ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 1, July 2013

Construction of Trend Free Run Orders for Orthogonal Arrays Using Linear Codes

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Abstract: ***Sometimes when the experimental runs are carried out in a time order sequence, the response can depend on the run order. To avoid unwanted time effect, one may be interested to select a run order in such a way that all effects are orthogonal to the trend effect. These types of designs are known as trend free designs. There are many techniques to construct trend free run orders. In this paper, we construct some trend free run orders for symmetric and asymmetric orthogonal arrays using the results due to [5], [8] and the property that any d-1 columns in the parity check matrix of a linear $[n,k,d]_q$ code are linearly independent.***

Keyword: **Linear codes, Minimum Distance, Orthogonal arrays, Trend free run orders.**

I. INTRODUCTION

In some experimental situations where the treatments are to be applied sequentially to experimental units over space or time, there may be an unknown or uncontrollable trend effect which is highly correlated with the order in which the observations are obtained. In such situations, one may prefer to assign treatments to experimental units in such a way that the usual estimates for the factorial effects of interest are not affected by unknown trend. Such run orders are called trend free run orders. When trend free effects are considered in factorial experiments, the order of experimental runs is essential. Under such conditions it is helpful to use orthogonal arrays to arrange experimental runs and their order simultaneously. For this, one needs to derive the trend free property in the columns of array to gain an appropriate order.[11]introduced the concept of orthogonal arrays in the context of fractional factorial experiments. Orthogonal arrays are related to combinatorics, finite fields, geometry and error-correcting codes. Asymmetric orthogonal arrays, introduced by [10] have received great attention in recent years. Many researchers have dealt with the problem of construction of trend free designs. See [4], [12] and [5]. A good deal of work has been done on the construction of trend free run orders of factorial designs when all factors have same number of levels. [6], [2] and [9] considered this problem for mixed level factorial designs. In this paper, we construct some trend free run orders for symmetric and asymmetric orthogonal arrays, using the property that any d-1 columns in the parity check matrix of a linear $[n,k,d]_q$ code are linearly independent and the results due to [5] and [8] respectively. Section 1 gives the preliminaries required. In section 2 a brief introduction of coding theory is given. Section 3 presents the method to construct trend free run orders for symmetric orthogonal arrays. Trend free run orders for asymmetric orthogonal arrays are constructed in Section 4.

II. PRELIMINARIES

***Definition 1*:**An orthogonal array OA$(N,n,q_1 \times q_2 \times \ldots\ldots \times q_n,g)$ of strength g, $2 \le g \le n$ is an N×n matrix having q_i ($\ge$2) distinct symbols in the i^{th} column, i=1,2,…,n such that in every N×g sub matrix, all possible combinations of symbols appear equally often as a row.In particular, if $q_1 = \cdots\cdots = q_n = q$, the orthogonal array is called symmetric orthogonal arrayand is denoted by OA(N,n,q,g), otherwise the array is called asymmetric orthogonal array.

***Definition 2*:**The system of orthogonal polynomials on m equally spaced points *l*=0,1,2,…,m-1is the set$\{P_{sm},\ s = 0,1,2,\ldots,m-1\}$ of polynomials satisfying

$$\sum_{l=0}^{m-1} P_{sm}(l) = 0 \quad for\ all\ s \quad \ge 1 \quad (2.1)$$

$$\sum_{l=0}^{m-1} P_{sm}(l)P_{s'm}(l) = 0 \quad for\ all\ s \quad \ne s'$$

where $P_{0m}(l) = 1$ and $P_{sm}(l)$ is a polynomial of degree s. We assume that each polynomial in the system is scaled so that its values are always integers.

***Definition 3*:**(Trend vector **t**). The N values of a polynomial trend of degree *s* , $1 \le s \le N-1$ are $P_{sN}(l),\ 0 \le l \le N-1,$ the values of the orthogonal polynomial of degree s on N equally spaced points in definition 2.

From the above definition, the linear trend vector **t** can be expressed as follows:

$$t = \begin{pmatrix} -(k-1), -(k-1)+2, \ldots, -3, \\ -1,1,3,\ldots,(k-1)-2,(k-1) \Big/ c \end{pmatrix}' \quad \textbf{for k even}$$

and

$$t = \begin{pmatrix} -\frac{k-1}{2}, -\frac{k-1}{2}+1, \ldots -2,-1,0, \\ 1,2,\ldots,-\frac{k-1}{2}-1,-\frac{k-1}{2} \Big/ c \end{pmatrix}' \quad \textbf{for k odd}$$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 1, July 2013

where $C = \sqrt{\sum_{i=1}^{k} t_i^2}$ is a normalizing constant.

For a given ordered allocation of the treatments in d_N(N run orthogonal design) to experimental units ,let

$$\mathbf{Y} = (y_1, y_2, \dots y_N)'$$

denotes the ordered vector of observations. Suppose these observations are influenced by a time trend that can be represented by a polynomial of degree s (1≤s≤N-1). The model for d_N can be written as

$$\mathbf{Y} = (\mathbf{X}\ \mathbf{T})\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{pmatrix} + \epsilon$$

where ϵ is a N-vector of uncorrelated random errors with zero means, $\mathbf{X}$ is N×n matrix of factor effect coefficients (we consider only main effects for trend freeness) and **T** is the N×k matrix of polynomial trend coefficients. The vectors $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are the factor and trend parameter effects respectively.

A run order is optimal for the estimation of the factor effects of interest β_1 in thepresenceof nuisance s-degree polynomial trendif

$$\mathbf{X}'\mathbf{T} = 0 \qquad (2.4)$$

If(2.4) is satisfied we say that the run order is s trend free. If **x** is any column of $\mathbf{X}$ and **t** is any column of **T** then the usual inner product $\mathbf{x}'\mathbf{t}$ is called the time count between **x** and **t**. Criterion (2.4) states that all the time count are zero for optimal run order.

III. CODING THEORY

A linear $[n,k,d]_q$ code C over GF(q),where q prime or prime power, n is the length, k is the dimension and d is the minimum distance, is a k-dimensional subspace of the n-dimensional vector space V(n,q) over GF(q). The elements of C are called code words. The minimum distance d of the code is the smallest number of positions in which two different code words of C differ. Equivalently, d is the smallest number of nonzero symbols in any nonzero codeword of C. A linear code may be concisely specified by giving a k×n generator matrix G whose rows form a basis for the code. The standard form of the generator matrix is

$$G = [\ \mathbf{I}_k \mid A\ ]$$

Where A is an k × (n – k) matrix with entries from GF(q).

The dual code $C^\perp$ of an $[n,k,d]_q$ code C is $C^\perp = \{ v \in V(n,q) / v.w=0 \text{ for all } w \in C\}$. This is an $[n,n-k,d^\perp]_q$ code and an (n-k)×n generator matrix H of $C^\perp$ is called a parity check matrix of C. If the generator matrix is given in the standard form, a corresponding parity check matrix is given as

$$H = [-\mathbf{A}^{\mathbf{T}} \mid \mathbf{I}_{\mathbf{n-k}}]$$

Any $d^\perp$-1 columns in generator matrix G of C are linearly independent and any d-1 columns in parity check matrix H are linearly independent.

Trend free run orders for Symmetric Orthogonal Arrays

In this section, we construct trend free run orders for symmetric orthogonal arrays. [5]used Generalized Fold over Scheme (GFS) to construct trend free designs and also discussed conditions for linear trend free effects in GFS. These conditions involve the generator matrix. They also provided a method for construction of generator matrix so that the systematic run order for a design is constructed by GFS. However this method is difficult to use. The generator matrices for the construction of systematic run order are obtained from linear codes.

We present here the main result due to [5] in the form of following theorem. (For more details see [7] Th.7.3.3). The method of construction in Theorem 1 is called the Generalized Foldover Technique.

Theorem 1: Let q (≥2) be a prime or prime power. Suppose that there exists an r × n matrix M , with elements from GF(q), such that every r × g submatrix of M has rank g and every column of M has at least (s+1) non zero elements. Then there exists a symmetric orthogonal array OA(q^r , n , q ,g) in which all main effects are s-trend free.

We use the result due to [5] to construct some trend free run orders for orthogonal arrays obtainable from the linear codes.

Example 1: Consider the linear code $[8,4,4]_2$ given in [3] with parity check matrix H where

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0\\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Consider those columns of H which have weight ≥ 2, where, the weight of a column is defined as the number of nonzero elements in the column. We get the following matrix M

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 0 & 0 & 1 & 1\\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The matrix M satisfy the condition of Theorem 1 with q=2, k=4, g= 3, s + 1 = 2, n = 7. Let ξ denote a 4 ×1 vector with entries from GF(2). Considering all the 2^4 possible distinct choices of ξ, we form a $2^4 \times 7$ array with rows of the form $\xi' M$.

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 1, July 2013

ξ' M=

$$\begin{bmatrix} 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{bmatrix}$$

$$\begin{bmatrix} 1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1 \\ 0&1&1&0&0&1&1 \\ 1&0&1&0&1&0&1 \end{bmatrix}$$

The array generated is given in Table I. From the table I, we observe that the time counts for all the main effects are zero. Hence we get a trend free run orders for OA(2^4,7,2,3) having all the main effects linear trend free.

Example 2: Consider a linear $[10,5,4]_2$ code given in [3]. The parity check matrix H of the linear code is given as

$$H = \begin{bmatrix} 1&0&0&0&0&1&1&1&0&0 \\ 0&1&0&0&0&1&1&0&1&0 \\ 0&0&1&0&0&1&1&0&0&1 \\ 0&0&0&1&0&1&0&1&1&1 \\ 0&0&0&0&1&0&1&1&1&1 \end{bmatrix}$$

From the matrix H if we retain all the columns with weight ≥ 3, then we get the following matrix

$$L = \begin{bmatrix} 1&1&1&0&0 \\ 1&1&0&1&0 \\ 1&1&0&0&1 \\ 1&0&1&1&1 \\ 1&0&1&1&1 \end{bmatrix}$$

The matrix L satisfy the conditions of Theorem 1 with q=2, k=5, g=3, s+1=3, n=5. Let ζ denote a 5×1 vector with entries from GF(2). Considering all the 2^5 possible distinct choices of ζ we form $2^5 \times 5$ array with rows of the form ζ' L. A trend free run order for OA (2^5,5 ,2,3) is obtained having all main effects linear and quadratic trend free. Table II lists the trend free run orders for symmetric orthogonal arrays (with degrees of trend freeness of all main effects obtained from a class of linear codes given by [3].

IV. TREND FREE RUN ORDERS FOR ASYMMETRIC ORTHOGONAL ARRAYS

Trend free run orders for asymmetric orthogonal arrays can also be obtained from the parity check matrix of a linear $[n,k,d]_q$ code. Using the generator matrices obtained from linear codes in Section 3, we construct trend free run order for asymmetric orthogonal arrays.

Consider an OA(N, n, $q_1 \times q_2 \times \ldots\ldots \times q_n$, g) whose columns are called as factors denoted by F_1, F_2, ..., F_n. Also consider GF(q), of order q, where q is a prime or prime power. For the factor F_i $(1 \leq i \leq n)$ define u_i columns, say $p_{i_1}, p_{i_2}, \ldots, p_{i_{u_i}}$, each of order k× 1 with elements from GF(q). Thus for the n factors we have in all $\sum_{i=1}^{n} u_i$ columns. The following result was proved in [1].

Theorem 2: Let M be the k × n matrix, where n = $\sum u_{i_j}$ and k $\geq \sum u_{i_j}$ such that any d – 1 columns of C are linearly independent. Then M can be partitioned as M= $[A_1 \; A_2 \; \ldots \; A_n]$, where A_i = $[p_{i_1} \; p_{i_2} \; \ldots \; p_{i_{u_i}}]$, $1 \leq i \leq n$. Then for each of the matrices $A_{i_1}, A_{i_2}, \ldots, A_{i_g}$; where $g \leq d-2$, out of $A_1, A_2, \ldots, A_n$; the $k \times \sum u_{i_j}$ matrix $\left[A_{i_1} \; A_{i_2} \; \ldots \; A_{i_g}\right]$ has full column rank over GF(q), Then an OA(q^k, n, $(q^{u_1}) \times \ldots \times (q^{u_n})$, g) can be constructed.

Using Theorem 1 and Theorem 2 we present a method in the form of a theorem given below. The proof of this theorem follows from the proofs given in [7] Th.7.3.3) and [1].

Theorem 3: Let q be a prime or prime power. If there exist an k × n matrix M with entries from GF(q^k) such that

i) Any d-1 columns, where $d \leq n$ be any positive integer, are linearly independent over GF(q^k) and

ii) Every column of M has at least s+1 non zero elements then

an OA(q^k, n, $(q^{u_1}) \times \ldots \times (q^{u_n})$, g) can be constructed in which all the main effects are trend free of order s.The method is explained in the following example.

Example 4: Consider the matrix M obtained in Example 1. Represent Mas $[M_1 \; M_2 \; \ldots \; M_7]$, where M_i ; $1 \leq i \leq 7$ denotes the i^{th} column of matrix M.

To construct an orthogonal arrayOA(2^4, 6, $(2^2) \times 2^5$, 2) we choose the following matrices, corresponding to the factors of the array.

$$A_1 = \begin{bmatrix} 1&1 \\ 0&0 \\ 0&1 \\ 1&0 \end{bmatrix} = [\, M_1 M_2 \,] ,\; A_i = M_{i+1} \; 2 \leq i \leq 5$$

The rank condition of the Theorem 2 is always satisfied for g = 2 by the above matrix M. This can also be shown with above choices of A_i matrices corresponding to the 6

(i) Let i, j ∈ { 2, 3, ... , 6} ; i ≠ j. For this choice of the indices i and j, the matrix $[A_i , A_j]$ will always have rank 2, because any 3 columns of the matrix M are linearly independent (Theorem 2).

IJEIT
ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 1, July 2013

(ii) Let i = 1 and j ∈ { 2, 3, …, 6} ; i ≠ j. For this choice of indices i, j the matrix $[A_i, A_j]$ will always have rank 2 since any 3 or fewer columns of M are linearly independent.

Thus in each case the rank condition of Theorem 2 is satisfied and the desired orthogonal array can be constructed by Computing $\xi' M$ where ξ is a $2^4 \times 4$ matrix whose rows are all possible 4-tuples over GF(2) and replacing the 4 combinations (00), (01), (10), (11) under the first two columns by 4 distinct symbols 0, 1, 2, 3 respectively we get an OA(2^4 ,6, (2^2) $\times 2^5$ 2).Here we observe that the run order for column (factor) with 4 symbols say F_1 is also linear trend free with the other remaining five columns (factors). Thus we get trend free run order for asymmetric orthogonal array OA(2^4 ,6, $(2^2) \times 2^5$, 2) in which all the main effects are linear trend free.

In Table III we list the trend free run orders (with degree of trend free of all the main effects) for asymmetric orthogonal arrays obtained from a class of linear codes given by [3]

Table IOA $(2^4,7,2,3)$

	A	*B*	*C*	*D*	*E*	*F*
	0	0	0	0	0	0
	1	1	1	1	1	1
	0	0	0	1	1	1
	1	1	1	0	0	0
	0	1	1	0	0	1
	1	0	0	1	1	0
	0	1	1	1	1	0
	1	0	0	0	0	1
	1	0	1	0	1	0
	0	1	0	1	0	1
	1	0	1	1	0	1
	0	1	0	0	1	0
	1	1	0	0	1	1
	0	0	1	1	0	0
	1	1	0	1	0	0
	0	0	1	0	1	1
Time count	**0**	**0**	**0**	**0**	**0**	**0**

Linear codes	Asymmetric Orthogonal array	Degree of trend free for main effects in Asymmetric orthogonal array
$[8,4,4]_2$	$(2^4,6,(2^2)\times2^5,2)$	Linear
$[10,5,5]_2$	$(2^5,4,(2^2)\times2^3,3)$	Linear,
	$(2^5,3,(2^3)\times2^2,2)$	Quadratic
$[12,6,4]_2$	$(2^6,5,(2^2)\times2^4,2)$	Linear, Quadratic
$[16,8,5]_2$	$(2^8,7,(2^2)\times2^6,3)$ $(2^8,6,(2^3)\times2^5,2)$	Linear, Quadratic
$[18,9,6]_2$	$(2^9,8,(2^2)\times2^8,4)$ $(2^9,7,(2^3)\times2^6,2)$	Linear, Quadratic
$[22,11,6]_2$	$(2^{11},10,(2^2)\times2^9,4)$	Linear, Quadratic
$[24,12,7]_2$	$(2^{12},11,(2^2)\times2^{10},5)$ $(2^{12},10,(2^3)\times2^9,3)$	Linear, Quadratic
$[28,14,8]_2$	$(2^{14},13,(2^2)\times2^{12},6)$ $(2^{14},12,(2^3)\times2^{11},4)$	Linear, Quadratic

Linear codes	Trend free symmetric OA's	Degree of trend free for main effects in symmetric OA's
$[8,4,4]_2$	$(2^4,7,2,3)$	Linear
$[10,5,4]_2$	$(2^5,5,2,3)$	Linear, Quadratic
$[12,6,4]_2$	$(2^6,6,2,3)$	Linear, Quadratic
$[16,8,5]_2$	$(2^8,8,2,4)$	Linear, Quadratic
$[18,9,6]_2$	$(2^9,9,2,5)$	Linear, Quadratic
$[22,11,6]_2$	$(2^{11},11,2,6)$	Linear, Quadratic
$[24,12,7]_2$	$(2^{12},12,2,6)$	Linear, Quadratic
$[28,14,8]_2$	$(2^{14},14,2,7)$	Linear, Quadratic

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 1, July 2013

VI. CONCLUSION

The generator matrices for the construction of trend free run orders for orthogonal arrays can easily be obtained from linear codes as given in this article. Further this technique can also be used to construct trend free run orders for orthogonal arrays of higher/mixed levels and for higher strength.

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