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A New Algebraic Method for Computing the Pfaffian

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***Abstract*— In this paper we postulate a new decomposition theorem of a matrix $\mathbf{A} \subset \mathbf{R}^{\mathbf{n}}$ into two matrices, namely, a lower triangular one, M, all whose entries are determinants, and an upper triangular matrix U whose entries are also in determinant form. Applying this new linear transformation to skew-symmetric matrix $\mathbf{A} \subset \mathbf{R}^{2\mathbf{n}}$ by means of partial pivoting strategy, we obtain the Pfaffian of A from the principal diagonal of matrix U, namely:**

$$Pf(\mathbf{A}) = \frac{\prod_{k=2}^{n} u_{k,k}}{\prod_{l=1}^{n-1} u_{l,l}}$$

Furthermore, if we apply this new linear transformation in Jordan version with the above pivot strategy to the augmented matrix $(\mathbf{A}\,|\,\mathbf{I}) \subset \mathbf{R}^{2\mathbf{n}}$, we obtain a "Pfaffian adjugate" G matrix. With it, we can solve systems of skew-symmetric linear equations in general, analogously to the "Cramer's Rule", which can be seen as a ratio of two Pfaffians. Both algorithms present an $O(n^3)$ computational complexity.

***Index Terms*— Pfaffian, LU decomposition, Pfaffian-Cramer rule, Pfaffian-Adjugate Matrix.**

I. INTRODUCTION

The determinant of a skew-symmetric matrix ($\mathbf{A} = -\mathbf{A}^{\mathbf{T}}$) is the square of another expression, which is called the Pfaffian [1]. For example,

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{vmatrix} = (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})^2$$

This property was proved by Cayley in 1849 [2]. The determinant of a skew-symmetric matrix of odd order is always 0.

Recently, Knuth [3], in a short history of Pfaffians, argues that these are in some way more fundamental than determinants, to which they are closely related. Johann Friedrich Pfaff introduced the functions that now bear his name in 1815 [1, pp. 396-401] while developing a general method to solve systems of first-order partial differential equations. He gave two procedures for listing all perfect matching's (subset of edges which covers all vertices of the graph), and observed that, when the matching are ordered lexicographically, the corresponding signs are strictly alternating +,-,+,…,+.

Formally speaking, the Pfaffian of a skew-symmetric matrix $\mathbf{A} \subset \mathbf{R}^{2\mathbf{n}}$ can be defined as follows [4]:

$$Pf(\mathbf{A}) = \sum_{perfect\ \ matchings\ \ M} sign\ \ M \cdot weight(M) \quad (1)$$

Here, a perfect matching M with $k = \frac{n}{2}$ edges is written as

$$M = (i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k) \quad (2)$$

Where, by convention, $i_k \langle j_k$ for each k. The sign of the matching M is defined as the sign of

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n-1 & n \\ i_1 & j_1 & i_2 & j_2 & \ldots & i_k & j_k \end{pmatrix}$$

When this is regarded as a permutation of $\{1,2,\ldots,n-1,n\}$. The weight of M is $(i_1, j_1)\cdot(i_2, j_2),\ldots,\cdot(i_k, j_k)$.

Because of their close connection with matching's, Pfaffians are of considerable interest in combinatory analysis, as seen in [5,6]. For example, for certain graphs, including the planar ones, it is possible to count the number of perfect matching's in a polynomial time by using Pfaffians.

The relation

$$|\mathbf{A}| = Pf\begin{pmatrix} 0 & \mathbf{A} \\ -\mathbf{A}^{\mathbf{T}} & 0 \end{pmatrix}$$

Shows that determinants are just special cases of Pfaffians that correspond to bipartite graphs. In this sense, the Pfaffian is a more basic notion, which would deserve a more thorough understanding, despite the traditional prevalence of the determinant in the curricula and in applications. This work expects to contribute in that direction.

A Linear System of Equations (LSE) can be defined as a set of *m* equations with *n* unknowns represented by a matrix **A**, a vector **b** and an unknown vector **x**, namely, $\mathbf{Ax} = \mathbf{b}$ .The Gaussian Transformation (GT) for solving an LSE has proved to be the best option for most practical applications. The new transformation proposed here can be obtained from it. Next, we briefly review this topic.

II. LU- MATRICIAL DECOMPOSITION WITH GT. NOTATION AND DEFINITIONS

The problem of solving a linear system of equations $\mathbf{Ax} = \mathbf{b}$ is central to the field of matrix computation. There are

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several ways to perform the elimination process necessary for its matrix triangulation. We will focus on the Doolittle-Gauss elimination method: the algorithm of choice when A is square, dense, and un-structured.

Let us assume that $\mathbf{A} \in \mathbf{R}^{nxn}$ is nonsingular and that we wish to solve the linear system $\mathbf{Ax} = \mathbf{b}$. Here we show how for an exact arithmetic and partial pivoting and column interchanges some Gauss transformations $\mathbf{M}_1, \ldots, \mathbf{M}_{n-1}$ can nearly always be found such that $\mathbf{M}_{n-1}, \ldots, \mathbf{M}_2\mathbf{M}_1\mathbf{A} = \mathbf{U}$ is upper triangular [7]. The original $\mathbf{Ax} = \mathbf{b}$ problem is then equivalent to the upper triangular system $\mathbf{Ux} = (\mathbf{M}_{n-1}, \ldots, \mathbf{M}_2\mathbf{M}_1)\ \mathbf{b}$ which can be solved through back-substitution.

Suppose, then, that $\mathbf{A} \in \mathbf{R}^{nxn}$ and that, for some k<n,

We have determined the Gauss transformations

$\mathbf{M}_1, \ldots, \mathbf{M}_{k-1} \in \mathbf{R}^{nxn}$ Such that

$$\mathbf{A}^{(k-1)} \equiv \mathbf{M}_{k-1} \cdots \mathbf{M}_1 \mathbf{A} = \begin{pmatrix} \mathbf{A}_{11}^{(k-1)} & \mathbf{A}_{12}^{(k-1)} \\ \mathbf{0} & \mathbf{A}_{22}^{(k-1)} \end{pmatrix} \begin{matrix} (k-1) \\ (n-k+1) \end{matrix}$$
$$(k-1) \quad (n-k+1)$$

where: $\mathbf{A}_{11}^{(k-1)}$ is an upper triangular matrix.

$$\mathbf{A}_{22}^{(k-1)} = \begin{pmatrix} a_{kk}^{(k-1)} & . & . & . & a_{kn}^{(k-1)} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ a_{nk}^{(k-1)} & . & . & . & a_{nn}^{(k-1)} \end{pmatrix}$$

Now, if

and $a_{kk}^{(k-1)} \neq 0$, then the multiplicators :

$$m_i = \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}; i = k+1, \ldots, n$$

; with $a_{kk} \neq 0$ are well defined.

So, we have the following

Definition. An elementary lower triangular matrix of order n and index k is a matrix of the form [8]

$$\mathbf{M}_k = \mathbf{I}_n - \mathbf{m}\mathbf{e}_k^T$$

where:

$$\mathbf{e}_k^T = (0, \ldots, 0, 1, 0, \ldots, 0)^T \qquad \mathbf{I}_n = \begin{pmatrix} 1 & . & . & . & 0 \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ 0 & . & . & . & 1 \end{pmatrix},$$
$$k$$

$$\mathbf{m}^T = (\underbrace{0,0,0,\ldots,0}_{k-times}, m_{k+1}, \ldots, m_n)$$

In general an elementary lower triangular matrix has the above form.

The computational significance of elementary lower triangular matrices is that they can be used to introduce zero components into a vector. Thus,

$$\mathbf{M}_k \cdot \begin{pmatrix} a_{11} \\ . \\ . \\ . \\ a_{k1} \\ a_{k+1,1} \\ . \\ . \\ . \\ a_{n1} \end{pmatrix} = \begin{pmatrix} a_{11} \\ . \\ . \\ . \\ a_{k1} \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}.$$

The matrix $\mathbf{M}_k$ is said to be a GT**.** The vector **m** is referred to as the Gauss vector. The components of **m** are known as multipliers.

Then it follows that

$$\mathbf{A}^{(k)} \equiv \mathbf{M}_k \mathbf{A}^{(k-1)} = \begin{pmatrix} \mathbf{A}_{11}^{(k)} & \mathbf{A}_{12}^{(k)} \\ \mathbf{0} & \mathbf{A}_{22}^{(k)} \end{pmatrix} \begin{matrix} (k) \\ (n-k) \end{matrix}$$
$$(k) \quad (n-k)$$

Where $\mathbf{A}_{11}^{(k)}$ is an upper triangular matrix.

This process illustrates the *k-th* step of the decomposition process, in which we used

$$(\mathbf{M}_k \cdots \mathbf{M}_1)^{-1} = \mathbf{M}_1^{-1} \cdots \mathbf{M}_k^{-1} = \prod_{i=1}^{k} (\mathbf{I}_n + \mathbf{m}^{(i)}\mathbf{e}_i^T) = \mathbf{I}_n + \sum_{i=1}^{k} \mathbf{m}^{(i)}\mathbf{e}_i^T$$

We find the final expression for the decomposition process as

$$\mathbf{A} = \begin{pmatrix} \mathbf{L}_{11}^{(k)} & \mathbf{0} \\ \mathbf{L}_{21}^{(k)} & \mathbf{I}_{n-k} \end{pmatrix} \begin{pmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22}^{(k)} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{11}^{(k)} & \mathbf{A}_{12}^{(k)} \\ \mathbf{0} & \mathbf{I}_{n-k} \end{pmatrix} \Rightarrow$$

$$(\mathbf{M}_k \ldots \mathbf{M}_1)^{-1} \equiv \begin{pmatrix} \mathbf{L}_{11}^{(k)} & \mathbf{0} \\ \mathbf{L}_{21}^{(k)} & \mathbf{I}_{n-k} \end{pmatrix} = \mathbf{I}_n + (\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}, 0, \ldots, 0).$$

In general, the forward elimination consists of *n-1* steps. At the *k-th* step, multiples of the *k-th* equations are subtracted from the remaining equations to eliminate the *k-th* variable. If the pivot element $a_{k,k}^{(k)}$ is null or "sufficiently small", it is advisable to interchange equations before this is done through **P**, a permutation matrix that records the row exchanges as detailed below.

III. LU DECOMPOSITION THEOREM

Using the above expression, the following can be established [9]:

Theorem. Let $\mathbf{A}_k$ denote the leader or main sub-matrix (*k* x *k*) of $\mathbf{A} \in \mathbf{R}^{nxn}$. If $\mathbf{A}_k$ is non-singular for *k=1,...,n* ; then there exist a lower triangular matrix $\mathbf{L} \in \mathbf{R}^{nxn}$ and an

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upper triangular matrix $\mathbf{U} \in \mathbf{R}^{nxn}$ so that **A=PLU.** Furthermore, $|\mathbf{A_k}| = u_{11}...u_{kk} \forall k = 1,..., n.$

IV. DERIVATION OF A NEW LINEAR TRANSFORMATION

In the previous section we have defined an elementary lower triangular matrix of order n and index k as $\mathbf{M_k} = \mathbf{I_n} - \mathbf{m}\mathbf{e_k^T}$. We shall find the above expression and the next theorem**,** whose proof, found in [8], can be useful.

Theorem. The pivot elements $a_{k,k}^{(k)}$ $(k = 1,..., n)$, are nonzero if and only if the leading principal sub matrices $\mathbf{A_k}$ $(k = 1,..., n)$, are non-singular.

On this basis, the following can be stated:

Corollary. Let $\mathbf{A} \in \mathbf{R}^{nxn}$ be a matrix with $\mathbf{A_k}$ $(k = 1,..., n)$ non-singular leading principal sub-matrices. Then, there exists a unique diagonal matrix for $k=1$ $(a_{00}^{(1)} = 1)$ whose entries are:

$$\mathbf{D_k} = \begin{pmatrix} 1_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & 1_k & & & & \\ & & & & \left(\dfrac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_{k+1} & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & \left(\dfrac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_n \end{pmatrix}$$

Where $a_{k,k}^{(k)}$ and $a_{k-1,k-1}^{(k)}$ are the pivots elements.

Now, if we scale the matrix $\mathbf{M_k}$ with that diagonal matrix, we will have

$$\mathbf{M_k^{'}} = \mathbf{D_k}\mathbf{M_k} =$$

$$= \begin{pmatrix} 1_1 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 1_k & & & \\ & & & -\dfrac{a_{k+1,k}^{(k)}}{a_{k-1,k-1}^{(k)}} & \left(\dfrac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_{k+1} & & \\ & & & \cdot & & \cdot & \\ & & & \cdot & & & \cdot \\ & & & \cdot & & & \\ & & & -\dfrac{a_{n,k}^{(k)}}{a_{k-1,k-1}^{(k)}} & \cdot & \cdot\ \cdot & \left(\dfrac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_n \end{pmatrix}$$

and, once simplified, it can be re-expressed as

$$\mathbf{M_k^{'}} = \frac{1}{a_{k-1,k-1}^{(k)}} \cdot \begin{pmatrix} \left(a_{k-1,k-1}^{(k)}\right)_1 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & \left(a_{k-1,k-1}^{(k)}\right)_k & & & \\ & & & -a_{k+1,k}^{(k)} & \left(a_{k,k}^{(k)}\right)_{k+1} & & \\ & & & \cdot & & \cdot & \\ & & & \cdot & & & \cdot \\ & & & \cdot & & & \\ & & & -a_{n,k}^{(k)} & \cdot & \cdot\ \cdot & \left(a_{k,k}^{(k)}\right)_n \end{pmatrix} \quad (3)$$

By using this new transformation and applying the elimination process to a matrix **A**, all of whose entries are integers, all intermediate results are integers too, forming a number ring [10], since they are obtained through additions and products. Thus, $\mathbf{A} \in \mathbf{I}^{nxn} \Rightarrow (\mathbf{M^{'}}, \mathbf{U}) \in \mathbf{I}^{nxn}$.

The products of these results times the factor $\dfrac{1}{a_{k-1,k-1}^{(k)}}$ are integers too, because previously, in the $k-1$ step of the elimination process, such results had been multiplied by $a_{k-1,k-1}^{(k)}$.This multiplication process leads to a simplification of the final result.

We can re-express this linear transformation as: $(a_{00}^{(1)} = 1)$

$$\mathbf{M'_k} = \begin{pmatrix} \dfrac{1}{a_{k,k}^{(k)}}\mathbf{I_k} & 0 \\ 0 & \dfrac{1}{a_{k-1,k-1}^{(k)}}\mathbf{I_{n-k}} \end{pmatrix} \left[a_{k,k}^{(k)}\mathbf{I} - \begin{pmatrix} 0_1 \\ \cdot \\ \cdot \\ \cdot \\ 0_k \\ a_{k+1,k}^{(k)} \\ \cdot \\ \cdot \\ \cdot \\ a_{n,k}^{(k)} \end{pmatrix} \mathbf{e_k^T} \right]; \forall k = 1,..., n-1 \quad (4)$$

V. NEW DECOMPOSITION THEOREM THROUGH DETERMINANTS

Using any of the above expressions, one can state this

Theorem. Let $\mathbf{A} \in \mathbf{R}^{nxn}$ and $\mathbf{M} = \prod_{\mathbf{k=n-1}}^{\mathbf{1}} \mathbf{M_k^{'}}$ .Then **M** is a lower triangular matrix all whose components are determinants and $\mathbf{U} = \mathbf{MA}$ also has all components in determinant form. Furthermore, $|\mathbf{A}| = u_{n,n}$

Proof. It follows from an induction on n. For $n=1$ the theorem is trivially true since $\mathbf{M} = \mathbf{M_1^{'}} = 1$ and $\mathbf{U} = \mathbf{MA} = a_{11}$. For the induction step $(k = 1,..., n-1)$ we have:

$n=2$

$k=1$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \ \mathbf{M_1^{'}} = \begin{pmatrix} 1 & 0 \\ -a_{21} & a_{11} \end{pmatrix}; \ \mathbf{M} = \mathbf{M_1^{'}};$$

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$$\mathbf{U}=\mathbf{MA}=\begin{pmatrix}1 & 0\\ -a_{21} & a_{11}\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ 0 & \begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\end{pmatrix}$$

n=3
k=1,2

$$\mathbf{A}=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix};\ \mathbf{M}_1^{'}=\begin{pmatrix}1 & 0 & 0\\ -a_{21} & a_{11} & 0\\ -a_{31} & 0 & a_{11}\end{pmatrix};$$

$$\mathbf{U}_1=\mathbf{M}_1^{'}\mathbf{A}=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ 0 & \begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}\\ 0 & \begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}\end{pmatrix};$$

$$\mathbf{M}_2^{'}=\frac{1}{a_{11}}\begin{pmatrix}a_{11} & 0 & 0\\ 0 & a_{11} & 0\\ 0 & -\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\end{pmatrix};$$

$$\mathbf{M}=\mathbf{M}_2^{'}\mathbf{M}_1^{'}=\begin{pmatrix}1 & 0 & 0\\ -a_{21} & a_{11} & 0\\ -\begin{vmatrix}a_{31} & a_{32}\\ a_{21} & a_{22}\end{vmatrix} & -\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\end{pmatrix};$$

The last row of the matrix **M** multiplied for the last column of the matrix **A** is equivalent to Laplace Expansion of **A** taking out the last column. Then we have $u_{33}=|\mathbf{A}|$:

$$\mathbf{U}=\mathbf{MA}=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ 0 & \begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}\\ 0 & 0 & \begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\end{pmatrix};$$

n
k=1,2,…,n-1

$$\mathbf{A}=\begin{pmatrix}a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1,n}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{1,n-1} & a_{2,n}\\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-1} & a_{3,n}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & a_{n-1,n}\\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-1} & a_{n,n}\end{pmatrix};$$

$$\mathbf{M}=\mathbf{M}_{n-1}^{'}\ldots\mathbf{M}_1^{'}=$$

$$=\begin{pmatrix}m_{11} & 0 & 0 & \cdots & 0 & 0\\ m_{21} & m_{22} & 0 & \cdots & 0 & 0\\ m_{31} & m_{32} & m_{33} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & & \\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n-1,n-1} & 0\\ m_{n,1} & m_{n,2} & m_{n,3} & \cdots & m_{n,n-1} & m_{n,n}\end{pmatrix}$$

Where

$m_{11}=1$; $m_{21}=-a_{21}$; $m_{22}=a_{11}$;

$$m_{31}=-\begin{vmatrix}a_{31} & a_{32}\\ a_{21} & a_{22}\end{vmatrix};\ m_{32}=-\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix};\ m_{33}=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix};$$

$$m_{n-1,1}=-\begin{vmatrix}a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2}\\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-2}\\ \vdots & \vdots & \vdots & & \vdots\\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2}\end{vmatrix};$$

$$m_{n-1,2}=-\begin{vmatrix}a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2}\\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2}\\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n-2}\\ \vdots & \vdots & \vdots & & \vdots\\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2}\end{vmatrix};$$

$$m_{n-1,3}=-\begin{vmatrix}a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2}\\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2}\\ \vdots & \vdots & \vdots & & \vdots\\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2}\end{vmatrix};$$

$$m_{n-1,n-1}=-\begin{vmatrix}a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-2}\\ a_{21} & a_{22} & a_{23} & \cdots & a_{2,n-2}\\ a_{31} & a_{32} & a_{33} & \cdots & a_{3,n-2}\\ \vdots & \vdots & \vdots & & \vdots\\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2}\end{vmatrix}$$

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$$m_{n,1} = -\begin{vmatrix} a_{n,1} & a_{n,2} & a_{n,3} & \cdot & \cdot & \cdot & a_{n,n-1} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3,n-1} \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdot & \cdot & \cdot & a_{n-1,n-1} \end{vmatrix};$$

$$m_{n,2} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1,n-1} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdot & \cdot & \cdot & a_{n,n-1} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3,n-1} \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdot & \cdot & \cdot & a_{n-1,n-1} \end{vmatrix};$$

$$m_{n,3} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2,n-1} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdot & \cdot & \cdot & a_{n,n-1} \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdot & \cdot & \cdot & a_{n-1,n-1} \end{vmatrix};$$

$$m_{n,n-1} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3,n-1} \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdot & \cdot & \cdot & a_{n,n-1} \end{vmatrix};$$

$$m_{n,n} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3,n-1} \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdot & \cdot & \cdot & a_{n-1,n-1} \end{vmatrix}$$

and

$$\mathbf{U} = \mathbf{MA} = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdot & \cdot & \cdot & u_{1,n-1} & u_{1,n} \\ 0 & u_{22} & u_{23} & \cdot & \cdot & \cdot & u_{2,n-1} & u_{2,n} \\ 0 & 0 & u_{33} & \cdot & \cdot & \cdot & u_{3,n-1} & u_{3,n} \\ \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & u_{n-1,n-1} & u_{n-1,n} \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & u_{n,n} \end{pmatrix}$$

Where

$u_{11} = a_{11}$; $u_{12} = a_{12}$; $u_{13} = a_{13}$; $u_{1,n-1} = a_{1,n-1}$;

$u_{1,n} = a_{1,n}$

$$u_{22} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}; \; u_{23} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix};$$

$$u_{2,n-1} = \begin{vmatrix} a_{11} & a_{1,n-1} \\ a_{21} & a_{2,n-1} \end{vmatrix}; \; u_{2,n} = \begin{vmatrix} a_{11} & a_{1,n} \\ a_{21} & a_{2,n} \end{vmatrix}$$

$$u_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}; \; u_{3,n-1} = \begin{vmatrix} a_{11} & a_{12} & a_{1,n-1} \\ a_{21} & a_{22} & a_{2,n-1} \\ a_{31} & a_{32} & a_{3,n-1} \end{vmatrix};$$

$$u_{3,n} = \begin{vmatrix} a_{11} & a_{12} & a_{1,n} \\ a_{21} & a_{22} & a_{2,n} \\ a_{31} & a_{32} & a_{3,n} \end{vmatrix}$$

The Laplace Expansion of sub-matrix $\mathbf{A_{n-1,n-1}}$ taking out the last column, is equivalent to multiply the *(n-1)- th* row of the matrix **M** by the *(n-1)- th* column of **A**. Then, we have $u_{n-1,n-1} = |\mathbf{A_{n-1,n-1}}|$ and

$$u_{n-1,n-1} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1,n-1} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2,n-1} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3,n-1} \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdot & \cdot & \cdot & a_{n-1,n-1} \end{vmatrix};$$

In a similar way, we have

$$u_{n-1,n} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1,n} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2,n} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3,n} \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdot & \cdot & \cdot & a_{n-1,n} \end{vmatrix};$$

Finally, having the last row of **M** multiplied by the last column of **A,** we have

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$$u_{n,n}=\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1,n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2,n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3,n} \\ . & . & . & . & & & . \\ . & . & . & & . & & . \\ . & . & . & & & . & . \\ a_{n,1} & a_{n,2} & a_{n,3} & . & . & . & a_{n,n} \end{vmatrix}$$

Now, if $\mathbf{M}$ is the new transformation, then we have

$$\mathbf{M}=\prod_{\mathbf{k=n-1}}^{\mathbf{1}}\mathbf{M}'_{\mathbf{k}}\ ;\ \mathbf{U}=\mathbf{MA}$$

In order to solve the linear system of equations $\mathbf{Ax}=\mathbf{b}$, we have

$$\mathbf{MAx}=\mathbf{Mb}$$

$$\mathbf{Ux}=\mathbf{Mb}$$

We can use the “backward process” and solve the linear system of equations using only determinants. For a matrix $\mathbf{A}$ with floating point entries this process requires

$$\frac{n(n-1)(2n-1)}{6}+\frac{(n-2)(n-1)}{2}=\frac{n^3}{3}-\frac{4}{3}n+1$$

Floating point multiplications.

VI. NEW DECOMPOSITION THEOREM WITH PARTIAL PIVOTING

Gauss elimination on real numbers is generally unstable due to the possibility of finding arbitrarily small pivots. This process can be alleviated, however, by exchanging rows during the elimination. In our case, only when a pivot is zero, we exchange the row in a similar way to the Gauss process. The following theorem is given without proof:

Theorem. Let $\mathbf{A}\in\mathbf{R}^{\mathbf{nxn}}$. Suppose that the New Transformation $\mathbf{M}'_{\mathbf{1}}\cdots\mathbf{M}'_{\mathbf{k-1}}$; row’s permutation matrices $\mathbf{P}=\mathbf{P_1}\cdots\mathbf{P_{k-1}}$ have been determined so that $\mathbf{U}=\mathbf{M}'_{\mathbf{k-1}}\mathbf{P_{k-1}}\cdots\mathbf{M}'_{\mathbf{1}}\mathbf{P_1}\mathbf{A}$. Then, the Upper Matrix $\mathbf{U}$ is obtained from $\mathbf{PA}$ without exchanging any rows and $\mathbf{U}=\mathbf{MPA}$. Furthermore, if *exch* ≡ *number of row’s exchanges*, we have $|\mathbf{A}|=(-1)^{exch}u_{n,n}$

VII. A NEW ALGEBRAIC METHOD FOR COMPUTING THE PFAFFIAN

Applying this new linear transformation to skew-symmetric matrix $\mathbf{A}\subset\mathbf{R}^{\mathbf{2n}}$ by means of partial pivoting strategy, we obtain the Pfaffian of $\mathbf{A}\subset\mathbf{R}^{\mathbf{2n}}$ from the principal diagonal of matrix $\mathbf{U}$, namely:

$$Pf(\mathbf{A})=\frac{\prod_{k=2}^{n}u_{k,k}}{\prod_{l=1}^{n-1}u_{l,l}}\quad(5)$$

, where $(k=2,4,6,..n)$ and $(l=1,3,5..n-1)$.

Proof. It follows from an induction on n:

n=2

$$\mathbf{A}=\begin{pmatrix}0 & a_{12}\\ -a_{12} & 0\end{pmatrix};\ \mathbf{P_1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\ \mathbf{P_1A}=\begin{pmatrix}-a_{12} & 0\\ 0 & a_{12}\end{pmatrix};\ \mathbf{M_1}=\begin{pmatrix}1 & 0\\ 0 & -a_{12}\end{pmatrix};$$

$$\mathbf{M_1P_1A}=\begin{pmatrix}1 & 0\\ 0 & -a_{12}\end{pmatrix}\begin{pmatrix}-a_{12} & 0\\ 0 & a_{12}\end{pmatrix}=\begin{pmatrix}-a_{21} & 0\\ 0 & -a_{12}^2\end{pmatrix};$$

If $\mathbf{M}=\mathbf{M_1}$ and $\mathbf{P}=\mathbf{P_1}$, then

$$\mathbf{U}=\mathbf{MPA}=\begin{pmatrix}1 & 0\\ 0 & -a_{12}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & a_{12}\\ -a_{12} & 0\end{pmatrix}=$$

$$=\begin{pmatrix}1 & 0\\ 0 & -a_{12}\end{pmatrix}\begin{pmatrix}-a_{12} & 0\\ 0 & a_{12}\end{pmatrix}=\begin{pmatrix}-a_{12} & 0\\ 0 & -a_{12}^2\end{pmatrix}$$

With this new method, the Pfaffian is the result of dividing the last value among the first of the main diagonal of the matrix $\mathbf{U}$:

$$Pf(\mathbf{A})=\frac{-a_{12}{}^2}{-a_{12}}=a_{12}$$

n=4

$$\mathbf{A}=\begin{pmatrix}0 & a_{12} & a_{13} & a_{14}\\ -a_{12} & 0 & a_{23} & a_{24}\\ -a_{13} & -a_{23} & 0 & a_{34}\\ -a_{14} & -a_{24} & -a_{34} & 0\end{pmatrix};\quad \mathbf{P_1}=\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix};$$

$$\mathbf{P_1A}=\begin{pmatrix}-a_{12} & 0 & a_{23} & a_{24}\\ 0 & a_{12} & a_{13} & a_{14}\\ -a_{13} & -a_{23} & 0 & a_{34}\\ -a_{14} & -a_{24} & -a_{34} & 0\end{pmatrix}$$

$$\mathbf{M_1P_1A}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -a_{12} & 0 & 0\\ a_{13} & 0 & -a_{12} & 0\\ a_{14} & 0 & 0 & -a_{12}\end{pmatrix}\begin{pmatrix}-a_{12} & 0 & a_{23} & a_{24}\\ 0 & a_{12} & a_{13} & a_{14}\\ -a_{13} & -a_{23} & 0 & a_{34}\\ -a_{14} & -a_{24} & -a_{34} & 0\end{pmatrix}=$$

$$=\begin{pmatrix}-a_{12} & 0 & a_{23} & a_{24}\\ 0 & -a_{12}^2 & -a_{12}a_{13} & -a_{12}a_{14}\\ 0 & a_{12}a_{23} & a_{13}a_{23} & \begin{vmatrix}a_{13} & a_{34}\\ a_{12} & a_{24}\end{vmatrix}\\ 0 & a_{12}a_{24} & \begin{vmatrix}a_{14} & a_{34}\\ a_{12} & a_{23}\end{vmatrix}^{+} & a_{14}a_{24}\end{pmatrix};$$

Where $\begin{vmatrix}a_{14} & a_{34}\\ a_{12} & a_{23}\end{vmatrix}^{+}=a_{14}a_{23}+a_{12}a_{34}$ is the *permanent*, similar to determinant, except that each term in the development is written with a plus sign.

Next:

$$\mathbf{M_2}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & a_{23} & a_{12} & 0\\ 0 & a_{24} & 0 & a_{12}\end{pmatrix};$$

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$\mathbf{M_2M_1P_1A} =$

$$=\begin{pmatrix} -a_{12} & 0 & a_{23} & a_{24} \\ 0 & -a_{12}^2 & -a_{12}a_{23} & -a_{12}a_{14} \\ 0 & 0 & 0 & a_{12}\left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right) \\ 0 & 0 & a_{12}\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right) & 0 \end{pmatrix};$$

$$\mathbf{P_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

$\mathbf{P_2M_2M_1P_1A} =$

$$=\begin{pmatrix} -a_{12} & 0 & a_{23} & a_{24} \\ 0 & -a_{12}^2 & -a_{12}a_{23} & -a_{12}a_{14} \\ 0 & 0 & a_{12}\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right) & 0 \\ 0 & 0 & 0 & a_{12}\left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right) \end{pmatrix};$$

$$\mathbf{M_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\dfrac{\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right)}{a_{12}} \end{pmatrix};$$

$\mathbf{U = M_3P_2M_2M_1P_1A} =$

$$=\begin{pmatrix} -a_{12} & 0 & a_{23} & a_{24} \\ 0 & -a_{12}^2 & -a_{12}a_{23} & -a_{12}a_{14} \\ 0 & 0 & a_{12}\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right) & 0 \\ 0 & 0 & 0 & \left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right)^2 \end{pmatrix};$$

The Pfaffian is obtained starting from a process that could be described as a "backward simplification". This is because it starts from the ultimate diagonal element of the matrix **U**, and then divided among the penultimate one and, in turn, among the antepenultimate one and, this way, until arriving to the first value of the diagonal:

$$Pf(\mathbf{A}) = \frac{\left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right)^2}{\dfrac{a_{12}\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right)}{\dfrac{-a_{12}{}^2}{-a_{12}}}} = \frac{\left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right)^2}{\dfrac{-a_{12}{}^2\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right)}{-a_{12}{}^2}} =$$

$$= \frac{\left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right)^2}{-\left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right)} = a_{14}a_{23} + a_{12}a_{34} - a_{13}a_{24}$$

The Pfaffian can be expressed by *permanents* too:

$$Pf(\mathbf{A}) = \frac{\left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right)^2}{\dfrac{a_{12}\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right)}{\dfrac{-a_{12}{}^2}{-a_{12}}}} = \frac{\left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right)^2}{\dfrac{-a_{12}{}^2\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right)}{-a_{12}{}^2}} =$$

$$= \frac{\left(-\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right)\right)^2}{\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right)} = a_{14}a_{23} + a_{12}a_{34} - a_{13}a_{24}$$

On the other hand one has that:

$\overline{\mathbf{M}} = \mathbf{M_3P_2M_2M_1P_1} =$

$$=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -a_{12} & 0 & 0 & 0 \\ -a_{12}a_{24} & a_{12}a_{14} & 0 & -a_{12}^2 \\ a_{23}\left(\begin{vmatrix} a_{14} & a_{13} \\ a_{24} & a_{23} \end{vmatrix} + a_{12}a_{34}\right) & a_{13}\left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right) & a_{12}\left(\begin{vmatrix} a_{14} & a_{13} \\ a_{24} & a_{23} \end{vmatrix} + a_{12}a_{34}\right) & 0 \end{pmatrix};$$

$\mathbf{U} = \overline{\mathbf{M}}\mathbf{A} =$

$$=\begin{pmatrix} -a_{12} & 0 & a_{23} & a_{24} \\ 0 & -a_{12}^2 & -a_{12}a_{23} & -a_{12}a_{14} \\ 0 & 0 & a_{12}\left(\begin{vmatrix} a_{14} & a_{34} \\ a_{12} & a_{23} \end{vmatrix}^{+} - a_{13}a_{24}\right) & 0 \\ 0 & 0 & 0 & \left(\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} - a_{12}a_{34}\right)^2 \end{pmatrix}$$

Finally, for the general case, we have:

$$\mathbf{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots & a_{1n} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} & \cdots & a_{2n} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} & \cdots & a_{3n} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} & \cdots & a_{4n} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} & \cdots & a_{5n} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 & \cdots & a_{6n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & -a_{3n} & -a_{4n} & -a_{5n} & -a_{6n} & \cdots & 0 \end{pmatrix};$$

Then, for $(a_{00}^{(1)} = 1)$ and $k = 1, \ldots, n-1$

$\mathbf{M_k} =$

$$\begin{pmatrix} 1_1 & & & & & & \\ & \ddots & & & & & \\ & & 1_{k-1} & & & & \\ & & & 1_k & & & \\ & & & -\dfrac{a_{k+1k}^{(k)}}{a_{k-1k-1}^{(k)}} & \dfrac{a_{kk}^{(k)}}{a_{k-1k-1}^{(k)}} & & \\ & & & \vdots & & \ddots & \\ & & & -\dfrac{a_{nk}^{(k)}}{a_{k-1k-1}^{(k)}} & & & \dfrac{a_{kk}^{(k)}}{a_{k-1k-1}^{(k)}} \end{pmatrix}.$$

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So:

$$\overline{\mathbf{M}} = \mathbf{M}_{n-1}\mathbf{P}_{n-3}\mathbf{M}_{n-2}\mathbf{M}_{n-3}\mathbf{P}_{n-4}\mathbf{M}_{n-4}\mathbf{M}_{n-5} \ \dots \ \mathbf{P}_3\mathbf{M}_4\mathbf{M}_3\mathbf{P}_2\mathbf{M}_2\mathbf{M}_1\mathbf{P}_1$$

and we can proceeded to obtain $\mathbf{U} = \overline{\mathbf{M}}\mathbf{A}$
The Pfaffian is the ultimate value of the diagonal of matrix **U** divided among the elements of the main diagonal corresponding to the sequence *l= 1,3,5,7,...* and multiplied by the elements corresponding to the sequence *k=2,4,6,...* namely:

$$Pf(\mathbf{A}) = \frac{\prod_{k=2}^{n} u_{kk}}{\prod_{l=1}^{n-1} u_{ll}} \ ; \ k = 2,4,6,8,... \ ; \ l = 1,3,5,7,... \quad (5)$$

VIII. CALCULATION OF THE "PFAFFIAN ADJUGATE MATRIX" ($\mathbf{G}$) WITH ANOTHER NEW LINEAR TRANSFORMATION $\mathbf{M}_{J_k}$

The adjugate is also called the adjoint. We avoid this usage because, in functional analysis, it refers to the equivalent of the conjugate transpose of a matrix. The adjugate $\mathbf{A}^{\mathbf{Adj}}$ of a matrix $\mathbf{A} \subset \mathbf{R}^{2n}$ is the transpose of the co-factor´s matrix of the elements of $\mathbf{A}$. Computing the adjugate from its definition, involves the calculation of n^2 determinants of order $(n-1)$. On the other hand, the calculation from the formula $\mathbf{A}^{\mathbf{Adj}} = |\mathbf{A}| \cdot \mathbf{A}^{-1}$ breaks down when **A** is singular and is potentially unstable when **A** is ill-conditioned with respect to inversion.

Expressing the new Linear Transformation in Jordan version we have:

For $(a_{00}^{(1)} = 1)$ and $k = 1,\dots,n$

$$\mathbf{M}_{\mathbf{J}_k} = \frac{1}{a_{k-1k-1}^{(k)}} \cdot \begin{pmatrix} a_{kk}^{(k)} & & & & -a_{1k}^{(k)} & & & \\ & \ddots & & & \vdots & & & \\ & & \ddots & & \vdots & & & \\ & & & a_{kk}^{(k)} & \vdots & & & \\ & & & & a_{k-1k-1}^{(k)} & & & \\ & & & & & a_{kk}^{(k)} & & \\ & & & & \vdots & & \ddots & \\ & & & & \vdots & & & \ddots \\ & & & & -a_{nk}^{(k)} & & & a_{kk}^{(k)} \end{pmatrix} \quad (6)$$

It is sufficient to do the operations indicated by the new Linear Transformation $\mathbf{M}_\mathbf{J} : \mathbf{I}^{\mathbf{nxn}} \rightarrow \mathbf{I}^{\mathbf{nxn}}$ to find the adjugate matrix of size n : $\mathbf{M}_\mathbf{J} = \prod_{\mathbf{k=n}}^{\mathbf{1}} \mathbf{M}_{\mathbf{J}_k} = \mathbf{A}^{\mathbf{Adj}}$.

Calculating the matrix with $\mathbf{M}_\mathbf{J}$ proves highly efficient when working with the augmented matrix **(A|I) :**

$$\mathbf{M}_{\mathbf{J}_n} \cdot \mathbf{M}_{\mathbf{J}_{n-1}} \cdot \dots \cdot \mathbf{M}_{\mathbf{J}_1} (\mathbf{A}_{\mathbf{n,n}} \,|\, \mathbf{I}_{\mathbf{n,n}}) = (|\mathbf{A}| \cdot \mathbf{I}_{\mathbf{n,n}} \,|\, \mathbf{A}_{\mathbf{n,n}}^{\mathbf{Adj}}).$$

This process demands as many as

$$\frac{(n^2-n)(2n-1)}{6} + \sum_{i=1}^{n-1} i(n-i-1) + \frac{(n-2)(n-1)}{2} = \frac{n^3}{2} - \frac{n^2}{2} - \frac{3}{2}n + 1$$

Floating point multiplications.
Another form to express the new transformation is

$$\mathbf{M}_{\mathbf{J}_k} = \begin{pmatrix} \frac{1}{a_{k-1k-1}^{(k)}} \cdot \mathbf{e}_1^{\mathbf{T}} \\ \vdots \\ \frac{1}{a_{kk}^{(k)}} \cdot \mathbf{e}_k^{\mathbf{T}} \\ \vdots \\ \frac{1}{a_{k-1k-1}^{(k)}} \cdot \mathbf{e}_n^{\mathbf{T}} \end{pmatrix} \cdot \left[a_{kk}^{(k)}\mathbf{I} - \begin{bmatrix} a_{1k}^{(k)} \\ a_{2k}^{(k)} \\ \vdots \\ 0_{kk}^{(k)} \\ a_{k+1}^{(k)} \\ \vdots \\ a_{nk}^{(k)} \end{bmatrix} \cdot \mathbf{e}_k^{\mathbf{T}} \right] \quad (7) \quad (k = 1,\dots,n);$$

where $a_{00}^{(1)} = 1$

$$\mathbf{A}^{\mathbf{Adj}} = \mathbf{M}_\mathbf{J} = \prod_{\mathbf{k=n}}^{\mathbf{1}} \mathbf{M}_{\mathbf{J}_k}$$

Additionally, with a minimum and partial pivoting strategy, we have

$\mathbf{A}^{\mathbf{Adj}} = \mathbf{M}_\mathbf{J}\mathbf{P}$

Proof. Let it suffice to explain how the algorithm works with the following case.

Let $\mathbf{A} \subset \mathbf{R}^{2n}$ and n=2:

$$\mathbf{A} = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}; \quad \mathbf{P}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$\mathbf{M}_{\mathbf{J}_3}\mathbf{M}_{\mathbf{J}_2}\mathbf{M}_{\mathbf{J}_1}\mathbf{PA} = |\mathbf{A}|\mathbf{I}$$
$$\mathbf{M}_\mathbf{J} = |\mathbf{A}|(\mathbf{PA})^{-1}$$
$$\mathbf{M}_\mathbf{J} = |\mathbf{A}|\mathbf{A}^{-1}\mathbf{P}^{-1}$$
$$\mathbf{M}_\mathbf{J}\mathbf{P} = \mathbf{A}^{\mathbf{Adj}}$$

On the other hand, for k=1,…,4 ; $(a_{00}^{(1)} = 1)$ we have:

$$\mathbf{M}_{\mathbf{J}_4}\mathbf{M}_{\mathbf{J}_3}\mathbf{P}_2\mathbf{M}_{\mathbf{J}_2}\mathbf{M}_{\mathbf{J}_1}\mathbf{P}_1\mathbf{A} = |\mathbf{A}|\mathbf{I}$$
$$\mathbf{M}_{\mathbf{J}_4}\mathbf{M}_{\mathbf{J}_3}\mathbf{P}_2\mathbf{M}_{\mathbf{J}_2}\mathbf{M}_{\mathbf{J}_1}\mathbf{P}_1\mathbf{A} = (\mathbf{Pf}(\mathbf{A}))^2\mathbf{I}$$
$$\mathbf{Pf}(\mathbf{A})\mathbf{G}\mathbf{A} = (\mathbf{Pf}(\mathbf{A}))^2$$
$$\mathbf{G}\mathbf{A} = \mathbf{Pf}(\mathbf{A})$$
$$\mathbf{G} = \mathbf{Pf}(\mathbf{A})\mathbf{A}^{-1} = \mathbf{Pf}(\mathbf{A}) \cdot \frac{\mathbf{A}^{\mathbf{Adj}}}{|\mathbf{A}|} = \frac{\mathbf{A}^{\mathbf{Adj}}}{\mathbf{Pf}(\mathbf{A})} \quad (8)$$

SOLUTION OF AX=B WITH THE G MATRIX

The simultaneous linear equations systems can also be solved with this new **G** matrix.

Since: $\mathbf{G} = \frac{\prod_{\mathbf{k=n}}^{\mathbf{1}} \mathbf{M}_{\mathbf{J}_k}}{\mathbf{Pf}(\mathbf{A})} = \frac{\mathbf{A}^{\mathbf{Adj}}}{\mathbf{Pf}(\mathbf{A})}$, then:

$$\frac{\mathbf{A}^{\mathrm{Adj}}\mathbf{Ax}}{\mathbf{Pf}(\mathbf{A})} = \frac{\mathbf{A}^{\mathrm{Adj}}\mathbf{b}}{\mathbf{Pf}(\mathbf{A})}; \quad \frac{|\mathbf{A}|\mathbf{Ix}}{\mathbf{Pf}(\mathbf{A})} = \mathbf{Gb} \ ; \ \mathbf{x} = \frac{\mathbf{Gb}}{\mathbf{Pf}(\mathbf{A})} \quad (9)$$

This new result is a Pfaffian-Cramer-type solution in $O(n^3)$.

EXAMPLE OF SOLUTION OF AX=B WITH THE G MATRIX : VOLTERRA-LOTKA EQUATION FOR PREDATOR-PREY DYNAMICS

Suppose we have two competing species of fish, labeled prey (small fish) and predator (large fish). Let $\mathbf{N}_1$ and $\mathbf{N}_2$ denote the number of small and large fish, respectively. The differential equations satisfied by $\mathbf{N}_1$ and $\mathbf{N}_2$, in the classic Volterra-Lotka model [12] are

$$\frac{d\mathbf{N}_1}{dt} = \alpha_1\mathbf{N}_1 - \lambda_1\mathbf{N}_1\mathbf{N}_2 \quad (10)$$

$$\frac{d\mathbf{N}_2}{dt} = -\alpha_2\mathbf{N}_2 + \lambda_2\mathbf{N}_1\mathbf{N}_2 \quad (11)$$

Where

α_1 = natural growth rate of small fish in the absence of large fish

λ_1 = death rate per encounter of small fish due to large fish

α_2 = natural death rate of large fishes in the absence of small fish

λ_2 = efficiency of turning predated small fishes due to predation

$-\lambda_1\mathbf{N}_1\mathbf{N}_2$ = gives the rate of small fish being lost

$\lambda_2\mathbf{N}_1\mathbf{N}_2$ = gives the growth rate of the population of large fish

For n species, the system of differential equations is in the form

$$\frac{d\mathbf{N}_i}{dt} = k_i\mathbf{N}_i + \frac{1}{\beta_i}\sum_{j=1}^{n} a_{ij}\mathbf{N}_i\mathbf{N}_j \ , \ (i = 1,\ldots,n) \quad (12)$$

Where

$a_{ij} = -a_{ji}$ and $a_{ii} = 0$ for all i ; the constants k_i and β_i are positive and are called equivalence numbers. Equations (12) are called the Volterra-Lotka equations, and play an important role in Ecology.

Suppose we are interested in the equilibrium solutions of system (12), in which case

$$\frac{d\mathbf{N}_i}{dt} = 0 \quad \forall \quad i \quad (13)$$

System (12) becomes

$$\mathbf{N}_i\left(k_i\beta_i + \sum_{j=1}^{n} a_{ij}\mathbf{N}_j\right) = 0$$

or, provided that $\mathbf{N}_i \neq 0$ for *(i=1,…,n)*

$$\sum_{j=1}^{n} a_{ij}\mathbf{N}_j = -k_i\beta_i \ , \ (i = 1,\ldots,n) \quad (14)$$

If the number of species is odd, n = *2N+1*, no equilibrium solutions can exist, because a_{ij} is a skew-symmetric matrix of odd order.

Simple and interesting algebraic considerations permit us to study and describe the situation when the number of species is even, n = *2N*.

Writing (14) in matrix notation, we have

$$\mathbf{Ax} = \mathbf{b} \quad (15)$$

Where $\mathbf{A}$ is an even skew-symmetric matrix $\left(\mathbf{A} = -\mathbf{A}^{\mathbf{T}}\right)$ and $\mathbf{x}$ and $\mathbf{b}$ are 2N-dimensional vectors. Equation (15) is readily solved by the Pfaffian-Cramer´s Rule, provided that $\mathbf{Pf}(\mathbf{A}) \neq \mathbf{0}$

An order *2N* Pfaffian is a triangular array of *N(2N-1)* elements (i,j):

$$Pf = \backslash(i,j) = \left| \begin{array}{ccccc} a_{12} & a_{13} & a_{14} & \cdots & a_{1,2N} \\ & a_{23} & a_{24} & \cdots & a_{2,2N} \\ & & a_{34} & \cdots & a_{3,2N} \\ & & & \cdots & \cdots \\ & & & & a_{2N-1,2N} \end{array} \right| \quad (16)$$

which may be expanded by its first row in the same way as a determinant, except that the minor of an element *(i,j)* is a Pfaffian of order *2(N-1)* obtained from *Pf* by deleting *both* the *i*-th row and column *and* the *j*-th row and column [13,14,15]

Thus, if *n=4* we have

$$\mathbf{Pf}(\mathbf{A}) = \left| \begin{array}{ccc} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{array} \right| = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$$

The Cramer´s Rule reduces, in this case, to the ratio of two pfaffians. So, we will calculate the Pfaffians

$$\mathbf{Pf}(\mathbf{A}_1) = \left| \begin{array}{ccc} -b_2 & -b_3 & -b_4 \\ & a_{23} & a_{24} \\ & & a_{34} \end{array} \right| = -b_2a_{34} + b_3a_{24} - b_4a_{23}$$

$$\mathbf{Pf}(\mathbf{A}_2) = \left| \begin{array}{ccc} b_1 & a_{13} & a_{14} \\ & -b_3 & -b_4 \\ & & a_{34} \end{array} \right| = b_1a_{34} + b_4a_{13} - b_3a_{14}$$

$$\mathbf{Pf}(\mathbf{A}_3) = \left| \begin{array}{ccc} a_{12} & b_1 & a_{14} \\ & b_2 & a_{24} \\ & & -b_4 \end{array} \right| = -b_4a_{12} - b_1a_{24} + b_2a_{14}$$

$$\mathbf{Pf}(\mathbf{A}_4) = \left| \begin{array}{ccc} a_{12} & a_{13} & b_1 \\ & a_{23} & b_2 \\ & & b_3 \end{array} \right| = b_3a_{12} - b_2a_{13} + b_1a_{23}$$

Finally

$$\mathbf{x_i} = \frac{\mathbf{Pf}(\mathbf{A_i})}{\mathbf{Pf}(\mathbf{A})} \quad (\mathbf{i} = \mathbf{1},\ldots,\mathbf{4}) \quad (17)$$

Otherwise, we have

$$\mathbf{G} = \begin{pmatrix} 0 & -a_{34} & a_{24} & -a_{23} \\ a_{34} & 0 & -a_{14} & a_{13} \\ -a_{24} & a_{14} & 0 & -a_{12} \\ a_{23} & -a_{13} & a_{12} & 0 \end{pmatrix}$$

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If $\mathbf{g}_i^T \in \mathbf{G}$ is the *i*-th row vector of **G**, and $\mathbf{a}_i \in \mathbf{A}$ is the column vector of **A**, we have

$$\mathbf{x}_i = \frac{\mathbf{g}_i^T\mathbf{b}}{\mathbf{g}_i^T\mathbf{a}_i} = \frac{\mathbf{Pf}(\mathbf{A}_i)}{\mathbf{Pf}(\mathbf{A})} \quad (\mathbf{i}=\mathbf{1},...,\mathbf{4}) \qquad (18)$$

IX. CONCLUSION

In this paper we have introduced a new theorem on the decomposition into determinants of matrix $\mathbf{A} \in \mathbf{R}^n$ and the new linear transformations, expressed as equations (3) and (4). Most simultaneous linear equation systems can also be solved with these new linear transformations. The result is Cramer-type solutions in $O(n^3)$. Furthermore, we have obtained the Pfaffian of **A**, expressed as equation (5) from the **U** matrix.

On the other hand, we have proposed a modified Doolittle-Gauss-Jordan elimination process in two versions: the first one applied to the augmented matrix **(A|I)** and the second to the augmented matrix **(A|b)**. The first version is an algorithm to calculate the new **G** matrix from $\mathbf{A} \in \mathbf{R}^{2n}$, expressed as equation (8). The second version is a new direct method to solve linear system, equation (9), if $(\mathbf{A,b}) \subset \mathbf{R}^{2n}$. The above algorithms calculate Pfaffian-Cramer-type solutions of the linear systems.

Gaussian elimination is usually the most economical way to solve $\mathbf{Ax} = \mathbf{b}$. Nevertheless there is one reason why this new method might be relevant when $\mathbf{A} \in \mathbf{R}^{2n}$: while the Gaussian elimination is oriented to traditional prevalence of the determinants, this new method is focused to a pffafian approach.

Finally, when referring to the Cramer's rule, it has been affirmed by G. Strang [11] that: "…Thus each component of *x* is a ratio of two determinants, a polynomial of degree *n* divided by another polynomial of degree *n*. This fact might have been recognized from Gauss elimination, but it never was". Not only has this fact been made evident in the present paper but also it has been shown that the Pfaffian-Cramer´s Rule, is actually a ratio of two Pfaffians, derived from the Gauss elimination process.

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