ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 10, Issue 5, November 2020

# Deflection Analysis of Multi-Stepped Beams under Parallel Forces

Kazem Abhary
School of Engineering, University of South Australia, AUSTRALIA

***Abstract:-*** ***This paper explains an accurate computational method, within the limits of conventionally accepted assumptions in Solid Mechanics, for deflection analysis of multi-stepped beams under parallel forces. First the deflection analysis of a cantilever multi-stepped beam under the application of a force and moment at its free end is developed. Then the deflected beam is modeled as two cantilever beams, joined at the section right under the force, and deflected by the reaction forces. The modeling procedure is implemented by hypothetically cutting the beam right under the point of application of the force and treating each portion of it as a cantilever multi-stepped beam clamped at the cut section. Then it is illustrated how the deflection of the beam at steps and any other point of interest is to be calculated. Finally, this method is used to analyze the deflection of a stepped beam under the application of multiple loads. For this purpose the beam is analyzed under each force separately and then the corresponding deflections are superimposed to achieve the overall deflection and slope of the beam at each step and at any other point of interest, including the point of application of the forces. A numerical procedure is established to determine the magnitude and the location of maximum deflection.***

***Keywords:*** **deflection analysis, stepped beams, computational mechanics**.

## I. INTRODUCTION

Deflection analysis of multi-stepped beams, unlike their stress analysis which is very much the same as that of uniform beams, is a complex procedure. It is due to this very reason that textbooks on Mechanics of Solids/Materials, Structures and Mechanical Design have rarely touched this topic.

The literature on this topic is very limited. A few researchers in the past have developed different approaches to the problem. Halaz developed a method, based on bending theory of uniform beams, to find the deflection equation of each span of the beam and used them, by trial and error, to search for the location and the magnitude of maximum deflection of the beam [5]. Niemann introduced a method for calculating the deflection of the point of application of a single force applied upon a stepped beam by visualising it as two cantilever stepped beams joined at the section under the force [7]. Dearth used general energy method in conjunction with bending theory and developed a rather lengthy but fairly accurate formulation of the problem [4]. Application of bending theory of uniform beams to stepped beams introduces some error in calculation.

**Manuscript received: 10 October 2020**
**Manuscript received in revised form: 04 November 2020**
**Manuscript accepted: 24 November 2020**
**Manuscript Available online: 15 December 2020**

Sanderson and Kitching demonstrated that where a beam is stepped with large and abrupt change of section the calculations incur errors [8]. Also Mischke demonstrated the error that is organic to trapezoidal approximation of integration. He showed that remedy is simple and exact using Simpson's rule [6].

Chen and Wang developed a general expression for determining accurate deflection and slope of stepped shafts using a method combining singularity functions with Laplace transformations [2]. Datta and Bandyopadhyay analysed deflection, slope and critical speeds of simply supported shafts by developing computer software [3]. Caddemi et al developed a closed-form solution to deflection function of stepped Timoshenko beams with internal singularities and along-axis internal supports via modelling the internal singularities as concentrated reductions in the flexural and the shear stiffness by making use of the distribution theory [1].

The method cares for any number of concentrated external parallel forces applying on the beam, and distributed forces could be approximated by a suitable number of concentrated forces. The method developed here is mathematically simpler and more straightforward than all other above methods.

In the following calculations, downward forces and clockwise moments are considered positive and upward forces negative.

Maximum deflection of a cantilever uniform beam under a single load at its free end. Figure 1a shows a cantilever beam under the application of force P at its free end and Figure 1b shows the same beam under the application of bending moment M. From fairly established theories of Mechanics of Solids, the linear and angular deflection of the former is

$$\delta = PL^3/3EI \tag{1}$$

$$\alpha = PL^2/2EI \tag{2}$$

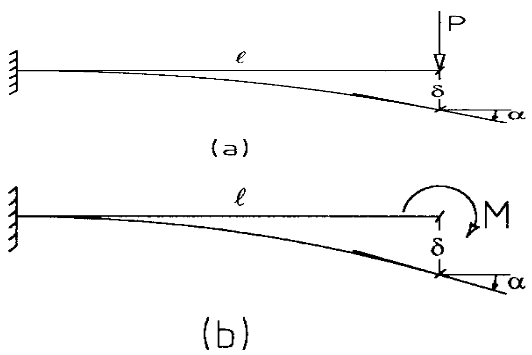


**Fig. 1. Deflection of a cantilever beam**

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 10, Issue 5, November 2020

Respectively and those of the latter are

$$\delta = ML^2/2EI \quad (3)$$

$$\alpha = ML/EI \quad (4)$$

Where E is the *Young modulus* of the beam and I is its *area moment of inertia*. The positive direction of loads and displacements are as shown in Figure 1. This piece of well-known information will be used in the next section to pave the road for deflection analysis of multi-stepped shafts.

## II. DEFLECTION ANALYSIS OF A SIMPLY SUPPORTED BEAM UNDER A SINGLE FORCE

To pave the way for deflection analysis of stepped beams the deflection of a simply supported uniform beam AC under force P, Figure 2, is calculated herein different from the standard methods in the textbooks on Mechanics of Solids by visualising the beam as two cantilever beams fixed at section B under the force. Because of the small elastic deflections of the beam, this is achieved by drawing the tangent $A'C'$ to the deflected beam at the point of application of the load, i.e. point $B'$. According to Figure 2, $B'A'$ and $B'C'$ are two cantilever beams under reaction forces $R_1$ and $R_2$ at their free ends respectively. Therefore, from Equation 1

$$\delta_1 = R_1 L_1^{\ 3}/3EI = PL_1^3 L_2/3EIL \quad (5)$$

$$\delta_2 = R_2 L_2^{\ 3}/3EI = PL_1 L_2^3/3EIL \quad (6)$$

and from trapezium $ACC'A'$, the slope $\beta$ (see Figure 3) and the deflection y of the beam at $B'$ are

$$\beta = (\delta_2 - \delta_1)/L \quad (7)$$

$$y = (L_2\delta_1 + L_1\delta_2)/L \quad (8)$$

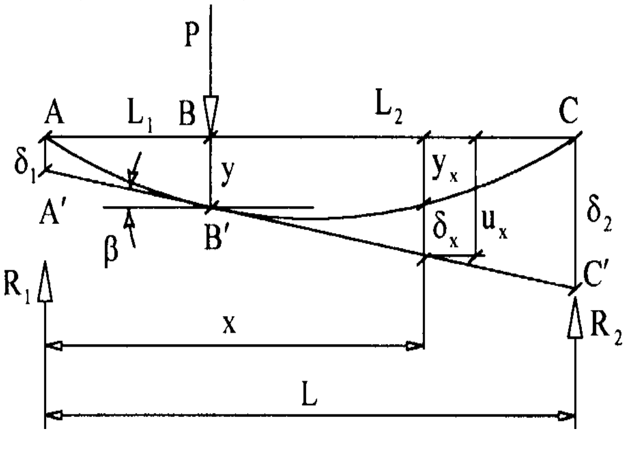


**Fig. 2. Deflection of a simply supported beam**

Displacement of the beam at a point positioned at distance x from the left end, $y_x$, can be calculated from trapezium $BCC'B'$

$$y_x = u_x - \delta_x \quad (9)$$

Where

$$u_x = [(L-x)\delta_1 + x\delta_2]/L \quad (10)$$

and (using Equation 1 and 3)

$$\delta_x = R_1(L_1 - x)^3/3EI + R_1 x(L_1 - x)^2/2EI \quad \text{for } x < L_1 \quad (11)$$

$$\delta_x = R_2(x - L_1)^3/3EI + R_2(L - x)(x - L_1)^2/2EI \quad \text{for } x > L_1 \quad (12)$$

The slope of the beam at distance x from the left end, $\theta$, according to Figure 3

$$\theta = \alpha + \beta \quad \text{for } x < L_1 \quad (13.1)$$

$$\theta = \alpha - \beta \quad \text{for } x > L_1 \quad (13.2)$$

where $\beta$ is obtained from Equation 7 and $\alpha$(using Equation 2 and 4) is

$$\alpha = R_1(L_1 - x)^2/2EI + R_1 x(L_1 - x)/EI \quad \text{for } x < L_1 \quad (14)$$

$$\alpha = R_2(x - L_1)^2/2EI + R_2(L - x)(x - L_1)/EI \quad \text{for } x > L_1 \quad (15)$$

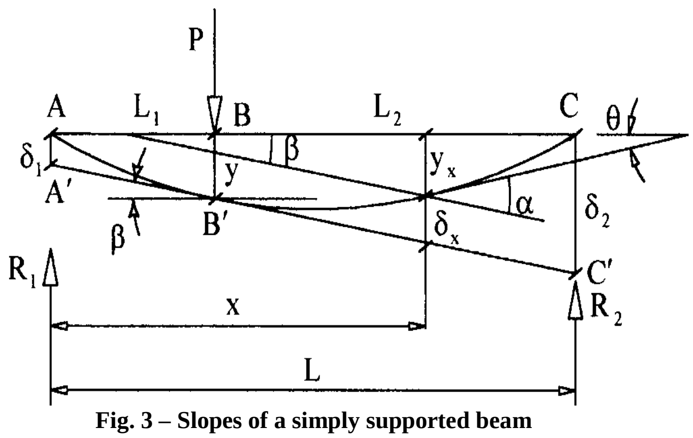


**Fig. 3 – Slopes of a simply supported beam**

## III. DEFLECTION OF A MULTI-STEPPED CANTILEVER BEAM UNDER A SINGLE FORCE AT ITS FREE END

To analyse the deflection of a multi-stepped cantilever beam of n spans under a single force P at its free end, the length and area moment of inertia of span i is denoted by $L_i$ and $I_i$ respectively. To formulate the deflection, the beam is cut at step i from which it can be seen that span i functions as a cantilever beam, fixed at step $i-1$, such that

$$\delta_i = \delta_{i-1} + \delta_{i/i-1}$$

or

$$\delta_i = \delta_{i-1} + h_i + v_i \quad (16)$$

Where $\delta_i$ = displacement of the beam at step i

$\delta_{i-1}$ = displacement of the beam at step i-1

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 10, Issue 5, November 2020

$\delta_{i/i-1}$ = displacement of the beam at step i with respect to step i-1

$h_i$ = rigid displacement of the beam at step i with respect to step i-1

$v_i$ = elastic displacement of the beam at step i with respect to step i-1 and using Equation 2 and 4

$$\Delta\alpha_i = PL_i^2/2EI + ML_i/EI_i$$
$$= PL_i^2/2EI + P(\ell - \ell_i)L_i/EI_i$$

Hence

$$\Delta\alpha_i = PL_i[L_i + 2(\ell - \ell_i)]/2EI_i \quad (17)$$

$$\alpha_i = \alpha_{i-1} + \Delta\alpha_i \quad (18)$$

$$h_i = L_i\alpha_{i-1} \quad (19)$$

and using Equation 1 and 3

$$v_i = PL_i^3/3EI + P(\ell - \ell_i)L_i^2/2EI_i$$

or

$$v_i = PL_i^2[2L_i + 3(\ell - \ell_i)]/6EI_i \quad (20)$$

Now Equations 17 to 20 are employed in Equation 16 to calculate the displacement of the beam at step i. To reach this stage, computation starts at step 1 (knowing that at the fixed end, i.e. step 0, $\alpha_0 = h_0 = \delta_0 = v_0 = 0$):

$$\alpha_1 = \alpha_0 + \Delta\alpha_1 = PL_1[L_1 + 2(\ell - \ell_1)]/2EI_1 \quad (21)$$

$$h_1 = L_1\alpha_0 = 0 \quad (22)$$

$$v_1 = PL_1^2[2L_1 + 3(\ell - \ell_1)]/6EI_1 \quad (23)$$

$$\delta_1 = 0 + 0 + v_1 \quad (24)$$

and proceeds step by step to the free end, step n.

Generally, at a point positioned at distance x from the left end between steps i-1 and i, i.e. $\ell_{i-1} < x < \ell_i$ displacement can be calculated simply by replacing $\ell_i$ by x and $L_i$ by $x - \ell_{i-1}$ in Equations 17 to 20:

$$\Delta\alpha_x = P(x - \ell_{i-1})[x - \ell_{i-1} + 2(\ell - x)]/2EI_i \quad (25)$$

$$\alpha_x = \alpha_{i-1} + \Delta\alpha_x \quad (26)$$

$$h_x = (x - \ell_{i-1})\alpha_i \quad (27)$$

$$v_x = P(x - \ell_{i-1})^2[2(x - \ell_{i-1})_i + 3(\ell - x)]/6EI_i \quad (28)$$

Therefore, from Equation 16

$$\delta_x = \delta_{i-1} + h_x + v_x \quad (29)$$

## IV. DEFLECTION OF A SIMPLY SUPPORTED MULTI-STEPPED BEAM UNDER A SINGLE LOAD

A simply supported multi-stepped beam is treated as two cantilever beams, as shown in Figure 3, each one analysed according to Equations 16 to 20. The deflection under the load is obtained from Equation 8, the deflection at any other point including steps, from Equation 9 and its angular deflection from Equation 13.

In general case, a number of parallel forces $P_j$, j = 1, 2, …, p, may be applied on the beam. In this case, the contribution of force j to the linear and angular deflection of the beam at step i, i.e. $y_{ij}$ and $\theta_{ij}$ respectively, is obtained by applying the method developed herein and then superimposing them to obtain the overall linear and angular deflection of the beam:

$$\theta_i = \sum_{j=1}^{p} \theta_{ij} \quad (30)$$

$$y_i = \sum_{j=1}^{p} y_{ij} \quad (31)$$

***Example***

The multi-stepped beam of Young modulus *E = 210 GPa* , dimensions (in mm) and span diameters *40, 50, 60, 50* and *40mm* from the left respectively, is used to demonstrate the application of the method developed herein.

The algorithm was performed manually to determine the contribution of each load to the deflection of the beam. The manual calculation showed that the slope sign of step 2 and 3 are different. This implies that the maximum deflection occurs between these two steps. At this stage the calculation was repeated for a few points along span 3 (i.e. between steps 2 and 3) which led to the maximum deflection at *x = 1,262.5 mm* as follows:

| | |
|---|---|
| Under $P_1$: | $y_{1max} = 5.14mm$ |
| Under $P_2$: | $y_{2max} = 2.41mm$ |
| Total: | $y_{max} = 7.55mm$ |

The algorithm was programmed in Excel and run for this example. The computer results were quite consistent with the above-mentioned manual calculations.

## V. CONCLUSION

A method has been developed for deflection analysis of multi-stepped beams under parallel forces. The method is specifically designed to be computational; therefore it easily lent itself to computerisation. The advantage of the method is that it is easily extendable to general three-dimensional loading (the next phase of this project). The method is straightforward; this was justified even by manual application.

## REFERENCES

[1] Caddemi, S., Calio `, I. and Cannizzaro, F.,"Closed-form solutions for stepped Timoshenko beams with internal singularities and along-axis external supports", Archive of Applied Mechanics, v 83, n 4, PP.559-577, 2013.

[2] Chen, Lian and Wang, Yuanwen, "General expression for determining deflections of the stepped shaft in bending",

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 10, Issue 5, November 2020

Journal of Mechanical Strength, v 22, n 2,PP.153-155, 2000.

[3] Datta, D. and Bandyopadhyay, P.,"An in-house developed software for generalised deflection analysis of a stepped shaft", Journal of the Institution of Engineers (India): Mechanical Engineering Division, v 82, n 2,PP.47-51, 2001.

[4] Dearth, D. R.,"Finding Deflection of Complex Beams", Journal of Machine Design, October 11, pp.98-99, 1979.

[5] Halaz, S. T.,"Minicalculators Find Stepped Beam Deflections", Journal of Machine Design, June 27,pp.78-82, 1974.

[6] Mischke, C.R.,"An exact numerical method for determining the bending deflection and slope of stepped beams", Proceedings of Advances in reliability and stress analysis conference, ASME, San Francisco, California, December, pp.101-115,1979.

[7] Niemann, G.,"Machine Elements Design and Calculation in Mechanical Engineering",Spring-Verlag, Vol 1, pp.317-318,1978.

[8] Sanderson, N. and Kitching, R., "Flexibility of Beams with Abrupt Change of Section", Int. J. Mech. Sci., Vol. 20, pp.189-199,1978.