Measuring Chaos in Some Discrete Nonlinear Systems

L.M. Saha, Niteesh Sahni, Til Prasad Sarma

Abstract—Lyapunov exponents and correlation dimensions, as measuring tools for chaos in a dynamical system, are explained in detail. Some discrete systems exhibiting chaotic motion have been discussed for the application of these tools. Bifurcation diagrams are drawn for such systems which provide clear ideas of regular and chaos at various set of parameter values. Also, chaotic attractors for these systems have been obtained for certain orbits at certain set of parameter values. Then, numerical calculations have been carried out for each model and plot of Lyapunov exponents and plot of correlation integral curve have been obtained. Least square linear fit method has been applied to find correlation dimension to the obtained data for correlation integrals. Some interesting graphics obtained through numerical simulation indicate very interesting results.

Index Terms—Lyapunov Exponents, Correlation Dimensions, Bouncing Ball.

I. INTRODUCTION

All evolutionary systems come under the domain of dynamical system. A detailed discussion on the subject can be obtained from pioneer articles on the subject by Devaney (1989), Hao Bai-Lin (1984), Smale (1967), Moon (1987), Stewart (1989), Gleich (1987), Sarkovskii (1964), May (1974, 1976), Mandelbrot (1983), Hénon (1976), Chirikov (1979) etc. Chaos is exhibited in nonlinear systems and can be viewed by observing bifurcations by varying a parameter of the system. As natural systems are mostly nonlinear, existence of chaos in nature is also quite natural. We say a system evolve chaotically if it shows divergence in behavior of two trajectories initiated at slightly different initial conditions. Such sensitivity to initial condition was first noticed by Henri Poincaré (1913), and later termed as chaos, Lorenz (1963). Lyapunov characteristic exponents, (LCE), are considered as very effective tools to distinguish regular and chaotic motions and provide a clear measure of chaos. If the divergence is exponential in time, with the constant factor, say \( \lambda \), in the exponent, then \( \lambda \) is a LCE of the system and if \( \lambda > 0 \), then the system becomes chaotic. The system is regular as long as \( \lambda \leq 0 \), (Ref. Grassberger and Procaccia (1983), Sandri (1996), Martelli (1999), Nagashima and Baba (2005), Saha et al (2006), Litak et al (2009)). A chaotic system display a set called strange attractor which is composed of a complex pattern. It is a dense set within which all periodic motions are unstable and has fractal structure i.e. having self-similar property. Dimension of such a set is positive fractional, (not an integer), and provides a measure of chaos of the system. Chaos measurement in a system can be given by the measure of positivity of the LCEs for every orbit and also, by the measurement of correlation dimension.

The objective of the present work is to observe regular and chaotic motions in certain discrete dynamical systems, which have large applications in different areas. Bifurcation diagrams are obtained for each system and then, proceeded further to calculate numerically LCEs and correlation dimension of certain chaotic orbit. Our investigation would confined to the following discrete systems: one dimensional epidemic model, one dimensional exponential (Salmon) map, the gross national product (GNP) model, prey-predator population model, neural networks model and Ushiki map. Appearance of chaos is shown through bifurcation diagrams which indicate the qualitative change in behavior of the system due to certain variations of system parameters.

II. SOME DEFINITION

A. Bifurcation Phenomena

In ordinary sense, bifurcation means splitting into two. Bifurcation in a dynamical system occurs when a small smooth change made to the parameter values of the system causing a sudden ‘qualitative’ or topological change in its behavior. It is the sudden change in behavior due to sudden change of set of parameter values of the system. The point, where qualitative change in behavior occur, is known as the bifurcation point. The name "bifurcation" was first introduced by Henri Poincaré in 1885. Bifurcations occur in both continuous systems (described by ODEs, DDEs or PDEs), and discrete systems (described by maps).

B. Lyapunov Exponents

Relative stability of typical orbits of a system is measured by numbers called Lyapunov exponents, named after the Russian Engineer Alexander M. Lyapunov. A system can have as many Lyapunov exponents as there are dimensions in its phase space. Lyapunov exponents are less than zero, signifies nearby initial conditions all converge on one another, and initial small errors decrease with time. However, if any of the Lyapunov exponents is positive, then infinitesimally nearby initial conditions diverge from one another exponentially fast; means the errors in initial conditions will grow with time. This condition, known as sensitive dependence on initial conditions, is one of the few universally agreed-upon conditions defining chaos.

Since the eigenvalues of a limit cycle characterize the rate at which nearby trajectories converge or diverge from the cycle, Lyapunov exponents can be considered generalizations of the eigenvalues of steady-state and limit-cycle solutions to differential equations. Calculation of Lyapunov exponents involves (for nonlinear systems) numerical integration of the underlying differential/difference equations of motion, together with their associated equations of variation. The
equations of variation govern how the tangent bundle attached to a system trajectory evolves with time. The largest eigenvalue of a complex dynamical system is an indicator of chaos, Saha and Budhraja (2007). For, a one dimension map the separation at nth iteration can be given by

\[ 1 x_n - y_n \approx \left( \prod_{t=0}^{n-1} f'(t) \right) \left| x_0 - y_0 \right|, \]

(2.1)

where \( x_n \) and \( y_n \) are respectively the n\textsuperscript{th} iterations of orbits of \( x_0 \) and \( y_0 \) under f. Then, the exponential separation rate \( \log |f'(x)| \) of two nearby initial conditions, averaged over the entire trajectory, can be given by

\[ \lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \log \left( \prod_{t=0}^{n-1} f'(x_t) \right), \]

(2.2)

where \( \prod_{t=0}^{n-1} f'(x_t) \approx e^{\lambda(x_0)n} \), for \( n >> 1 \).

The \( \lambda(x_0) \) defined in (2.2) is the LCE of orbit of \( x_0 \). Quantitatively, two trajectories in phase space with initial separation \( \delta x_0 \) diverge (provided that the divergence (can be treated within the linearized approximation)

\[ \delta x(t) \approx e^{\lambda t} \delta x(0) \]

(2.5)

Where \( \lambda > 0 \) is the Lyapunov exponent.

C. Correlation Dimension

Correlation dimension describes the measure of dimensionality of the chaotic attractor. It is a positive fractional (non-integer) number. The correlation dimension was introduced in a work by Grassberger and Procaccia (1983) and letter followed by various researchers. Actually, LCE be a positive measure of sensitivity of the initial. A statistical measure provides us more authenticity in analysis of the behavior of the models discussed in the previous sections, Alseda and Costa (2008).

Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. It is a very practical and efficient method of counting dimension then other methods, like box counting etc. The procedure to obtain correlation dimension follows the following steps, Martelli (1999):

Consider an orbit \( O(x_1) = \{x_1, x_2, x_3, \ldots \} \), of a map \( f: U \to U \), where U is an open bounded set in \( \mathbb{R}^n \). To compute correlation dimension of \( O(x_1) \), for a given positive real number r, we form the correlation integral, Grassberger and Procaccia (1983).

\[ C(r) = \lim_{n \to \infty} \frac{1}{n(n-1)} \sum_{i \neq j} H \left( r - \| x_i - x_j \| \right). \]

(2.6)

Where, \( H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \)

is the unit-step function, (Heaviside function). The summation indicates the the number of pairs of vectors closer to r when \( 1 \leq i, j \leq n \) and \( i \neq j \). \( C(r) \) measures the density of pair of distinct vectors \( x_i \) and \( x_j \) that are closer to r.

The correlation dimension \( D_c \) of \( O(x_1) \) is defined as

\[ D_c = \lim_{r \to 0} \frac{\log C(r)}{\log r} \]

(2.7)

To obtain \( D_c \), \( \log C(r) \) is plotted against \( \log r \) and then we find a straight line fitted to this curve. The y- intercept of this straight line provides the value of the correlation dimension \( D_c \).

III. DYNAMICAL SYSTEMS EXPLORED

In this paper we consider the following six dynamical systems: the one dimensional epidemic model, one dimensional Salmon map, Gross National Product model, Predator-Prey population model, a neural network model, and the Ushiki map. The study of the above six models forms the subject matter of sections A-F that follow.

A. Discrete One Dimensional Epidemic Model

Biological models are of great importance in real life science, May (1974). The following equation represents the spread of epidemic in society,

\[ x_{n+1} = k x_n^2 - 1 \]

(3.1)

This model is used quite frequently to model the spreading of measles. The bifurcation diagram for this model, when \( 0 \leq k \leq 2.0 \) is represented by left figure of Fig.1. The figure showing period doubling phenomena of Feigenbaum (1979) followed by chaos. Also, a plot of Lyapunov exponents, for same range of parameter k, is shown by right side figure of Fig.1. This plot provides the range of values of k for the regular (periodic) motion as well as the chaotic motion. The indication of periodic windows appearing in bifurcation diagram, are nicely shown in the LCE diagram. Exponents (LCEs) of epidemic model. We have also drawn the correlation integral curve, Fig. 2, by numerically calculating the data of correlation integral. By using the method of least square linear fit to the data of correlation integrals, one obtains the equation of the straight line fitting the above curve as

\[ y = 0.410138 + 0.133082 x \]
The y-intercept of this curve is $0.410138 \approx 0.41$. Thus the correlation dimension of the chaotic set emerging in the epidemic model is approximately 0.41.

**Bifurcation of $f(x) = kx^2 - 1$, $0 \leq k \leq 2$**

![Bifurcations diagram and plot of Lyapunov Exponents](image)

**Fig. 1:** Bifurcations diagram and plot of Lyapunov Exponents

**Fig. 2:** Correlation integral curve of the epidemic model for the orbit of $x_0 = 0.5$ with $k = 1.25$.

**B. One dimensional Salmon map.**

$$x_{n+1} = x_n \exp[\mu (1 - x_n)]$$

(3.2)

Fixed points of this map are given by $x_1^* = 0$, $x_2^* = 1$ and roots of equation

$$1 + \frac{\mu (1-x)}{2!} + \frac{\mu^2 (1-x)^2}{3!} + \frac{\mu^3 (1-x)^3}{4!} + \ldots = 0,
\mu \geq 0.$$  

The point is $x_1^*$ neutral and $x_2^*$ is stable if when $0 < \mu < 2$. At $\mu > 2$, $x_2^*$ becomes unstable. So, one gets one cycle for $0 < \mu < 2$ and at $\mu = 2$, two cycles born. The point $\mu = 2$ thus becomes a bifurcation point. Looking at the bifurcation diagram of map (3.2), Fig. 3(a), one realizes this fact. Fig. 3(b) is drawn for the LCEs computed for $1.5 \leq \mu \leq 3.5$ for $x_0 = 0.1$ and Fig. 3(c) is just a superposition of these two. A detailed discussion on bifurcation can be found in Sinha and Das (1997).

![Repeating the numerical calculations, as in case (a) of epidemic model, we have obtained the values of correlation integrals. Then, applying the method of least square linear fit, obtained the correlation dimension as $1.082 \approx 1.1$ for an orbit of $x = 0.5$ with $\mu = 2.9$.]

**Fig 3.: Plots of Salmon map for $1.5 \leq \mu \leq 3.5$: (a) Bifurcation diagram, (b) Plot of LCEs and (c) Superposition of plots (a) and (b).**

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**C. The Gross National Product (GNP) Model**

Gross National Product, GNP, means the total of all business production and service sector industry in a country plus its gain on overseas investment. Thus, GNP model measures the economic activity based on labor and production output within a country and is represented by the map

$$k_{n+1} = \frac{s(k_n) f(k_n)}{(1+\lambda)}.$$  

(3.3)
where $k_n$ is the capital-labor ratio at generation $n$, $s$ is the savings ratio function and $\lambda$ is the natural rate of population growth.

This GNP map takes a simplified form by assuming, Day (1982),

$$ s(k) = \sigma, \quad f(k) = B k^\beta (m - k)^\gamma, $$

where $\beta, \gamma, m > 0$.

Then, we obtain a simplified GNP model for a country.

$$ k_{n+1} = \sigma \frac{B k_n^\beta (m - k_n)^\gamma}{(1 + \lambda)}. \quad (3.4) $$

We have plotted the bifurcation diagram of this map by varying $B$, $3.35 \leq B \leq 3.6$, and for parameter values $\sigma = 0.5, \beta = 0.5, \gamma = 0.2, \lambda = 0.2$, and $m = 1$ shown in Fig. 4. On the right of Fig. 4 we have drawn the cobwebs for this map taking $B = 3.63$ and other parameters same. For both the figures the initial value is $x_0 = 0.1$. Looking carefully at the graphs, one can say that the GNP model evolve chaotically after $B$ reaches the value 3.63. The Lyapunov exponents plot (Fig.5) confirms this argument.

**D. Predator-Prey Population model**

$$ x_{n+1} = a x_n (1 - x_n) - x_n y_n, $$

$$ y_{n+1} = b x_n y_n \quad (3.5) $$

Here $x_n$ and $y_n$ denote, respectively, the prey and predator population at time $n$. The growth rate of the predator is assumed to be proportional to the number of preys. The fixed points of the map are $(0, 0)$, $((a - 1)/a, 0)$ and $(1/b, (a b - a - b)/b)$. The map evolve into a bifurcation sequences and show variety of strange attractors for different sets of values of parameters $a$ and $b$. Below, in Fig. 6, a stable limit cycle and a strange attractor are drawn given parameter values.

**Fig. 4**: Bifurcation diagram and cobwebs of GNP model $(3.4)$.

**Fig. 5**: Plot of LCEs for GNP map for $3.35 \leq B \leq 3.64$ and $\sigma = 0.5, \beta = 0.5, \gamma = 0.2, \lambda = 0.2, m = 1$.

Using the procedure of cases (a) and (b), the correlation dimension for this map is calculated as $0.477058 \approx 0.48$.

**Fig. 6**: Limit cycle and chaotic attractor of prey-predator population model $(3.5)$; the parameter $b = 3.226$. Bifurcation diagram for map $(3.5)$, Fig. 7 left, drawn for $b = 3.226$ and $2.5 \leq a \leq 4.0$ show period doubling phenomena analogous to that of logistic map. Also we have drawn the LCEs, for $a = 3.6545$ and $b = 3.226$, for an orbit $(x_0, y_0) = (0.1, 0.1)$ of this map, Fig.7 right.
E. Model for Neural Networks

As in Paseman and Stollenwerk (1998), the activity of a recurrent two-neuron model, shown in Fig.8, at time n is given by vectors $x_n = (x_n, y_n)^T$.

![Diagram of a chaotic module consisting of a self-inhibitory neuron with activity $X_n$ and an excitatory neuron with activity $Y_n$ at time step $n$.]

The discrete dynamical system representing the neuromodel is given by

$$
x_{n+1} = \theta_1 + \omega_{11} \sigma (x_n) + \omega_{12} \sigma (y_n)
$$

$$
y_{n+1} = \theta_2 + \omega_{21} \sigma (x_n)
$$

(3.6)

with $\sigma (x) = \frac{1}{1 + e^{-x}}, \theta_1, \theta_2,$ are the neuron biases and $\omega_{ij}$ are weights, Lynch (2007).

By taking numerical values of parameters as $\theta_2 = 3, \omega_{11} = -20, \omega_{12} = 6, \omega_{21} = -6$ and changing $\theta_1$ within $-4.5 \leq \theta_1 \leq 1.0$, one obtains the interesting bifurcation curves along $x$ and $y$ direction as shown in figure below, Fig.9.

![Bifurcation diagram of neural system with $\theta_1$ in $-4.5 \leq \theta_1 \leq 1.0$. Other parameters are $\theta_1 = 3, \omega_{11} = -20, \omega_{12} = 6, \omega_{21} = -6$. Again, by taking $\theta_1 = -2, \theta_2 = 3, \omega_{11} = -20, \omega_{12} = 6, \omega_{21} = -6$ and changing $\theta_2$ within $0.5 \leq \theta_2 \leq 6.5$, one obtains bifurcations along $x$ and $y$ direction as shown in figure below, Fig.10.]

![Bifurcation diagram of neural system with $\theta_2$ in $0.5 \leq \theta_2 \leq 6.5$. Other parameters are $\theta_1 = -2, \omega_{11} = -20, \omega_{12} = 6, \omega_{21} = -6$.]

For $a = 3.6545$ and $b = 3.226$ and $(x_0, y_0) = (0.5, 0.5)$, correlation dimension of the chaotic orbit is obtained as $1.21849 \approx 1.22$. Fig.7: Bifurcation diagram and the plot of LCEs of map (3.5).
For values of parameters $\theta_1 = -2.1, \theta_2 = 3.7, \omega_{11} = -20, \omega_{12} = 6, \omega_{21} = -6$, the neuron model evolves into a chaotic attractor shown in figure, Fig.10.

This map starts evolving chaotically when parameter $a$ exceeds the value 3.588 keeping $b = 0.1$ and $c = 0.15$ fixed. Attractor for this map for values of $a = 3.8, b = 0.1, c = 0.15$ is shown below in Fig.12. It appears as a multiply folded two-dimensional surfaces projected in a plane.

Correlation integrals have been evaluated for above two cases and repeating the method explained earlier the correlation dimensions have been obtained respectively as $3.02614 \approx 3.03$ and $3.07366 \approx 3.07$.

**F. Ushiki map**

Discrete equations of the Ushiki map is defined as

\[
\begin{align*}
  x_{n+1} &= (a - x_n - b y_n) x_n \\
  y_{n+1} &= (a - y_n - c x_n) y_n
\end{align*}
\]

(3.7)

The bifurcation diagrams of this map for $b = 0.1, c = 0.15$ and varying $a$, $2.5 \leq a \leq 4.0$ are shown in Fig. 13.
IV. DISCUSSION

Evolutionary properties of six discrete maps represented through eqns. (3.1) - (3.7) and are explained by drawing bifurcation diagrams, strange attractors, Lyapunov exponents, correlation integral curves etc. and calculating correlation dimension of each model corresponding to chaotic orbits. Significant revelation obtained through such graphics that how the stable solutions change into chaotic when parameters varied. Numerical calculations of Lyapunov characteristic exponents for models (3.2) - (3.5) confirm regular and chaotic evolutionary behavior. The positive LCE indicates the chaotic regions whereas its negative value indicates the regular regions of evolution. The dimension of the chaotic attractor obtained for each map calculated numerically by using least square linear fit method. Dimensions are positive non-integers. Mathematical codes have been used for numerical calculations. As nonlinear models do not follow a single rule, plots for different models may show very dissimilar properties.

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