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Transient Thermo Elastic Problem of a Circular Plate With Heat Generation

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Abstract- ***We apply integral transformation techniques to study thermoelastic response of a circular plate, in general in which sources are generated according to the linear function of the temperature, with boundary conditions of the radiation type. The results are obtained as series of Bessel functions. Numerical calculations are carried out for a particular case of a plate made of Aluminum metal and the results are depicted in figures.***

Keywords- **Transient response, circular plate, temperature distribution, thermal stress, integral transform**

I. INTRODUCTION

Nowacki [5] has determined steady-state thermal stresses in a thick circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face circular edge. Wankhede [7] has determined the quasi-static thermal stresses in circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature. However there aren't many investigations on transient state. Roychoudhary [6] has succeeded in determined the quasi static thermal stresses in a circular place subjected to transient temperature along the circumference of circular upper face with lower face at zero temperature and the fixed circular edge thermally insulated. In a recent work, same problems have been solved by Noda et al. [1] and Deshmukh et al. [2]. In all aforementioned investigations an axisymmetrically heated plate has been considered.

Recently Nasser [4] proposed the concept of heat sources in generalized Thermo elasticity and applied to a thick plate problem. They have not however considered any thermoelastic problem with boundary conditions of radiation type in which source are generated according to the linear function of the temperature, which satisfies the time dependent heat conduction equation. From the previous literature regarding circular solid cylinder as considered, it was observed by the author that no analytical procedure has been established considering internal heat source generation within the body.

This paper is concerned with the transient thermoelastic problem of a circular plate in which sources are generated according to the linear function of temperature occupying the space $D=\{(x,y,z)\in R^3 \ : \ 0\le (x^2+y^2)^{1/2}\le a, \quad -h\le z\le h\}$ where $r=(x^2+y^2)^{1/2}$ with radiation type boundary conditions.

II. STATEMENT OF THE PROBLEM

Consider a circular plate in which sources are generated according to the linear function of temperature. The material is isotropic, homogeneous and all properties are assumed to be constant. Heat conduction with internal heat source and the prescribed boundary conditions of the radiation type is considered. The equation for heat conduction is given by

$$k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right]+\chi(r,z,t)=\frac{\partial T}{\partial t} \tag{1}$$

Where k is the thermal diffusivity of the material of the cylinder (which is assumed to be constant), subject to the initial and boundary conditions

$$T(r,z,t)=f(r,z) \quad \text{at} \quad t=0 \quad \text{for all} \quad 0\le r\le a,\ -h\le z\le h \tag{2}$$

$$T(r,z,t)=0 \text{ at } r=a \text{ for all } -h\le z\le h,\ t>0 \tag{3}$$

$$\left[T+k_1\frac{\partial T}{\partial z}\right]_{z=h}=\exp(-\omega t)\delta(r-r_0)\ ,\quad 0\le r\le a \tag{4}$$

$$\left[T+k_2\frac{\partial T}{\partial z}\right]_{z=-h}=\exp(-\omega t)\delta(r-r_0)\quad 0\le r\le a \tag{5}$$

Where k_1 and k_2 are radiation constants on the upper and lower surface of the cylinder respectively.

The Navier's equations without the body forces for axisymmetric two-dimensional thermoelastic problem can be expressed as [3]

$$\nabla^2 u_r-\frac{u_r}{r^2}+\frac{1}{1-2\nu}\frac{\partial e}{\partial r}-\frac{2(1+\nu)}{1-2\nu}a_t\frac{\partial\theta}{\partial r}=0$$
$$\nabla^2 u_z-\frac{1}{1-2\nu}\frac{\partial e}{\partial z}-\frac{2(1+\nu)}{1-2\nu}a_t\frac{\partial\theta}{\partial z}=0 \tag{6}$$

where u_r and u_z are the displacement components in the radial and axial directions, respectively and the dilatation e as

$$e=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}$$

The displacement components are given by as [**3**]:

$$u_r=\frac{\partial\phi}{\partial r}-\frac{\partial^2 M}{\partial r\partial z} \tag{7}$$

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$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \qquad (8)$$

In which Goodier's thermoelastic potential must satisfy the equation

$$\nabla^2 \phi = \frac{1+\nu}{1-\nu} a_t \theta \qquad (9)$$

and the Michel's function M must satisfy the equation

$$\nabla^2(\nabla^2 M) = 0 \qquad (10)$$

where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$

The stress functions are given by

$$\sigma_{rr} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\nu\nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right)\right\} \qquad (11)$$

$$\sigma_{\theta\theta} = 2G\left\{\left(\frac{1}{r}\frac{\partial \phi}{\partial r} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\nu\nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right)\right\}$$
(12)

$$\sigma_{zz} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right\} \qquad (13)$$

$$\sigma_{rz} = 2G\left\{\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}\left((1-\nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right\} \qquad (14)$$

where G and ν are the shear modulus and Poisson's ratio respectively. The boundary conditions on the traction free surface of a solid cylinder are

$$\sigma_{rr} = \sigma_{rz} = 0 \text{ at } r = a \qquad (15)$$

The equations (1) to (15) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

Applying finite Hankel transform and finite Marchi Fasulo transform [3] we get the expression for temperature distribution as

$$T = \frac{2}{a^2}\sum_{m,n=1}^{\infty}\frac{J_0(r\xi_m)}{[J_0'(a\xi_n)]^2}\frac{P_m(z)}{\lambda_m}e^{-kp^2 t}$$

$$\times[B(m)e^{-(\omega-kp^2)t} + 2kp^2\int \bar{\chi}^* e^{kp^2 t}dt + \bar{f}^*] \qquad (16)$$

Where λ_m is the Marchi-Fasulo transform parameter,

$$P_m(z) = Q_m\cos(a_m z) - W_m\sin(a_m z)$$

$$Q_m = a_m(k_1+k_2)\cos(a_m h)$$

$$W_m = 2\cos(a_m h) + (k_2-k_1)a_m\sin(a_m h)$$

The Eigen values a_m are the positive roots of the characteristic equation

$$[k_1 a\cos(ah) + \sin(ah)][\cos(ah) + k_2 a\sin(ah)]$$

$$= [k_2 a\cos(ah) - \sin(ah)][\cos(ah) - k_1 a\sin(ah)]$$

$$B(m) = \frac{-k}{kp^2 + w}\left(\frac{P_m(h)}{k_1} - \frac{P_m(-h)}{k_2}\right)$$

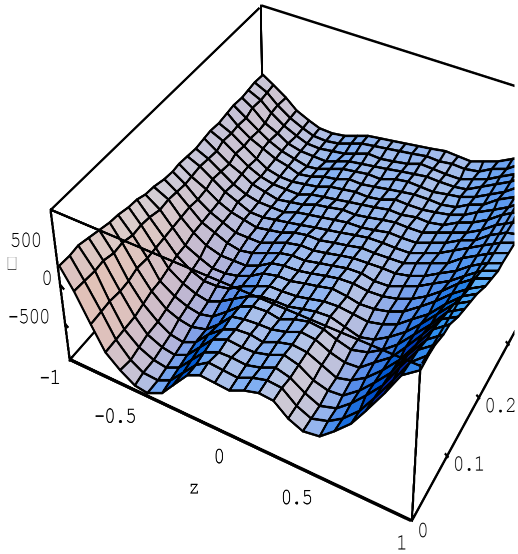

Fig 1: Temperature distribution along *r*- and *z*-direction for *t* =3

Fig. 1 shows the temperature distribution along the r and z direction of the circular plate at t = 3. It is observed that due to the width of the plate, increase in temperature was found at the beginning. Finally temperature distribution further increase and attains zero value at the extreme end.

The solution for the displacement function are represented by the Goodier's thermoelastic displacement potential ϕ governed by equation (8) is given by

$$\phi = \left(\frac{1+\nu}{1-\nu}\right)\frac{r^2 a_t}{2a^2}\sum_{m,n=1}^{\infty}\frac{J_0(r\xi_m)}{[J_0'(a\xi_n)]^2}\frac{P_m(z)}{\lambda_m}e^{-kp^2 t}$$

$$\times[B(m)e^{-(\omega-kp^2)t} + 2kp^2\int \bar{\chi}^* e^{kp^2 t}dt + \bar{f}^*] \qquad (17)$$

The solution for Michel's function M are assumed so as to satisfy the governed condition of equation (9) as

$$M = \left(\frac{1+\nu}{1-\nu}\right)\frac{r^2 a_t}{2a^2}\sum_{m,n=1}^{\infty}\frac{J_0(r\xi_m)}{[J_0'(a\xi_n)]^2}e^{-kp^2 t}$$

$$\times[A_n J_0(\beta_n r) + C_n(\beta_n r)J_1(\beta_n r)]\cosh(\beta_n z)$$

$$\times[B(m)e^{-(\omega-kp^2)t} + 2kp^2\int \bar{\chi}^* e^{kp^2 t}dt + \bar{f}^*] \qquad (18)$$

Using equations (16) and (17) in equations (6) and (7), the displacement components are obtained as

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$$u_r = -\left(\frac{1+\nu}{1-\nu}\right)\frac{a_t}{2a^2}\sum_{m,n=1}^{\infty}\frac{\beta_n \sinh(\beta_n z)}{[J_0'(a\xi_n)]^2}\ e^{-kp^2 t}$$

$$\times[B(m)e^{-(\omega-kp^2)t} + 2kp^2\int \bar{\chi}^* e^{kp^2 t}dt + \bar{f}^*]$$

$$\times\Big\{\ r^2 J_0(r\xi_n)[A_n J_0'(\beta_n r) + C_n(\beta_n r)J_1'(\beta_n r)] + C_n\beta_n J_1(\beta_n r)]$$

$$\times\left[A_n J_0(\beta_n r) + C_n(\beta_n r)J_1(\beta_n r)] - \frac{P_m(z)}{\lambda_m \beta_n \sinh(\beta_n z)}\right]$$

$$\times[r^2 J_0'(r\xi_n) + 2rJ_0(r\xi_n)]\ \Big\} \qquad (19)$$

$$u_z = \left(\frac{1+\nu}{1-\nu}\right)\frac{a_t}{2a^2}\sum_{m,n=1}^{\infty}\frac{\cosh(\beta_n z)}{[J_0'(a\xi_n)]^2}\ e^{-kp^2 t}$$

$$\times[B(m)e^{-(\omega-kp^2)t} + 2kp^2\int \bar{\chi}^* e^{kp^2 t}dt + \bar{f}^*]$$

$$\times\left\{\frac{r^2 J_0(r\xi_n)P_m'(z)}{\lambda_m \cosh(\beta_n z)} + 2r^2(1-\nu)J_0(r\xi_n)\right.$$

$$\times[A_n J_0''(\beta_n r) + C_n(\beta_n r)J_1''(\beta_n r)] + 2C_n\beta_n J_1'(\beta_n r)]$$

$$+[4r^2(1-\nu)J_0'(r\xi_n) + 10r(1-\nu)J_0(r\xi_n)]$$

$$\times[A_n J_0'(\beta_n r) + C_n(\beta_n r)J_1'(\beta_n r) + C_n\beta_n J_1(\beta_n r)]$$

$$+[2r^2(1-\nu)J_0''(r\xi_n) + 10r(1-\nu)J_0'(r\xi_n)$$

$$+(8-8\nu+\beta_n^2 r^2 - 2\nu\beta_n^2 r^2)J_0(r\xi_n)]$$

$$\left.\times[A_n J_0(\beta_n r) + C_n(\beta_n r)J_1(\beta_n r)]\right\} \qquad (20)$$

The stress components can be calculated by substituting the values of thermoelastic displacement potential ϕ from equation (16) and Michel's function M from equation (17) in equations (10) to (13), one obtain the stress functions as

$$\sigma_{rr} = -2G\left(\frac{1+\nu}{1-\nu}\right)\frac{a_t}{2a^2}\sum_{m,n=1}^{\infty}\frac{\sinh(\beta_n z)}{[J_0'(a\xi_n)]^2}\ e^{-kp^2 t}$$

$$\times[B(m)e^{-(\omega-kp^2)t} + 2kp^2\int \bar{\chi}^* e^{kp^2 t}dt + \bar{f}^*]$$

$$\times\left\{\frac{1}{\lambda_m \sinh(\beta_n z)}[rP_m(z)J_0'(r\xi_n) + (2P_m(z) + r^2 P_m''(z))J_0(r\xi_n)\right.$$

$$-\beta_n r^2(\nu-1)J_\mu(r\xi_n)\begin{bmatrix} A_n J_0''(\beta_n r) + C_n(\beta_n r)J_1''(\beta_n r)] \\ +2C_n\beta_n J_1'(\beta_n r)\end{bmatrix}$$

$$-[2(\nu-1)\beta_n r^2 J_0(r\xi_n) + r\beta_n(5\nu-4)J_0(r\xi_n)]$$

$$\times[A_n J_0'(\beta_n r) + C_n(\beta_n r)J_1'(\beta_n r)] + C_n\beta_n J_1(\beta_n r)]$$

$$-[(\nu-1)\beta_n r^2 J_0''(r\xi_n) + r\beta_n(5\nu-4)J_0'(r\xi_n)$$

$$+\beta_n(\nu\beta_n^2 r^2 + 4\nu - 2)J_0(r\xi_n)]$$

$$\left.\times[A_n J_0(\beta_n r) + C_n(\beta_n r)J_1(\beta_n r)]\right\} \qquad (21)$$

Similarly σ_{rz}, $\sigma_{\theta\theta}$ and σ_{zz} can be calculated

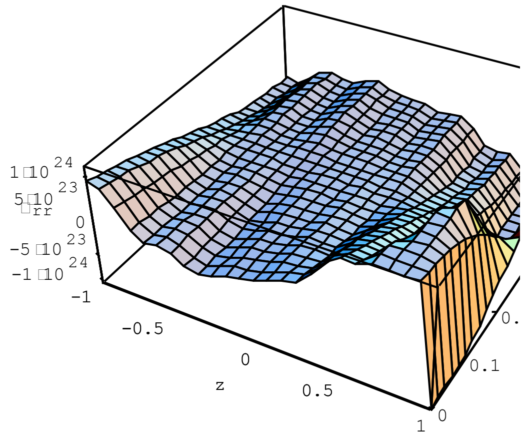

Fig. 2: Radial stress distribution along *r*- and *z*- direction

Fig. 2 shows the radial stress distribution σ_{rr} along the r and z direction of the plate with varying time. From the figure, the location of points of minimum stress occurs at the end points, while the thermal stress response is maximum at the inner surface.

IV. SPECIAL CASE

Set

$$f(r,z) = z^2(a^2 - r^2) \qquad (22)$$

The temperature distribution is given as

$$T = \frac{2}{a^2}\sum_{m,n=1}^{\infty}\frac{J_0(r\xi_n)}{[J_0'(a\xi_n)]^2}\frac{P_m(z)}{\lambda_m}\ e^{-kp^2 t}$$

$$\times[B(m)e^{-(\omega-kp^2)t}$$

$$+2kp^2\int \bar{\chi}^* e^{kp^2 t}dt + \frac{4aQ_n}{\beta_n^2 a_n^3}\left[\frac{2}{\beta_n}J_1(\beta_n a) - aJ_0(\beta_n a)\right]$$

$$\times[(a_n^2 h^2 - 2)\sin(a_n h) + 2a_n h\cos(a_n h)] \qquad (23)$$

V. NUMERICAL RESULTS, DISCUSSION AND REMARKS

Set $k_1 = k_2 = 0.5$, $r_0 = 0.20$, $z_0 = 0.5$ and $\omega = 0.5$ in equations (22) to (30).

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Table 1: Material properties and parameters used in this study

Modulus of Elasticity, E	6.9×10^{11}
Shear modulus, G	2.7×10^{11}
Poisson ratio, υ	0.281
Thermal expansion coefficient α_t	25.5×10^{-6}
Thermal diffusivity, κ	1
Length, h	1 cm
Outer radius, b	0.5 m

VI. CONCLUSION

In this paper, the temperature distributions, displacement components and thermal stresses of a circular plate have been derived with the help of finite Hankel transform and finite Marchi Fasulo transform techniques. The expressions are obtained in terms of Bessel's functions in the form of infinite series and depicted graphically. Any particular case of special interest can be assigned by choosing suitable values to the parameters and functions. The results that are obtained can be used in the construction of engineering models particularly in the field of Aeronautical space.

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