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Fuzzy Graph Applications of Job Allocation

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Abstract — ***In this paper we generalize the concept of the chromatic joins and chromatic sum of a graph to fuzzy graphs and define the fuzzy chromatic sum of fuzzy graph. Here we consider fuzzy graph by taking fuzzy set of vertices and fuzzy set of edges. This concept of obtaining fuzzy sum of fuzzy colorings problem has a natural application in scheduling theory. In this paper we consider the problem of scheduling N jobs on a single machine and obtain the minimum value of the job completion times which is equivalent to finding the fuzzy chromatic sum of the fuzzy graph modeled for this problem.***

Index Terms— **Fuzzy Graphs, Chromatic Number, Chromatic Sum, α - cut of Graph, k-fuzzy colouring of Graph, Γ -chromatic sum of Graph, chromatic fuzzy sum of Graph.**

I. Introduction

The colouring problem consists of determining the chromatic number of a graph and an associated colouring function. Let G be a simple graph with n vertices. A colouring of the vertices of G is a mapping $f : V(G) \to N$, such that adjacent vertices are assigned different colours. The chromatic sum of a graph introduced in [5] is defined as the smallest possible total over all vertices that can occur among all colourings of G. In this paper we generalize these concepts to fuzzy graphs. Here we define fuzzy graphs with fuzzy vertex set and fuzzy edge set.

II. Main Definitions and Results

Defintion-1: A graph G that requires different color for its proper colorings and the number k is called the chromatic number of G.

Definition-2: A fuzzy set A defined on a non-empty set X is the family $A = \{(x, \mu_A(x)) / x \in X\}$, where $\mu_A : X \to I$ is the membership function. In classical fuzzy set theory the set I is usually defined as the interval [0,1] such that $\mu_A(x) = 0$, if x does not belongs to A, $\mu_A(x) = 1$, if x strictly belongs to A and any intermediate value represents the degree in which x could belongs to A. The set I could be discrete set of the form I= {0, 1, 2, k}, where $\mu_A(x) < \mu_A(x')$ indicates that the degree of membership of x to A is lower than the degree of membership of x'.

Definition-3: Let V be a finite nonempty set. The triple $G = (V, \sigma, \mu)$ is called a fuzzy graph on V, where σ and μ are fuzzy sets on V and E (VxV) respectively, such that $\mu(\{x, y\}) \le \min\{\sigma(x), \sigma(y)\}$ for all x, y $\in$ V.

We use $\mu(xy)$ for $\mu(\{x, y\})$. The fuzzy sub graph of G is the graph $G' = (V, \sigma', \mu')$ such that for x, y $\in$ V, we have $\sigma'(x) \le \sigma(x)$ and $\mu'(xy) \le \mu(xy)$.

Definition-4: Given $\alpha \in (0,1]$, the α - cut of G is the graph $G^\alpha = (V^\alpha, E^\alpha)$, where $G^\alpha = \{x \in V / \sigma(x) \ge \alpha\}$ and $\{xy \in E / \mu(xy) \ge \alpha\}$. The fuzzy graph have a finite number of different α - cuts. The α - cuts do not change through the following intervals: $(0, \alpha_1], \ldots, (\alpha_{k-1}, \alpha_k]$.

Definition-5: Let $\Gamma = \{\gamma_1, \gamma_2, \ldots\ldots, \gamma_k\}$ be a finite family of fuzzy sets on V. The fuzzy set Γ on V is defined by $\wedge\Gamma(x) = \max_i \gamma_i(x)$. $\Gamma = \{\gamma_1, \gamma_2, \ldots\ldots, \gamma_k\}$ is called a k-fuzzy colouring of $G = (V, \sigma, \mu)$, if (i) $\wedge\Gamma = \sigma$, (ii). $\gamma_i \wedge \gamma_j = 0$ and (iii). For every strong edge xy of G, $\min\{\gamma_i(x), \gamma_j(y) = 0, (1 \le i, j \le k)$ the least value of k for which G has a fuzzy colouring, denoted by $\chi^f(G)$, is called the fuzzy chromatic number of G.

Definition-6: For a k-fuzzy colouring $\Gamma = \{\gamma_1, \gamma_2, \ldots\ldots, \gamma_k\}$ of the fuzzy graph G, Γ -chromatic sum of G, denoted by $\Sigma_\Gamma(G)$, is defined as $\Sigma_\Gamma(G) = 1\sum_{x \in C_1}\theta_1(x) + \ldots\ldots + k\sum_{x \in C_k}\theta_k(x)$, where C_1= support of γ_1 and $\theta_1(x) = \max\{\sigma(x) + \mu(xy) / y \in C_i\}$.

Definition-7: The chromatic fuzzy sum of G denoted by $\Sigma(G)$ is defined as follows: $\Sigma(G) = \min\{\Sigma_\Gamma(G) / \Gamma$ is a fuzzy colouring of G} the number of fuzzy colourings of G is finite and so there exists a fuzzy colouring Γ_0 which is called minimal fuzzy colouring of G such that $\Sigma(G) = \Sigma_{\Gamma_0}(G)$.

III. Theorems

Theorem-1: Let G be a fuzzy graph and $\Gamma_0 = \{\gamma_1, \gamma_2, \ldots\ldots, \gamma_k\}$ a minimal fuzzy sum colouring of G. Then the

$$\Sigma_{x \in C1}\theta_1(x) \ge \Sigma_{x \in C2}\theta_2(x) \ge \ldots\ldots \ge \Sigma_{x \in Cs}\theta_s(x).$$

Proof: Suppose that for some i<j we have $\Sigma_{x \in Ci}\theta_i(x) < \Sigma_{x \in Cj}\theta_j(x)$.

Consider the fuzzy colouring $\Gamma_0' = \{\gamma'1, \gamma'2, \ldots\ldots, \gamma's\}$ defined by $\gamma'_r = \begin{cases} \gamma_r, r \notin \{i, j\}, \\ \gamma_j, r = i, \\ \gamma_i, r = j. \end{cases}$

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Now we have

$$\Gamma'_0(G)-\Gamma_0(G)=(i-j)[\Sigma_{x\in Cj}\theta_j(x)-\Sigma_{x\in Ci}\theta_i(x)]<0.$$

Therefore, $\_\Gamma'_0(G)<\_\Gamma_0(G)$ this contradicts the minimality of Γ_0.

Theorem-2: For a fuzzy graph $G=(V,\sigma,\mu)$ $\Sigma(G)\le\frac{3}{4}(\chi^f(G)+1)h(\sigma)|V|$.

Proof: Let Γ_1 be a – colouring of G where $k=\chi^f(G)$, such that $\Sigma(G)=\Sigma_{\Gamma1}(G)$, by Theorem-1, we have $\Sigma_{x\in C1}\theta_1(x)\ge\Sigma_{x\in C2}\theta_2(x)\ge.......\ge\Sigma_{x\in Ck}\theta_k(x)$. Hence, for each i, $1\le i\le k$ we have

$$i\Sigma_{x\in Ci}\theta_i(x)+(\chi^f(G)-i+1)(\Sigma_{x\in Ck}\theta_k(x)$$
$$\le\sum_{1\le i\le k}\{\frac{(\chi^f(G)+1)}{2}[\Sigma_{x\in C1}\theta_1(x)+\Sigma_{x\in C1}\theta_1(x)]\}.$$

Then,

$$\sum_{1\le i\le k}i\Sigma_{x\in Ci}\theta_i(x)\le\frac{(\chi^f(G)+1)}{2}\sum_{1\le i\le k}\Sigma_{x\in Ci}\theta_i(x).$$

But since we have

$$\sum_{1\le i\le k}\Sigma_{x\in Ci}\theta_i(x)\le(3h(\sigma)/2)|V|.$$

Thus the upper bound for $\Sigma(G)$ is.

$$\frac{3}{4}(\chi^f(G)+1)h(\sigma)|V|$$

Definition-8: For the fuzzy graph $G=(V,\sigma,\mu)$, define $w=\min\{\sigma(x)+\mu(xy)>0/x\in V$, xy is a weak edge of G$\}$.

Remark:

1. Let $G=(V,\sigma,\mu)$ be a connected fuzzy graph with e strong edges. Then the lower bound for $\sum(G)=w\sqrt{8}e$.

2. The fuzzy chromatic sum lies between $w\sqrt{8}e$ and $\frac{3}{4}(\chi^f(G)+1)h(\sigma)|V|$.

IV. EXAMPLE

Let us consider the example of scheduling 6 jobs on a single machine. At any given time the machine is capable to perform any number of tasks, as long as these tasks are independent or the conflicts between them are less than 1. The consuming time of tasks 1 and 5 is 1 hr. and that of tasks 3 and 4 and task 2 are 0.2 hrs. and 0.3 hrs. Respectively. Tasks {1,2} and {2,5} can be performed together with a conflict of 0.3 hrs. and the tasks {1,3},{2,3},{3,4},{2,4},{3,5} can be performed together with a conflict of 0.1 hrs. Suppose the machine is capable to perform on x and y simultaneously the amount of time that machine spends on x (or) y depends on the individual amount of time which was previously spent on x (or y) together with the measure of conflict between x and y. Our goal is to minimize the average response time, or equivalently to minimize the sum of the task completion time. In order to solve this problem we define the fuzzy graph

$\hat{G}=(V,\sigma,\mu)$, where X is the set of all tasks, $\sigma(x)$ is the amount of the consuming time of the machine for each $x\in X$ and $\mu(xy)$ is the measure of the conflict between the tasks x and y. Finding the minimum value of the job completion times for this problem is equivalent to the chromatic fuzzy sum of $\hat{G}$ which we will study in this section. The fuzzy graph $G=(V,\sigma,\mu)$ corresponding our example is defined as follows,

$$\text{Let } V=\{v_1,v_2,.....,v_6\},\ \sigma(v_i)=\begin{cases}1 & for\ i=1,5\\0.3 & for\ i=2\\0.2 & for\ i=3,4\end{cases},$$

$$\mu(v_i,v_j)=\begin{cases}0.3, & i,j\in\{12,35\},\\0.1, & i,j\in\{13,23,24,25,34\},\\0, & otherwise.\end{cases}$$

Let $\Gamma=\{\gamma_1,\gamma_2,\gamma_3\}$ be a family of fuzzy sets defined on v, where

$$\gamma_1(v_i)=\begin{cases}1, & i=1\\0.2, & i=4\\0, & otherwise\end{cases},$$

$$\gamma_2(v_i)=\begin{cases}0.3, & i=3\\0, & otherwise\end{cases},$$

$$\gamma_3(v_i)=\begin{cases}1, & i=5\\0.3, & i=1\\0.2, & i=3\\0, & otherwise\end{cases}$$

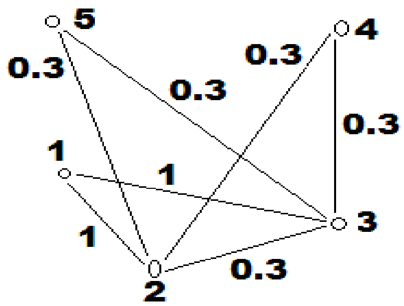

Fig. 1. The fuzzy graph corresponding to example

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vertex	γ_1	γ_2	γ_3	Max
1	1	0	0.3	1
2	0	0.3	0	0.3
3	0	0	0.2	0.2
4	0.2	0	0	0.2
5	0	0	1	1

We can see that condition

(i). i.e. $\max_i \gamma_i(v_j) = \sigma(v_j),\ j = 1,2,3,4,5$

(ii). $\gamma_i \wedge \gamma_j = 0$

(iii). for every strong edge xy of $\hat{G}$,

$\min\{\gamma_i(x), \gamma_j(y) = 0, (1 \le i \le k)$.

Therefore G has a 3-colouring and $\chi^f(G) = 3$, for this 3-colouring $\Gamma -$ chromatic number can be calculated as follows:

Let $C_i = \text{supp.}\,\gamma_1$ then C_1= {1, 4} C_2= {2, 3} C_3= {1, 3, 5}

θ_1 (1) = max {1+0, 1+0} = 1,
θ_1 (4) = max {0.2+0, 0.2+0} = 0.2,
θ_2 (2) = max {0.3+0} = 0.3,
θ_2 (2) = max {0.3+0, 0.3+0} = 0.3.
θ_3 (1) = max {1+0.1, 1+0, 1+0} = 1,
θ_2 (3) = max {0.2+0.1, 0.2+0, 0.2+0.1} = 0.3,
θ_3 (5) = max {1+0, 1+0.1, 1+0} = 1.1

Then $\Gamma -$ chromatic fuzzy sum of G,

$\sum_\Gamma$ = 1(1+0.2) +2(0.3) +3(1+0.3+1.1) = 9.

Now, let $\Gamma = \{\gamma_1, \gamma_2\}$ be a family of fuzzy sets defined on V given by

$$\gamma_1(x_i) = \begin{cases} 1, & i \in \{1,5\} \\ 0.2, & i = 4 \\ 0, & otherwise \end{cases},$$

$$\gamma_1(x_i) = \begin{cases} 0.2, & i = 3 \\ 0.3, & i = 2 \\ 0, & otherwise \end{cases}$$

We can see that condition

(i). i.e. $\max_i \gamma_i(v_j) = \sigma(v_j),\ j = 1,2,3,4,5$

(ii). $\gamma_i \wedge \gamma_j = 0$

(iii).for every strong edge xy of $\hat{G}$, $\min\{\gamma_i(x), \gamma_j(y) = 0$, $(1 \le i \le k)$, therefore G has a 3-colouring and $\chi^f(G) = 3$, for this colouring $\Gamma -$ chromatic number can be calculated as follows:

Let $C_i = \text{supp.}\,\gamma_1$ then C_1= {1, 4} C_2= {2} C_3= {1, 3, 5}

θ_1 (1) = max {1+0, 1+0} = 1,
θ_1 (4) = max {0.2+0, 0.2+0} = 0.2,

θ_2 (2) = max {0.3+0} = 0.3

Then $\Gamma -$ chromatic fuzzy sum of G, $\sum_\Gamma$ = 1(1+0.2) +2(0.3) =1.8.

Let us find the lower bound for $\sum$ (G). i.e, $w\sqrt{8}\,e$ Where, e is no of strong edges of G.

The set of weak edges are {15, 14, and 45}

$w = \min\{\sigma(x) + \mu(xy) > 0 / x \in V,$ XY is a weak edge of G}.

= min {1+0, 1+0, 0.2+0} = 0.2

$$\sum(G) \ge w\sqrt{8e} = 0.2\sqrt{5}\,6 = 1.4967$$

The chromatic fuzzy sum of G, $\sum$ (G) = min {9, 1, 8} =1.8

$$\sum(G) = \frac{3}{4}(\chi^f(G)+1)h(\sigma)|V| = \frac{3}{4}(3)(1.5) = 11.25$$

The fuzzy chromatic number lies between 1.4967 and 11.25. But in our problem $\sum$ (G) =1.8 since it is not possible to find a k-colouring such that the corresponding fuzzy chromatic sum is less than 1.8.Therefore the minimum time of completion of jobs in our examples is 1.8 hrs.

V. CONCLUSION

This paper Demonstrate the problem of scheduling N jobs on a single machine and obtain the minimum value of the job completion times which is equivalent to finding the fuzzy chromatic sum of the fuzzy graph modelled for this problem.

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