thermal Stresses of a Semi Infinite Rectangular Beam

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***Abstract-** This paper is concerned with thermal stresses of a semi infinite Rectangular beam. Here the results are obtained in terms of Bessel's functions in the form of infinite series by using Marchi-Fasulo transform and Fourier sine transform. The numerical solutions are depicted graphically.*

Keywords **- Thermo elastic problem, Semi infinite rectangular beam, Thermal stresses, Integral transform.**

I. INTRODUCTION

Adams and Bert [1], Tanigawa and Komatsubara [4] and Vihak et al. [5] have studied the direct problem of thermo elasticity in a rectangular plate under thermal shock. Khobragade et al. [2] has studied the inverse steady-state thermo elastic problem and determined the temperature distribution, unknown temperature gradient, displacement function and thermal stresses at the edges of a thin rectangular plate. Very recently Khobragade et al [3] have discussed thermo elastic solution of a semi-infinite rectangular beam due to heat generation. In the present paper, an attempt has been made to determine the temperature distribution, displacement function and thermal stresses of a semi infinite rectangular beam occupying the space $D: -a \le x \le a;\ -b \le y \le b;\ 0 \le z < \infty$ with known boundary conditions. Here Marchi-Fasulo transforms and Fourier sine transform techniques have been used to find the solution of the problem.

II. STATEMENT OF THE PROBLEM

Consider a thin rectangular beam occupying the space $D: -a \le x \le a;\ -b \le y \le b;\ 0 \le z < \infty$. The displacement components u_x, u_y, u_z in the x, y and z directions respectively as Noda et. al.[3] are

$$u_x = \int_{-a}^{a}\left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \nu\frac{\partial^2 U}{\partial x^2}\right) + \lambda T\right]dx \tag{1}$$

$$u_y = \int_{-b}^{b}\left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} - \nu\frac{\partial^2 U}{\partial y^2}\right) + \lambda T\right]dy \tag{2}$$

$$u_z = \int_{0}^{\infty}\left[\frac{1}{E}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \nu\frac{\partial^2 U}{\partial z^2}\right) + \lambda T\right]dz \tag{3}$$

Where E, ν and λ are the Young's modulus, Poisson's ratio and the linear coefficient of thermal expansion of the material of the beam respectively and $U(x, y, z, t)$ is the Airy's stress functions which satisfy the differential equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^2 U(x, y, z, t) = -\lambda E\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)T(x, y, z, t)$$

where $T(x, y, z, t)$ denotes the temperature of a rectangular beam satisfying the following differential equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g(x, y, z, t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{5}$$

where k is the thermal conductivity and α is the thermal diffusivity of the material,
subject to the initial condition

$$T(x, y, z, 0) = f(x, y, z) \tag{6}$$

The boundary conditions are

$$\left[T(x, y, z) + k_1\frac{\partial T(x, y, z, t)}{\partial x}\right]_{x=-a} = f_1(y, z, t) \tag{7}$$

$$\left[T(x, y, z) + k_2\frac{\partial T(x, y, z, t)}{\partial x}\right]_{x=a} = f_2(y, z, t) \tag{8}$$

$$\left[T(x, y, z, t) + k_3\frac{\partial T(x, y, z, t)}{\partial y}\right]_{y=-b} = f_3(x, z, t) \tag{9}$$

$$\left[T(x, y, z, t) + k_4\frac{\partial T(x, y, z, t)}{\partial y}\right]_{y=b} = f_4(x, z, t) \tag{10}$$

$$\left[T(x, y, z, t)\right]_{z=0} = 0 \tag{11}$$

$$\left[T(x, y, z, t)\right]_{z=\infty} = 0 \tag{12}$$

The stress components in terms of $U(x, y, z, t)$ are given by

$$\sigma_{xx} = \left(\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) \tag{13}$$

$$\sigma_{yy} = \left(\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2}\right) \tag{14}$$

$$\sigma_{zz} = \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) \tag{15}$$

The equations (1) to (15) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

By applying finite Marchi- Fasulo transform defined in [2] and Fourier sine transform to (12), one obtains

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$$\bar{\bar{T}}^{*}(n,m,\eta,t)=\int_{0}^{t}\left\{\frac{\alpha\bar{\bar{g}}^{*}}{k}+\Psi\right\}e^{-\alpha q^{2}(t-t')}dt' \quad (16)$$

where $q^2 = a_n^{\ 2} + c_m^{\ 2} + \eta^2$,

$$\Psi = \alpha\Omega,\ \Omega = \bar{\phi} + \eta f_5 + \Phi$$

$$\phi = \frac{Q_n(a)}{k_1} f_1 - \frac{Q_n(-a)}{k_2} f_2$$

$$\Phi = \frac{P_m(b)}{k_3} f_3 - \frac{P_m(-b)}{k_4} f_4$$

Applying inversion of semi infinite Fourier sine transform and finite Marchi-Fasulo transform to the equation (16), one obtain the expression for temperature distribution as,

$$T(x,y,z,t)=\frac{2\eta}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)\Pi(z) \quad (17)$$

where

$$\Pi(z)=\int_0^{\infty}B(t)\sin\eta z dz,$$

$$B(t)=\int_0^t\left(\frac{\alpha\bar{\bar{g}}^{*}}{k}+\psi\right)e^{-\alpha q^2(t-t')}dt'$$

which is the required solution.

IV. AIRY'S STRESS FUNCTION

Substituting the value of temperature distribution $T(x, y, z, t)$ from equation (17) in equation (4) one obtains

$$U(x,y,z,t)=-\frac{2\eta\lambda E}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)\Pi(z) \quad (18)$$

Where $\Pi(z)=\int_0^{\infty}B(t)\sin\eta z dz$

V. DISPLACEMENT COMPONENTS

Substituting the value of Airy's stress function from equation (18) in the equations (1) to (3) one obtains

$$u_x=-\frac{2\eta\lambda}{\pi}\Pi(z)\int_{-a}^{a}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P''_m(y)}{\mu^2{}_m}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\eta^2\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)\right.$$

$$\left.-\nu\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q''_n(x)}{\lambda_n^2}\right)\left(\frac{P_m(y)}{\mu_m}\right)-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)\right]dx \quad (19)$$

$$u_y=-\frac{2\eta\lambda}{\pi}\Pi(z)\int_{-b}^{b}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q''_n(x)}{\lambda^2{}_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)\right.$$

$$\left.-\nu\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P''_m(y)}{\mu^2{}_m}\right)-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)\right]dy \quad (20)$$

$$u_z=-\frac{2\eta\lambda}{\pi}\Pi(z)\int_0^{\infty}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q''_n(x)}{\lambda^2{}_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P''_m(y)}{\mu^2{}_m}\right)\right.$$

$$\left.+\nu\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\eta^2\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)\right]dz$$

(21)

VI. DETERMINATION OF STRESS FUNCTIONS

Substituting the value of Airy's stress function $U(x, y, z, t)$ from equation (18) in the equations (13) to (15), one obtain the stresses functions as

$$\sigma_{xx}=-\frac{2\eta\lambda E}{\pi}\Pi(z)\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left[\left(\frac{P''_m(y)}{\mu^2{}_m}\right)-\sum_{m=1}^{\infty}\left(\frac{P_m(y)}{\mu_m}\right)\right] \quad (22)$$

$$\sigma_{yy}=-\frac{2\eta\lambda E}{\pi}\Pi(z)\sum_{m=1}^{\infty}\left(\frac{P_m(y)}{\mu_m}\right)\left[\frac{Q''_n(x)}{\lambda^2{}_n}-\sum_{l=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\right]$$

$$\sigma_{zz}=-\frac{2\eta\lambda E}{\pi}\Pi(z)\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q''_n(x)}{\lambda^2{}_n}\right)\left(\frac{P_m(y)}{\mu_m}\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P''_m(y)}{\mu^2{}_m}\right)\right] \quad (24)$$

VII. SPECIAL CASE

Set

$$f(x,y,z,t)=(x+a)^2(x-a)^2(y+b)^2(y-b)^2 z(1-e^{-t})$$

and

$$g(x,y,t)=\delta(x-x_1)\delta(y-y_1)\delta(z-z_1)\delta(t-t_1)$$

Applying twice finite Marchi- Fasulo transform and then Fourier sine integral transform, we get

$$\bar{\bar{g}}^{*}=A$$

$$\Omega=\bar{\bar{f}}=16(k_1+k_2)(k_3+k_4)z(1-e^{-t})$$

$$\times\left[\frac{(a_n a)\cos^2(a_n a)-\cos(a_n a)\sin(a_n a)}{a_n^{\ 2}}\right]$$

$$\times\left[\frac{(a_m b)\cos^2(a_m b)-\cos(a_m b)\sin(a_m b)}{a_m^{\ 2}}\right] \quad (26)$$

Substituting these values in equation (17), we get

$$T(x,y,z,t)=\frac{2\eta}{\pi}\Pi(z)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P_m(y)}{\mu_m}\right) \quad (27)$$

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VIII. NUMERICAL RESULTS

Se $a = 1, k = 1.2, \alpha = 0.86, b = 1, h = 1, t = 1$

A=1, $\delta = \frac{2\eta}{\pi}$, in the equation (27) to obtain

$$\frac{T(x, y, z, t)}{\delta} = \Pi(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{Q_n(x)}{\lambda_n}\right)\left(\frac{P_m(y)}{\mu_m}\right) \quad (28)$$

IX. MATERIAL PROPERTIES

The numerical calculation has been carried out for an Aluminum (Pure) rectangular beam with the material properties as,

Density ρ = 169 lb/ft3

Specific heat = 0.208 Btu/ lb0F

Thermal conductivity k = 117 Btu/(hr.ft0F)

Thermal diffusivity α = 3.33 ft2/hr

Poisson ratio ν = 0.35

Coefficient of linear thermal expansion α_t = 12.84 $\times 10^{-6}$ 1/F

Lame constant μ = 26.67

Young's Modulus of elasticity E =70GPa

X. DIMENSIONS

The constants associated with the numerical calculation are taken as

Length of rectangular beam x = 3ft

Breadth of rectangular beam y = 2ft

Height of rectangular beam z = 10^3ft

XI. CONCLUSION

The temperature distribution, displacements and thermal stresses at any point of a thin rectangular object have been obtained; when the boundary conditions are known with the aid of finite Marchi-Fasulo transform and semi infinite Fourier sine transform techniques. The results are obtained in the form of infinite series. Any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions.

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AUTHOR BIOGRAPHY

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APPENDIX

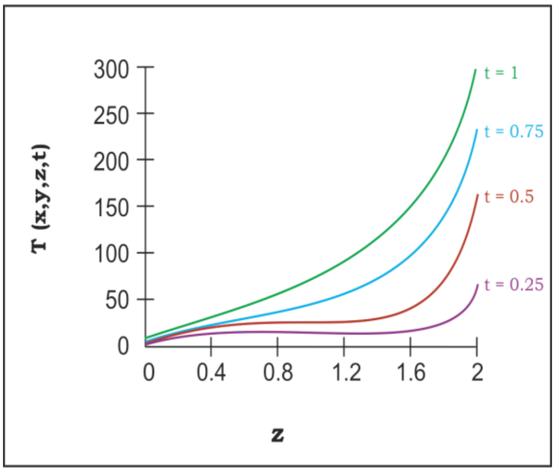

Fig. 1: Graph of temperature vs. z

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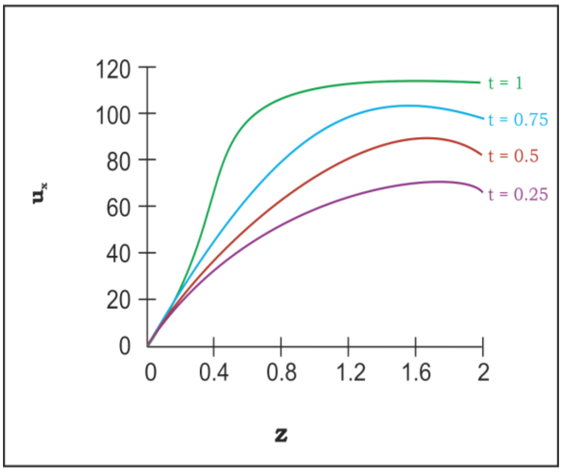

Fig. 2 : Graph of displacement component vs. z

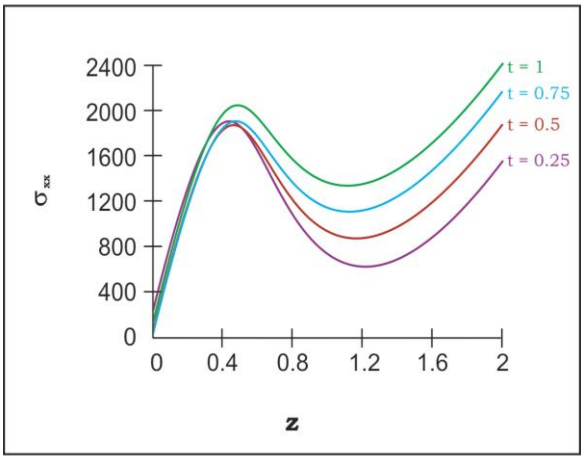

Fig. 3 : Graph of stress vs. z