Acoustic Performance of Reactive Central Inlet and Side Outlet Muffler by Analytical Approach

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Abstract—The expansion chamber is one of the most basic types of silencing elements used in intake and exhaust systems. The acoustic behavior of reactive mufflers with side outlet is investigated in detail by means of analytical approach for single chamber muffler. Analytical procedure is presented that allows the determination of transmission loss of a muffler, and a general formula is derived for the given muffler with side outlet. The results are verified by transfer matrix method and finite element method. Finally it is shown that by adopting suitable dimensions of the muffler it is possible to improve acoustic performance of mufflers at low to mid frequencies.

Keywords: Finite Element Method, Reactive Mufflers, Transfer Matrix Method, Transmission Loss.

I. INTRODUCTION

Accurate prediction of sound radiation characteristics from reactive mufflers is of significant importance in automotive exhaust system design. The most commonly used parameter to evaluate the sound radiation characteristics of muffler is transmission loss (TL). Transmission loss is one of the most frequently used criteria of muffler performance because it can be predicted very easily from the known physical parameters of the muffler. Many tools are available to simulate the transmission loss characteristics of a muffler. They vary in terms of complexity and assumptions. However analytical models if developed give accurate results especially in low and mid frequency region in no time by directly applying the developed formula. Such a developed formula can be very useful to represent performance characteristics of the muffler for relative comparison of design alternatives at design stage. The objective of the present work is to develop a generalized formula for the determination of transmission loss of a single chamber muffler with side outlet in a planar propagation range of frequency and validate the results by using transfer matrix method and finite element method. An acoustic filter theory approach for silencer design was first developed by Stewart and Lindsay [1]. In this theory the propagation of sound is considered one dimensional. Other features which are not considered in this theory are mean flow, effect of tailpipe reflections, mean temperature variation, interaction between mean gas flow and sound in the region of disturbed flow at the discontinuities. The theory fails to predict the transmission loss at higher frequencies where modes other than the plane wave are cut on. In spite of these drawbacks acoustic filter theory was extensively used by investigators to predict the transmission loss of several types of engine exhaust silencers. The NACA report by Davis et al. [2] was one of the first comprehensive attempts to model mufflers. They used the transmission line theory by assuming both continuity of pressure and continuity of volume velocity at discontinuities. A large amount of work has been published since then on the prediction of muffler performance. Performance of a reactive muffler (refer Fig. 1) is determined mainly by its geometric shape. When a sound wave travelling through a duct arrives at a discontinuity where the acoustical impedance is mismatched, only a portion of the acoustical energy can flow through the discontinuity. This mismatch of characteristic impedance results in a reflection of part of the acoustic energy back towards the source of sound, or back and forth among the different sections of duct until it is dissipated [3]. Thus the transmission of sound energy can be reduced by inserting appropriate discontinuities in the muffler system.

Fig 1. Single Chamber Central Inlet and Central Outlet Muffler

Acoustic performance of an expansion chamber in a duct as a sound attenuator is represented by the repeating dome shaped transmission loss curves when the plane wave theory is applied. But when the axial length of chamber is considerably reduced, this property changes remarkably and the chamber begins to act as a resonator muffler [4]. In order to predict the acoustic performance parameters of mufflers, such as transmission loss, the four pole parameters represent important characteristics of the muffler and can be combined with other four pole matrices when the muffler is connected to other elements in the exhaust system [3]. Munjal et al. [5, 6] used the mass velocity instead of volume velocity with a different definition.
for the acoustic impedance of filter elements and analyzed number of mufflers of arbitrary shapes and complex combinations. The major disadvantage of transfer matrix method (four pole parameters method) is that it suits only one dimensional system and the higher order mode effects in wave propagation are neglected. Kim and Soedel [7] developed a general method to formulate four poles parameters of muffler using modal expression. Selamet and Radavick [8] investigated the effect of \( \frac{d}{a} \) ratio on the acoustic performance of concentric chambers experimentally and theoretically. There are numerous modeling schemes for muffler performance analysis. The lumped parameter model is good for predicting very low frequency or very low kl range response. The method is very flexible and can model any small gas cavity, no matter how complicated the geometry, as long as the largest dimension of the cavity is much less than the smallest wavelength of interest. The 1-D travelling approach is the most traditional way of modeling a distributed muffler system. The advantage is that the method provides a closed form solution and is computationally faster. For the muffler elements with shorter length to diameter ratios, or for higher ranges of kl, multidimensional analysis methods should be used because the 1-D method may not predict all of the resonances or even any of the resonances, in the frequency range of interest [9]. Cheng and Wu used boundary element method for acoustic analysis of mufflers [10]. Sedamoto and Murakami [11] investigated resonant properties of short expansion chambers using the traditional analytical approach treating a duct system as a distributed parameter system and depending on chamber length which becomes half of axial wavelength of the incident \((m,n)\) mode; and the \((m,n+1)\) mode becomes cut on in the chamber. The higher order modes can significantly affect the performance of expansion chamber mufflers [12, 13]. The presence of these modes sharply reduces the attenuation above the cut-off frequency for the mode that is excited. The lengths of the chamber of an industrial muffler are not large enough; as a result, three dimensional evanescent modes are easily excited and alter the performance of the muffler considerably [13]. For reactive central inlet and side outlet single chamber muffler (refer Fig. 2) at the interface of main chamber and side outlet there is no problem of multidimensional wave propagation at low and mid frequency range but Wu et al. [14] have shown that the higher order mode effects manifest differences between the computed TL values and those calculated by the plane wave theory for the acoustically short chambers for higher frequencies. Bilawchuk and Fyfe [15] compared various numerical methods for calculating the transmission loss in silencers. Meh dizadeh and Paraschivoiu [16] described a faster three point method for the evaluation of the transmission loss and concluded that FEM can perform computations in the entire domain and is therefore more powerful. More specifically, FEMs are able to address problems in non homogeneous domains. Broatch et al. [17] analyzed simple geometries with well known acoustic behavior and concluded that good results may be obtained from any numerical method if the mesh spacing is sufficiently small but small meshes may imply an excessive computation time.

Fig 2. Single Chamber Side Outlet Muffler

Dowling and Peat [18] described an efficient algorithm for acoustic analysis of any general silencer system by recording the order in which all of the elements are analyzed and the sub systems are reduced. Selamet et al. [19] applied mode matching technique to predict the acoustic behavior of concentric circular dual-chamber mufflers and various effects have been studied including the presence of a baffle inside the muffler, the baffle hole radius, the axial location of the baffle and the extension of inlet and outlet ducts. Arenas and Crocker [20] studied the conical, exponential, parabolic, adenoideal and cosine shaped connectors to connect two pipes of different cross sectional area and proved that all the connectors possess a similar behavior for very low frequencies which is equal to TL of a single discontinuity, but at higher frequencies for all the connectors, the TL tends to zero. In the reactive muffler with side outlet at the centre of the chamber, the incident acoustic wave propagates inside the chamber and travels up to the closed end of chamber and gets reflected from that rigid end to the centre of the chamber creating destructive interference with the incident acoustic wave coming from the source thus cancelling waves which are propagating downstream (at the junction of side outlet), which drastically improves the transmission loss without changing much geometry. This enhancement of the transmission loss is primarily associated with the low and mid frequency range, where planar propagation dominates the acoustic field while keeping a reasonable level of attenuation in the high frequency range.

II. PLANE WAVE PROPAGATION

For a propagating plane wave the characteristic impedance \( \rho c \) is a real quantity, where \( \rho \) is density of air and \( c \) is speed of sound in air, and the acoustic pressure (p) and particle velocity (v) are in phase.

For the propagating plane wave (which is one dimensional wave) the wave equation is given by second order partial differential equation in \( x \) and \( t \) [21].
\[
\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}
\]  
(1)

The general solution to this equation is given by

\[
p(x,t) = F(x-ct) + G(x+ct)
\]
(2)

Where \( t \) is the time and the function \( F(x-ct) \) represents waves moving in the positive \( x \) direction and \( G(x+ct) \) represents waves moving in the opposite direction. When a sound wave is superimposed upon another wave of the same frequency but travelling in the opposite direction, a standing-wave sound field is generated.

For a given medium the acoustic impedance of a plane progressive wave is everywhere the same provided the size of wave front does not alter. The effect on the phase of a plane wave in a pipe when reflection occurs as a result of transmission to a larger cross section is analogous to that which is associated with reflection of sound going from dense medium to a rare medium. Likewise the transition from large cross section to small cross section is analogous to the passage of sound from a rare to dense medium.

In this model (Fig. 2) a harmonic plane wave travelling in the positive \( x \) direction interferes with another plane wave travelling in the negative \( x \) direction at the centre of the chamber.

The superimposition of two sound waves traveling in opposite directions is given by

\[
p_1(t) + p_2(t) = A_1 \sin(2\pi ft - kx) + A_2 \sin(2\pi ft + kx)
\]
(3)

where \( k \) is wave number

The first sine term in the equation represents a sound wave traveling in the positive \( x \) direction with amplitude \( A_1 \) and frequency \( f \). The second sine term in the equation represents a sound wave traveling in the negative \( x \) direction with amplitude \( A_2 \) and identical frequency \( f \).

Using trigonometric identities for the sum and difference of angles

\[
p_1(t) + p_2(t) = A_1 \sin(2\pi ft)\cos(kx) - A_2 \cos(2\pi ft)\sin(kx)
\]
\[
+ A_1 \sin(2\pi ft)\cos(kx) + A_2 \cos(2\pi ft)\sin(kx)
\]
(4)

For waves of equal amplitudes we obtain the following simplification:

\[
p_1(t) + p_2(t) = 2A_1 \sin(2\pi ft)\cos(kx)
\]
(5)

Equation may now be considered to be a simple sinusoidal function of time whose amplitude depends on the spatial location \( x \) of the observer. When the argument of the cosine assumes odd integer values of \( \frac{\pi}{2} \), i.e.,

\[
kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2n-1)\pi}{2}
\]
(6)

Where \( n=1,2,3,4,\ldots \)

The sound pressure vanishes, and there are nodal points in space where no sound exists. Solution of the preceding equation for \( x_n \) yields the spatial locations of these nodes:

\[
x_n = \frac{(2n-1)\pi}{2k}
\]
(7)

and since

\[
k = \frac{\omega}{c} = \frac{2\pi}{\lambda}
\]
(8)

\[
x_n = \frac{(2n-1)\lambda}{4}
\]
(9)

Thus the location of the nodes is simply related to the wavelength of the superimposed waves. By taking the difference between successive nodal locations, it can be demonstrated that the nodes occur every half wavelength, i.e.,

\[
x_{n+1} - x_n = \frac{2(n+1)\lambda}{4} - \frac{(2n-1)\lambda}{4} = \frac{\lambda}{2}
\]
(10)

The amplitude at the nodes is simply zero. These nodes or points of minimum sound pressure are stationary, located halfway between the antinodes and spaced one-half wavelength apart.

This important concept can be used to design the silencer. The length of the silencer of simple chamber is adjusted (take length of silencer \( \frac{\lambda}{2} \) ) in such a way that centre point of main chamber will be adjusted as node point of two waves which are propagating from two opposite direction where side outlet is located. The frequency at which maximum attenuation is desired (required depending upon the application) is fixed. From that frequency, wave length \( \lambda \) and hence length of silencer \( \frac{\lambda}{2} \) is calculated. Once the main chamber length is designed the sound pressure vanishes at the centre of the main chamber, and centre will act as node point where sound pressure is zero. With slight variation in frequency sound pressure is nearly zero at the centre. Again square of sound pressure is proportional to sound power; sound power also becomes very small. As transmission loss is 10 times the logarithm to base 10 of the ratio of sound power incident on the muffler to the sound power transmitted by the muffler. In this case sound power at the centre of chamber is small though the power incident is constant, which increases transmission loss.

### III. DERIVATION

Single chamber side outlet muffler In the present study, the assumptions and the basic continuity conditions are
briefly introduced. The basic assumptions for the theory of acoustic field in a circular duct are
1. The acoustic wave in the duct is a plane wave.
2. The viscosity effects and the gravity effects are neglected.
3. The radiation and convection of sound through the surface of the duct is negligible.
4. There is no reflection wave from the exit of the silencer.
5. Zero mean flow

Consider two points 1 and 2 as shown in Fig. 3 with sound travelling from the pipe of cross section \( S_1 \) \((S_1 = S_A)\) to that of cross section \( S_2 \) \((S_2 = S_B)\). Sound energy is supposed to pass through the system from left to right, at the junction of 1-2, it is expected to have an incident and reflected wave in \( S_1 \) and an incident reflected wave in \( S_2 \), and transmitted wave in \( S_3 \) \((S_3 = S_C)\).

The incident and reflected wave displacements in \( S_1 \) at point 1 will be denoted by
\[
\xi_{hi} = A_1 e^{i\omega t} \quad \text{(11)}
\]
\[
\xi_{ri} = B_1 e^{i\omega t} \quad \text{(12)}
\]

Respectively

The incident and reflected wave displacements in \( S_2 \) at point 2 will be denoted by
\[
\xi_{h2} = A_2 e^{i\omega t} \quad \text{(13)}
\]
\[
\xi_{r2} = B_2 e^{i\omega t} \quad \text{(14)}
\]

Where the coefficients are complex to take care of phase difference, and lastly the transmitted displacement in \( S_3 \) at point 3 can be written as
\[
\xi_{t3} = A_3 e^{i\omega t} \quad \text{(15)}
\]

The boundary conditions to be satisfied are
1. Continuity of pressure
2. Continuity of discharge rate or volume displacement at 1 (assuming \( x=0 \))
3. Continuity of pressure at point 1
4. Continuity of pressure at point 2
5. Continuity of pressure at point 3

The perturbed pressure at point 1 is given by
\[
p_{h1} = i\omega \rho c A_1 e^{i\omega t} \quad \text{(16)}
\]

Where \( p_{h1} \) is incident pressure at point 1

The perturbed pressure at point 2 is given by
\[
p_{r2} = i\omega \rho c A_2 e^{i\omega t} \quad \text{(18)}
\]

Where \( p_{r2} \) is incident pressure at point 2

The perturbed pressure at point 3 is given by
\[
p_{h3} = i\omega \rho c A_3 e^{i\omega t} \quad \text{(20)}
\]

Where \( p_{h3} \) is transmitted pressure at point 3

Where \( A_1, B_1, A_2, B_2, B_3 \) are the amplitudes. On substitution at the junction
\[
A_1 + B_1 = A_2 + B_2 \quad \text{(21)}
\]

\[
S_A(A_1 - B_1) = S_B(A_2 - B_2) \quad \text{(22)}
\]

Where \( \frac{S_B}{S_A} = m = \text{Area Ratio} \quad \text{(23)}
\]

After applying boundary conditions
\[
A_i = \frac{1}{2} [A_2(1 + m) + B_2(1 - m)] \quad \text{(24)}
\]

After simplification
\[
A_i = \frac{A_3}{4} [4\cos(kl_B/2) + 4i\sin(kl_B/2) + \frac{1}{m}\sin(kl_B/2)] \quad \text{(25)}
\]

\[
\frac{A_i}{A_3} = [\cos(kl_B/2) + \frac{i}{2}(2m + \frac{1}{2m})\sin(kl_B/2)] \quad \text{(26)}
\]

\[
\left| \frac{A_i}{A_3} \right|^2 = [1 + \frac{4}{4}(2m - \frac{1}{2m})^2\sin^2(kl_B/2)] \quad \text{(27)}
\]

\[
TL = 10\log_{10}\left( \left| \frac{A_i}{A_3} \right|^2 \right) \quad \text{(28)}
\]

Hence
\[
TL = 10\log_{10}[1 + \frac{4}{4}(2m - \frac{1}{2m})^2\sin^2(kl_B/2)] \quad \text{(29)}
\]

**IV. FOUR POLE TRANSFER MATRIX METHOD (TMM)**

Single chamber side outlet muffler the transmission loss (TL) of a muffler element can be computed by using transfer matrix approach. The development of four pole method is well known. Using the four pole method, a silencer system is evaluated at the inlet and outlet sections by the sound pressure and normal particle velocity and TL is only a property of the four pole parameters and independent of the
source and termination impedances. Four pole parameters of the muffler (refer Fig. 2) can be obtained as

$$T_{11} \quad T_{12} \quad T_{21} \quad T_{22} = \begin{bmatrix} a_{A_1} & a_{A_2} \\ a_{B_1} & a_{B_2} \\ a_{C_1} & a_{C_2} \end{bmatrix} \begin{bmatrix} \rho \sin(kl_A) \\ S_A \sin(kl_A) \cos(kl_A) \end{bmatrix}$$

(30)

where subscript $s$ refers to single chamber muffler and subscript $A$ refers to inlet pipe (A), B refers to main chamber (B) and C refers to outlet pipe (C)

$$X = \begin{bmatrix} \cos(kl_B)^2 \\ 2S_B \sin(kl_B)^2 \cos(kl_B)^2 \rho c \end{bmatrix} \begin{bmatrix} \rho \sin(kl_C) \\ S_C \sin(kl_C) \cos(kl_C) \end{bmatrix}$$

(31)

$$TL = 20\log \left( \frac{1}{2} \left( T_{11} + \frac{T_{12}}{Y_1} + T_{21}Y_3 + T_{22} \right) \right)$$

(32)

**V. FINITE ELEMENT METHOD**

Exhaust acoustics simulation is an important part of the exhaust system process. Comsol software is used for the finite element analysis of the silencers with side outlet. The analytical methodology is compared with transfer matrix method and both the analytical and transfer matrix method results are validated with FE calculations. The developed formula and the transfer matrix method (TMM) are suitable for plane wave propagation and TMM is widely used to calculate the TL below the cutoff frequency, FEM gives accurate results in entire frequency range. But when our requirement is to calculate transmission loss at low and mid frequency region this developed formula gives very accurate result in no time where as FEM is time consuming process. The numerical analysis was carried out using Comsol program (COMSOL Multiphysics) without fluid structure interaction. This program is capable of applying FEM solution to the muffler problem. In case this a frequency of 5-4000Hz is considered and the frequency response of the sound pressure (transmission loss) is observed. The air density and speed of sound are taken as 1.2kg m$^{-3}$ and 343m s$^{-1}$ respectively. Automatic meshing (free meshing) was used. Tetrahedral elements were used. The element size for the finite element domain was chosen to provide a minimum resolution of 12 elements per wavelength to ensure that the resolution requirements were met and consequently, accuracy was maintained.

**VI. RESULTS AND DISCUSSION**

The acoustic attenuation of single and dual chamber reactive muffler is expected to depend on the geometric characteristics. Three configurations are investigated. In first configuration (refer Fig. 1) transmission loss of simple muffler is predicted by FEM (refer Fig. 4). For second configuration (refer Fig. 2), these dimensions are chosen in such a way that length of chamber is exactly equal to $\frac{\lambda}{2}$. (\(\lambda\) wavelength corresponding to frequency of interest and trough occurs at that frequency in case of simple muffler) and the centre point of chamber is $\frac{\lambda}{4}$ where $\sin(\frac{kl_B}{2})=1$ which gives maximum value of transmission loss thereby producing a crest instead of trough. Due to that two domes are combined into one bigger dome as shown in Fig. 4, which drastically increases transmission loss at low and medium frequency range.

![Fig 4. Comparison of Performance of Simple Chamber Muffler with Single Chamber Side Outlet Muffler](image)

These results are investigated analytically (using the derived formula) and verified by transfer matrix method and finite element method. (refer Fig. 5).

![Fig 5. Comparison of Performance of Single Chamber Side Outlet Muffler by Analytical, TMM, FEM Method](image)

The developed generalized formula and transfer matrix method both assume planar wave without end correction, that’s why in the higher frequency range some cut on frequencies are excited which reduces transmission loss and FEM results differ from their (analytical and TMM) results. Thus 1-D model without end corrections fails even at much lower frequencies. It is well known that an area discontinuity generates evanescent higher order modes even at low frequencies within the plane wave limit. At low frequencies which are of interest in muffler design, only plane waves can propagate along a uniform duct and hence plane wave theory is generally adequate. However at the area discontinuities evanescent non planar modes are generated [13] and their effect causes a noticeable difference between the plane wave
analysis results and FEM. The evanescent modes generated at the first discontinuity do not decay completely owing to the short length of the chamber and leak into the outlet tube through the second discontinuity. In fact the Karal correction factor is obtained with the implicit assumption that the evanescent modes generated at the discontinuity die out completely. Obviously with the use of short chambers this assumption is violated [13]. That is why the 3D FEM model differs from analytical method and TMM. In the higher frequency range higher order modes are excited which reduces transmission loss and FEM results differ from their (analytical and TMM) results.

VII. CONCLUSION
Analytical study on the pressure loss of reactive muffler with side outlet is presented here. The aim of the study has been to establish the general formula for the prediction of TL of reactive single chamber muffler with side outlet. With this purpose, a muffler with an incompressible and stationary medium has been assumed. This analytical model has been formulated and TL was derived by using continuity equations for the side outlet. For the given model transmission loss by using transfer matrix method (TMM) has been calculated and compared with the developed analytical formula for single chamber mufflers which shows very good agreement. Same result is again validated by using three dimensional finite element model by using Comsol software.

REFERENCES