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The weak Hawkins-Simon condition verified by a Mexican's Algorithm through multi-core CPU-GPU equipment

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***Abstract.* A real square matrix satisfies the weak Hawkins-Simon condition if its leading principal minors are positive. From a book [32] published by first author, we define the coefficients of the characteristic polynomial of a matrix A as the alternate signed sum of all the traces of those sub-matrices obtained from A by two processes, namely, a partition process and a modification one. Precisely, we use the modification process in recurrent way and so we obtain all leading principal minor of Mexican Economy Matrix through multi-core CPU-GPU equipment.**

***Index Terms*— Characteristic polynomial, leading principal minors, Z matrix, M matrix, Leontief matrix, Inverse-positive matrix**

I. INTRODUCTION

A real square matrix is said to satisfy the weak Hawkins Simon [1] criterion, or to be of the **WHS** type, if all its **leading principal minors** are positive. In economics as well as other sciences, the inverse-positivity of real square matrices has been an important topic. The Hawkins-Simon condition, so called in economics, requires that every principal minor be positive, and they showed the **condition to be necessary and sufficient for a Z-matrix** (a matrix with nonpositive off-diagonal elements) **to be inverse-positive**.

One decade earlier, this was used by Ostrowski [2] to define an **M-matrix** (an inverse-positive Z-matrix), and was shown to be equivalent to some of other conditions; see Berman and Plemmons [3, Ch.6] for many equivalent conditions.

Georgescu-Roegen [4] argued that **for a Z matrix it is sufficient to have only leading (upper left corner) principal minors positive**, which was also proved in Fiedler and Ptak [5]. Nikaido's two books, [6] and [7], contain a proof based on mathematical induction. Dasgupta [8] gave another proof using an economic interpretation of indirect input.

II. NEW METHOD FOR COMPUTING A CHARACTERISTIC POLYNOMIAL OF A MATRIX THROUGH OF ALL PRINCIPAL MINORS

Many algorithms have been proposed in order to calculate the coefficients of the characteristic polynomial of a matrix **A**. A famous one is Leverrier's algorithm [9] which dates from 1840 [10]. It requires a greater number of operations than anyone of the more recent methods, albeit, it is utterly insensitive to the individual peculiarities of the matrix, in particular to "gaps" in the intermediate determinants. The computational process reduces itself to the successive calculation of the *n* powers of the matrix **A**, that of their traces and, finally, to the solution of a recurrence system. The number of multiplications necessary in the Leverrier's method is equal to $\frac{1}{2}(n-1)(2n^3-2n^2+n+2)$ [11] and the value of this method resides in its universality.

There exists another method, proposed by D.K. Faddeev and I.S. Sominsky [12], which, in addition to simplifying the computation of the coefficients of the characteristic polynomial, enables to determine the inverse matrix and the proper vectors of the original matrix. This algorithm has been rediscovered and modified several times. In 1840, U.J.J. Leverrier himself provided the basic connection with Newton´s identities. J.M. Souriau, also from France, and J. S. Frame, from Michigan State University, independently modified the algorithm to its present form. Souriau´s formulation was published in France in 1948, and Frame´s method appeared in the United States in 1949.

Yet, although the algorithm is intriguingly beautiful, it is not practical for floating-point computations. The number of operations necessary for obtaining the coefficients c_i is $(n-1)n^3$ products [11]. Recently, Shui-Hung Hou proved the above algorithm in [13]. From an analytical point of view, instead of the sequence of successive powers of the original matrix $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \ldots, \mathbf{A}^n$, let us compute the sequence $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ constructed in the following manner:

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$$\mathbf{A}_1 = \mathbf{A}, \quad tr\mathbf{A}_1 = c_1, \quad \mathbf{B}_1 = \mathbf{A}_1 - c_1\mathbf{I}$$

$$\mathbf{A}_2 = \mathbf{A}\mathbf{B}_1, \quad \frac{tr\mathbf{A}_2}{2} = c_2, \quad \mathbf{B}_2 = \mathbf{A}_2 - c_2\mathbf{I}$$

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$$\mathbf{A}_{n-1} = \mathbf{A}\mathbf{B}_{n-2}, \quad \frac{tr\mathbf{A}_{n-1}}{n-1} = c_{n-1}, \quad \mathbf{B}_{n-1} = \mathbf{A}_{n-1} - c_{n-1}\mathbf{I}$$

$$\mathbf{A}_n = \mathbf{A}\mathbf{B}_{n-1}, \quad \frac{tr\mathbf{A}_n}{n} = c_n, \quad \mathbf{B}_n = \mathbf{A}_n - c_n\mathbf{I}$$

where: $tr\mathbf{A} = \sum_{i=1}^{n} a_{ii}$, i.e., the sum of the diagonal elements and the characteristic polynomial of a matrix $\mathbf{A}$ is $P_f = (-1)^n\left(x^n - c_1x^{n-1} - c_2x^{n-2} ... - c_n\right)$.

Some other methods are quite relevant too, like Samuelson's algorithm. This allows the characteristic polynomial to be computed recursively without divisions, although we will not discuss it here in greater detail.

In an attempt to unravel useful ideas to design efficient parallel algorithms for the determinant, Valiant studied Samuelson's algorithm and interpreted its computation combinatorially [14]. Thus, he presented a theorem concerning closed walk (clows) in graphs, the correctness of which followed from that of Samuelson's algorithm. This was the first attempt to treat determinant computations as graph-theoretic rather than linear algebraic manipulations. Inspired by this, and the purely combinatorial and extremely elegant proof of the Cayley-Hamilton theorem due to Rutherford [15] also independently discovered by Straubing [16], Mahajan and Vinay [17] described a combinatorial algorithm for computing the characteristic polynomial. The proof of correctness of this algorithm is also purely combinatorial and does not rely on any linear algebra or polynomial arithmetic. In another paper [18], the authors follow up on the work presented in [14,16,17] and develop a unifying combinatorial framework in which to interpret and analyse a host of algorithms for computing the determinant and the characteristic polynomial. Their paper is thus a collection of new interpretations and proofs of known results and, in a sense, their paper trails the work done by a great number of combinatorialists in proving the correctness of matrix identities using the graph-theoretic setting.

III. DIRECT ANTECEDENTS OF THIS METHOD: QUASI-TRIANGULAR AND TRI-DIAGONAL MATRICES

Let $\mathfrak{R}$ be a domain in the form of a commutative ring with identity and without zero divisors. We assume that $\mathfrak{R}$ is endowed with an algorithm allowing *exact division*. This means that if any two elements *a* and *b* of $\mathfrak{R}$ are given (*a* being different from zero) such that *b*=*ac* with $c \in \mathfrak{R}$, then this algorithm can exhibit the exact quotient c . Let $\mathfrak{R}^{n \times m}$ denote the set of $n \times m$ matrices with entries in $\mathfrak{R}$, and $\mathfrak{M}$ be a *n*-rank free module over $\mathfrak{R}$.

Let *f* be an endomorphism of $\mathfrak{M}$ and p_f its characteristic polynomial. If $\mathbf{I}_n$ denote the *nxn* identity matrix and if $\mathbf{A} = (a_{ij}) \in \mathfrak{R}^{n \times m}$ is the matrix of f in terms of some given basis of $\mathfrak{M}$, then $p_f = \det(\mathbf{A} - x\mathbf{I})$. One of the important problems of computational commutative algebra is that to find effective methods for the computation of characteristic polynomials.

Until now, the best theoretical algorithms for solving this problem in a domain are:

a).- The sequential method [19,20] with $O(n^3)$ arithmetic operations in the fraction field $\mathfrak{K}$ of $\mathfrak{R}$.

b).- The Csanky parallel algorithm [21] improved by Preparata and Sarwate [22] and based on the method of the French astronomer Le Verrier .

In the case of an arbitrary *commutative* ring , the best parallel algorithms are the Chistov one [23] and the improved Berkowitz algorithm [24]. In fact, the sequential version of the last algorithm has the best practical behaviour for the computation of the characteristic polynomial in an arbitrary commutative ring [25].

Recently [26], two new more efficient sequential methods with $O(n^3)$ ring operations (addition, subtraccion, multiplication and exact division) are described. The first one is the Quasi-triangular method and the second one is the Tri-diagonal method.

The aim of this paper is to describe a new method with $O(2^n)$ ring operations. In this algorithm, all operations are executed with elements of the ring $\mathfrak{R}$ instead of the much more costly field operations of the other methods whose decomposition requires floating point arithmetic operations.

The method proposed in this paper operates on a new linear transformation that avoids the relative growth of the intermediate quantities in the calculations, improving the last two previous methods, namely: the Quasi-triangular and Tri-diagonal ones. At the end of this article, *Appendix B* contains a simple numeric example that shows the transformations which are performed during the reduction step in each sub-matrix.

As we need to express the characteristic polynomial coefficients in the form of determinants, in the next two sections we introduce special types of matrices.

IV. PARTITIONED COMBINATORY SYMMETRIC SQUARE MATRICES

Consider a diagonal matrix:

$$\mathbf{D_{n,n}} = \begin{pmatrix} a_{11} & & & & & & & & & \\ & a_{22} & & & & & & & & \\ & & a_{33} & & & & & & & \\ & & & a_{44} & & & & & & \\ & & & & a_{55} & & & & & \\ & & & & & . & & & & \\ & & & & & & . & & & \\ & & & & & & & . & & \\ & & & & & & & & a_{n-1,n-1} & \\ & & & & & & & & & a_{n,n} \end{pmatrix}$$

We shall form a partitioned combinatory symmetric square matrix of size $k \times k$:

$$\mathbf{A^i_{k,k}} \quad \forall \quad 2 \le k < n \quad ; i = 1,\dots,\binom{n}{k}$$

where $\binom{n}{k}$ = number of combinations of n objects taken k at a time. By appending up to the diagonal matrix $\mathbf{D_{k,k}}$ an upper triangular matrix with consecutive index numbers corresponding to diagonal elements and then, appending also a lower triangular matrix which is the transpose matrix of the above upper triangular one, we obtain a dense matrix $\mathbf{A^i_{k,k}}$.

Examples.

For $n=3$ and $k=2$ we have

$$\mathbf{D_{3,3}} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

Then

$$i = 1,\dots,\binom{3}{2} = 1,\dots,\frac{3!}{2!(3-2)!} = 1,\dots,3$$

So that

$$\mathbf{A^1_{2,2}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad ; \quad \mathbf{A^2_{2,2}} = \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} \quad ;$$

$$\mathbf{A^3_{2,2}} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

For $n \times n$ and $k=2,\dots,n-1$ we have

$$k=2 \; ; \quad i = 1,\dots,\binom{n}{2} = 1,\dots,\frac{n!}{2!(n-2)!}$$

$$\mathbf{A^1_{2,2}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad ,\dots, \quad \mathbf{A^{\binom{n}{2}}_{2,2}} = \begin{pmatrix} a_{11} & a_{1,n} \\ a_{n,1} & a_{n,n} \end{pmatrix}$$

$$k=3 \; ; \quad i = 1,\dots,\binom{n}{3}$$

$$A^1_{3,3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

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$$A^{\binom{n}{3}}_{3,3} = \begin{pmatrix} a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\ a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,n-2} & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

Finally

$$k=n-1 \; ; \quad i = 1,\dots,\binom{n}{n-1} = 1,\dots,n$$

$$A^1_{n-1,n-1} = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1,n-2} & a_{1,n-1} \\ a_{21} & a_{22} & . & . & . & a_{2,n-2} & a_{2,n-1} \\ . & . & . & & & . & . \\ . & . & & . & & . & . \\ . & . & & & . & . & . \\ a_{n-2,1} & a_{n-2,2} & . & . & . & a_{n-2,n-2} & a_{n-2,n-2} \\ a_{n-1,1} & a_{n-1,2} & . & . & . & a_{n-1,n-2} & a_{n-1,n-1} \end{pmatrix}$$

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$$A^n_{n-1,n-1} = \begin{pmatrix} a_{22} & a_{23} & . & . & . & a_{2,n-1} & a_{2,n} \\ a_{32} & a_{33} & . & . & . & a_{3,n-1} & a_{3,n} \\ . & . & . & & & . & . \\ . & . & & . & & . & . \\ . & . & & & . & . & . \\ a_{n-1,2} & a_{n-1,3} & . & . & . & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,2} & a_{n,3} & . & . & . & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

V. ISOVOLUMETRICS MATRICES

Let $\mathbf{A} \in \mathbf{R^{n \times n}}$, then we say that $\mathbf{A_G}$ is an isovolumetric matrix if $|\mathbf{A}| = |\mathbf{A_G}|$, where $|\mathbf{A}|$ represents the determinant of matrix $\mathbf{A}$ and $\mathbf{A_G}$ is obtained from $\mathbf{A}$ by permuting the main diagonal elements of $\mathbf{A}$ in any of their $n!$ possible positions. Finally, the diagonal matrix is appended to an upper triangular matrix with consecutive index members corresponding to diagonal elements, and then, appending also a lower triangular matrix wich is the transpose matrix of the above upper triangular one.

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Example

Let $\mathbf{A} \in \mathbf{R}^{3x3}$ then, there are six (*3!*) isovolumetric matrices:

$$\mathbf{A}_{\mathbf{G}_1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; \; \mathbf{A}_{\mathbf{G}_2} = \begin{pmatrix} a_{11} & a_{13} & a_{12} \\ a_{31} & a_{33} & a_{32} \\ a_{21} & a_{23} & a_{22} \end{pmatrix};$$

$$\mathbf{A}_{\mathbf{G}_3} = \begin{pmatrix} a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \\ a_{12} & a_{13} & a_{11} \end{pmatrix}; \; \mathbf{A}_{\mathbf{G}_4} = \begin{pmatrix} a_{22} & a_{21} & a_{23} \\ a_{12} & a_{11} & a_{13} \\ a_{32} & a_{31} & a_{33} \end{pmatrix};$$

$$\mathbf{A}_{\mathbf{G}_5} = \begin{pmatrix} a_{33} & a_{31} & a_{32} \\ a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \end{pmatrix}; \; \mathbf{A}_{\mathbf{G}_6} = \begin{pmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{pmatrix}.$$

We obtain these isovolumetric matrices by making use of diagonal permutation, which not change the sign of the determinant and it is appropriate to calculate a new linear transformation that we shall propose below.

VI. THE CHARACTERISTICS POLYNOMIAL COEFFICIENTS EXPRESSED IN DETERMINANT FORM

We shall deduce an expression of the characteristic polynomial by calculating its coefficients on the conmmutative ring with identity and without zero divisors of the integral domain.

Let us consider that the characteristic polynomial is on the left-hand side of the characteristic equation

$$|\mathbf{A} - x\mathbf{I}| = 0$$

Where $\mathbf{A}$ is a square matrix and $\mathbf{I}$ is the identity matrix of identical dimension.

So, if the characteristic polynomial of a matrix $\mathbf{A} \in \mathbf{I}^{nxn}$ is

$$P = (-1)^n \left(c_n x^n - c_{n-1} x^{n-1} + c_{n-2} x^{n-2} \ldots - c_0\right) = 0$$

where $c_n = 1$; and $c_0 = |\mathbf{A}|$ is the determinant of the matrix $\mathbf{A}$, then, it follows from an induction on n that, for $n=1$ we have

$$\mathbf{A} = (a_{11}) \text{ and } P = (-1)^1 \left(c_1 x^1 - c_0\right) = -x + |A| = 0$$

Then, the characteristic polynomial is

$$x - a_{11} = 0$$

For the induction steps i = 2,…,n; the characteristic equation of a 2×2 matrix is

$$x^2 - x(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

and it can be re-written in the particularly nice form

$$x^2 - xTr(\mathbf{A}) + |\mathbf{A}| = 0$$

where $Tr(\mathbf{A}) = \sum_{i=1}^{n} a_{i,i}$ is the trace of $\mathbf{A}$, i.e., the sum of its diagonal elements.

Similarly, the characteristic equation of a 3×3 matrix is

$$-x^3 + Tr(\mathbf{A})x^2 - \left(\sum_{i=1}^{\binom{3}{2}} \left|A_{2,2}^i\right| \right) x + |\mathbf{A}| = 0$$

Where

$$\left(\sum_{i=1}^{\binom{3}{2}} \left|A_{2,2}^i\right| \right) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

For the next induction step, n=4, we have

$$x^4 - Tr(\mathbf{A})x^3 + \left(\sum_{i=1}^{\binom{4}{2}} \left|A_{2,2}^i\right| \right) x^2 - \left(\sum_{i=1}^{\binom{4}{3}} \left|A_{2,2}^i\right| \right) x + |\mathbf{A}| = 0$$

where

$$\left(\sum_{i=1}^{\binom{4}{2}} \left|A_{2,2}^i\right| \right) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$$

$$\left(\sum_{i=1}^{\binom{4}{3}} \left|A_{3,3}^i\right| \right) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

Finally, for n, we have

$$(-1)^n \left[\begin{array}{l} x^n - Tr(\mathbf{A})x^{n-1} + \left(\sum_{i=1}^{\binom{n}{2}} \left|A_{2,2}^i\right| \right) x^{n-2} - \left(\sum_{i=1}^{\binom{n}{3}} \left|A_{3,3}^i\right| \right) x^{n-3} + \ldots + \\ + \left(\sum_{i=1}^{\binom{n}{n-1}} \left|A_{n-1,n-1}^i\right| \right) x - |\mathbf{A}| \end{array} \right] = 0 \quad (1)$$

and explicitly the coefficients are

$$c_n = 1$$

$$c_{n-1} = Tr(\mathbf{A})$$

$$c_{n-2} = \sum_{i=1}^{\binom{n}{2}} \left|A_{2,2}^i\right|$$

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$$c_{n-3} = \sum_{i=1}^{\binom{n}{3}} \left| A_{3,3}^{i} \right|$$

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$$c_1 = \sum_{i=1}^{\binom{n}{n-1}} \left| A_{n-1,n-1}^{i} \right|$$

$$c_0 = |\mathbf{A}|$$

We can observe very clearly that all the coefficients of the characteristic polynomial are calculated with additions, subtractions and products. In the section below, we shall obtain a better way to compute these determinants.

VII. DERIVATION OF A NEW LINEAR TRANSFORMATION ABLE TO CALCULATE THE CHARACTERISTIC POLYNOMIAL COEFFICIENTS EXPRESSED IN DETERMINANT FORM

Let us assume that $\mathbf{A} \in \mathbf{R}^{\mathbf{n}\times\mathbf{n}}$ is non-singular and that we wish to solve the linear system $\mathbf{Ax} = \mathbf{b}$. Here we show how, for exact arithmetic and partial pivoting and column exchanges, some Gauss transformations $\mathbf{M_1,..., M_{n-1}}$ can almost always be found such that $\mathbf{M_{n-1},..., M_2 M_1 A} = \mathbf{U}$ is upper triangular [27]. The original $\mathbf{Ax} = \mathbf{b}$ problem is then equivalent to the upper triangular system $\mathbf{Ux} = (\mathbf{M_{n-1},..., M_2 M_1})\ \mathbf{b}$ wich can be solved through back-substitution.

Suppose, then, that $\mathbf{A} \in \mathbf{R}^{\mathbf{n}\times\mathbf{n}}$ and that, for some $k<n$, we have determined the Gauss transformations $\mathbf{M_1,..., M_{k-1}} \in \mathbf{R}^{\mathbf{nxn}}$ such that

$$\mathbf{A^{(k-1)}} \equiv \mathbf{M}_{k-1} \cdots \mathbf{M}_1 \mathbf{A} = \begin{pmatrix} \mathbf{A_{11}}^{(\mathbf{k-1})} & \mathbf{A_{12}}^{(\mathbf{k-1})} \\ \mathbf{0} & \mathbf{A_{22}}^{(\mathbf{k-1})} \end{pmatrix} \begin{matrix} (k-1) \\ (n-k+1) \end{matrix}$$
$$\qquad (k-1) \quad (n-k+1)$$

where: $\mathbf{A_{11}}^{(\mathbf{k-1})}$ is an upper triangular matrix.

Now, if $\mathbf{A_{22}}^{(\mathbf{k-1})} = \begin{pmatrix} a_{kk}^{(k-1)} & . & . & . & a_{kn}^{(k-1)} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ a_{nk}^{(k-1)} & . & . & . & a_{nn}^{(k-1)} \end{pmatrix}$

and $a_{kk}^{(k-1)} \neq 0$, then the *multipliers* :

$m_i = \dfrac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} ; i = k+1,\dots,n$; with $a_{kk} \neq 0$ are well defined.

So, we have the following

Definition. An elementary lower triangular matrix of order *n* and index *k* is a matrix of the form [28]

$$\mathbf{M_k} = \mathbf{I_n} - \mathbf{m}\mathbf{e_k^T}$$

Where:

$$\mathbf{e_k^T} = (0,\dots,0,1,0,\dots,0)^T \underset{k}{} , \quad \mathbf{I_n} = \begin{pmatrix} 1 & . & . & . & 0 \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ 0 & . & . & . & 1 \end{pmatrix}$$

$$\mathbf{m^T} = (\underbrace{0,0,0,\dots,0}_{k-times}, m_{k+1},\dots,m_n)$$

In general, an elementary lower triangular matrix takes the above form. The computational significance of elementary lower triangular matrices is that they can be used to introduce zero components into a vector. Thus,

$$\mathbf{M_k} \cdot \begin{pmatrix} a_{11} \\ . \\ . \\ . \\ a_{k1} \\ a_{k+1,1} \\ . \\ . \\ . \\ a_{n1} \end{pmatrix} = \begin{pmatrix} a_{11} \\ . \\ . \\ . \\ a_{k1} \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}.$$

The matrix $\mathbf{M_k}$ is said to be a Gauss Transformation**.** The vector **m** is referred to as the Gauss vector. The components of **m** are known as multipliers.

Then it follows that

$$\mathbf{A^{(k)}} \equiv \mathbf{M_k A^{(k-1)}} = \begin{pmatrix} \mathbf{A_{11}}^{(\mathbf{k})} & \mathbf{A_{12}}^{(\mathbf{k})} \\ \mathbf{0} & \mathbf{A_{22}}^{(\mathbf{k})} \end{pmatrix} \begin{matrix} (k) \\ (n-k) \end{matrix}$$
$$\qquad (k) \quad (n-k)$$

Where $\mathbf{A_{11}}^{(\mathbf{k})}$ is an upper triangular matrix.

This process illustrates the *k-th* step of the decomposition process, in which we used

$$(\mathbf{M_k} \cdots \mathbf{M_1})^{-1} = \mathbf{M_1}^{-1} \cdots \mathbf{M_k}^{-1} = \prod_{i=1}^{k} (\mathbf{I_n} + \mathbf{m^{(i)}} \mathbf{e_i}^{\mathbf{T}}) = \mathbf{I_n} + \sum_{i=1}^{k} \mathbf{m^{(i)}} \mathbf{e_i^T}$$

We find the final expression for the decomposition process as

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$$\mathbf{A}=\begin{pmatrix}\mathbf{L}_{11}^{(k)} & \mathbf{0}\\ \mathbf{L}_{21}^{(k)} & \mathbf{I}_{n-k}\end{pmatrix}\begin{pmatrix}\mathbf{I}_{k} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_{22}^{(k)}\end{pmatrix}\begin{pmatrix}\mathbf{A}_{11}^{(k)} & \mathbf{A}_{12}^{(k)}\\ \mathbf{0} & \mathbf{I}_{n-k}\end{pmatrix}\Rightarrow$$

$$(\mathbf{M}_k\ldots\mathbf{M}_1)^{-1}\equiv\begin{pmatrix}\mathbf{L}_{11}^{(k)} & \mathbf{0}\\ \mathbf{L}_{21}^{(k)} & \mathbf{I}_{n-k}\end{pmatrix}=\mathbf{I}_n+(\mathbf{m}^{(1)},\ldots,\mathbf{m}^{(k)},0,\ldots,0).$$

In general, the forward elimination consists of *n-1* steps. At the *k-th* step, multiples of the *k-th* equations are subtracted from the remaining equations so to eliminate the *k-th* variable. Using the above expression, the following can be established [27]:

Theorem. Let $\mathbf{A}_k$ denote the leader or main sub-matrix ($k\times k$) of $\mathbf{A}\in\mathbf{R}^{n\times n}$. If $\mathbf{A}_k$ is non-singular for *k=1,...,n* then, there exists a lower triangular matrix $\mathbf{L}\in\mathbf{R}^{n\times n}$ and an upper triangular matrix $\mathbf{U}\in\mathbf{R}^{n\times n}$ so that **A=LU.** Furthermore, $|\mathbf{A}_k|=u_{11}\ldots u_{kk}\,\forall k=1,\ldots,n.$

Previously we have defined an elementary lower triangular matrix of order *n* and index *k* as $\mathbf{M}_k=\mathbf{I}_n-\mathbf{m}\mathbf{e}_k^T$. We shall find the above expression and the next theorem**,** whose proof can be found in [20], useful.

Theorem. The pivot elements $a_{k,k}^{(k)}$ $(k=1,\ldots,n)$, are nonzero if and only if the leading principal sub matrices $\mathbf{A}_k$ $(k=1,\ldots,n)$, are non-singular. On this basis, the following can be stated

Corollary. Let $\mathbf{A}\in\mathbf{R}^{n\times n}$ be a matrix with $\mathbf{A}_k$ $(k=1,\ldots,n)$ non-singular leading principal sub-matrices. Then, there exists a unique diagonal matrix for *k=1* $(a_{00}^{(1)}=1)$ whose entries are:

$$\mathbf{D}_k=\begin{pmatrix}1_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & 1_k & & & \\ & & & & & \left(\frac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_{k+1} & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & & \left(\frac{a_{k,k}^{(k)}}{a_{k-1,k-1}^{(k)}}\right)_n\end{pmatrix}$$

Where $a_{k,k}^{(k)}$ and $a_{k-1,k-1}^{(k)}$ are the pivot elements.

Now, if we scale the matrix $\mathbf{M}_k$ with that diagonal matrix, we will have, for *k=1* $(a_{00}^{(1)}=1)$,

$$\mathbf{M}_k^{'}=\frac{1}{a_{k-1,k-1}^{(k)}}\cdot\begin{pmatrix}\left(a_{k-1,k-1}^{(k)}\right)_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & \left(a_{k-1,k-1}^{(k)}\right)_k & & & \\ & & & & -a_{k+1,k}^{(k)} & \left(a_{k,k}^{(k)}\right)_{k+1} & & \\ & & & & \cdot & & \cdot & \\ & & & & \cdot & & & \cdot \\ & & & & \cdot & & & & \cdot \\ & & & & -a_{n,k}^{(k)} & & \cdot\;\cdot\;\cdot & & \left(a_{k,k}^{(k)}\right)_n\end{pmatrix}\quad(2)$$

Previously, we have obtained [29] this new linear transformation.

By modifying the previous expression, we obtain:

$$\mathbf{M}_k^{''}=\frac{1}{a_{k-1,k-1}^{(k)}}\cdot\begin{pmatrix}(0)_1 & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & (0)_k & & & \\ & & & & -a_{k+1,k}^{(k)} & \left(a_{k,k}^{(k)}\right)_{k+1} & & \\ & & & & \cdot & & \cdot & \\ & & & & \cdot & & & \cdot \\ & & & & \cdot & & & & \cdot \\ & & & & -a_{n,k}^{(k)} & & \cdot\;\cdot\;\cdot & & \left(a_{k,k}^{(k)}\right)_n\end{pmatrix}\quad(3)$$

If we use this new modified transformation and apply it to the elimination process on matrix **A**, all intermediate results are integers, forming a number ring [30], since it is carried out by means of additions, subtractions and products. Thus,

$$\mathbf{A}\in\mathbf{I}^{n\times n}\Rightarrow(\mathbf{M}_n^{''}\mathbf{A}_n,\mathbf{M}_{n-1}^{''}\mathbf{A}_{n-1},\ldots,\mathbf{M}_2^{''}\mathbf{A}_2)\in\mathbf{I}^{n\times n}.$$

The multiplication of these results by the factor $\frac{1}{a_{k-1,k-1}^{(k)}}$, is an integer too because previously, in the $k-1$ step of the elimination process, these results have been multiplied by $a_{k-1,k-1}^{(k)}$.

Also, we can re-express the above linear transformation as [31], for *k=1* $(a_{00}^{(1)}=1)$:

IJEIT
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ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 6, Issue 5, November 2016

$$\mathbf{M}''_{\mathbf{k}} = \begin{pmatrix} 0_k & 0 \\ 0 & \frac{1}{a^{(k)}_{k-1k-1}}\mathbf{I}_{\mathbf{n-k}} \end{pmatrix} \left[a^{(k)}_{kk}\mathbf{I} - \begin{pmatrix} 0_1 \\ \cdot \\ \cdot \\ \cdot \\ 0_k \\ a^{(k)}_{k+1k} \\ \cdot \\ \cdot \\ \cdot \\ a^{(k)}_{nk} \end{pmatrix} \mathbf{e}^{\mathbf{T}}_{\mathbf{k}} \right]; \forall k = 1,\ldots, n-1 \quad (4)$$

If we apply this re-modified Gauss linear transformation to a matrix **A** and also to all the sub-matrices obtained from it by eliminating of the first row and the first column until obtaining a 2x2 sub-matrix; we arrive at several transformed sub-matrices. Next we compute the traces of these and, by adding them, we obtain a coefficient of the characteristic polynomial of the matrix **A** on the domain.

Thus, the matrices obtained in this way are partitioned again and applied to the new linear transformation once more so to obtain a new sub matrix series whose added traces constitute the subsequent value of the coefficient of the characteristic polynomial and so forth. In summary, we define the coefficients of the characteristic polynomial as the alternate signed sum of all traces of these sub-matrices.

Then, if the characteristic polynomial of a matrix **A** is

$$P_f = (-1)^n \left(x^n - c_{n-1}x^{n-1} + c_{n-2}x^{n-2} \ldots - c_0 \right) = 0$$

and if ${}^{n-1}_{p}\mathbf{A}^{m}_{n,n}$ are the sub matrices obtained from matrix **A** by elimination of the first row and the first column and *n-1* is the dimension of the matrix whence it comes, *p* is the dimension of the sub-matrix which originates it, *m* represents the times that the original matrix or sub matrix has been modified and $n \times n$ it indicates the dimension of the current matrix or sub matrix, and, finally, if $\mathbf{A}_{\mathbf{n,n}} = \mathbf{A}$, then we have (*see Appendix A and figure 1*)

$$\begin{aligned}
c_n &= (1) \\
c_{n-1} &= Tr\left(\mathbf{A}_{\mathbf{n,n}}\right) \\
c_{n-2} &= Tr\left({}^{\mathbf{n}}_{\mathbf{n}}\mathbf{A}'_{\mathbf{n-1,n-1}}\right) + Tr\left({}^{\mathbf{n-1}}_{\mathbf{n-1}}\mathbf{A}'_{\mathbf{n-2,n-2}}\right) + \ldots + Tr\left({}^{3}_{3}\mathbf{A}'_{\mathbf{2,2}}\right) + Tr\left({}^{2}_{2}\mathbf{A}'_{\mathbf{1,1}}\right) \\
c_{n-3} &= Tr\left({}^{\mathbf{n-1}}_{\mathbf{n}}\mathbf{A}''_{\mathbf{n-2,n-2}}\right) + Tr\left({}^{\mathbf{n-2}}_{\mathbf{n}}\mathbf{A}''_{\mathbf{n-3,n-3}}\right) + \ldots + Tr\left({}^{3}_{\mathbf{n}}\mathbf{A}''_{\mathbf{2,2}}\right) + Tr\left({}^{2}_{\mathbf{n}}\mathbf{A}''_{\mathbf{1,1}}\right) + \\
&\quad + Tr\left({}^{\mathbf{n-2}}_{\mathbf{n-1}}\mathbf{A}''_{\mathbf{n-3,n-3}}\right) + Tr\left({}^{\mathbf{n-3}}_{\mathbf{n-1}}\mathbf{A}''_{\mathbf{n-4,n-4}}\right) + \ldots + Tr\left({}^{3}_{\mathbf{n-1}}\mathbf{A}''_{\mathbf{2,2}}\right) + \\
&\quad + Tr\left({}^{2}_{\mathbf{n-1}}\mathbf{A}''_{\mathbf{1,1}}\right) + \ldots + Tr\left({}^{2}_{3}\mathbf{A}''_{\mathbf{1,1}}\right) \\
&\cdot \\
&\cdot \\
&\cdot \\
c_0 &= Tr\left({}^{2}_{\mathbf{n}}\mathbf{A}^{(\mathbf{n-1})}_{\mathbf{1,1}}\right)
\end{aligned}$$

In other words, we define the coefficients of the characteristic polynomial as the alternate signed sum of all partial traces of these sub matrices. This algorithm has a numeric exponential complexity (*see Appendix A figure 2*)

Furthermore, if $\mathbf{M}''_{\mathbf{k}}$ is being applied to the matrix **A** or submatrix ${}^{n-1}_{p}\mathbf{A}^{m}_{n,n}$, the solution process is easier than by the others methods, because we use a number ring and all the elements of the diagonal matrix are integer numbers. It should be noted that, by each modification, the matrix or submatrix reduces **A** to a *n-1*×*n-1* dimension. The traces of this matrices are integer numbers too. By a numeric example for the case *n=5*, see *Appendix B*. By a numeric example for the case *n=8*, see [32].

It should be observed that the proposed algorithm may be limited if the pivot element is null. In this case, we obtain an iso-volumetric matrix in such a way that we can to compute the determinant of matrix **A** with no null pivots, see **Appendix C.**

Finally, the proposed algorithm is far better adapted to modular calculations. This can be achieved by applying the Chinese Remainder Theorem [33,34]. So, with the above new method we can calculate the coefficients when **A** has very large integer entries.

VIII. RESULTS: MEXICAN ECONOMY MATRIX

From Domestic symmetric matrix **A** of technical coefficients 2012, see *Appendix D(19x19)*, we obtain $(I - \mathbf{A})^{-1}$, see *Appendix E(19x19)* and next we calculate the characteristic polynomial applying the Mexican algorithm which it proposed in this paper and we get all principal minors of the inverse Leontief matrix, see *Appendix F*. **We note that all its *leading* principal minors are positive.**

IX. CONCLUSIONS

In this work we have proposed two new types of matrices, namely, a partitioned combinatory symmetric square matrix and an iso-volumetric matrix. Using the first one, we are enabled to express the coefficients of the characteristic polynomial in a determinant form. The second one is used to eliminate the null pivots. To this purpose we have made use of the new diagonal permutation.

We have also applied a new linear transformation obtained from previous work and expressed it as equation (2). Then, it was modified and re-expressed as equations (3) and (4) , in such a way that, when applied to the original matrix and sub-matrices obtained from the original one until reaching an elementary matrix 2×2, new sub-matrices are achieved. The traces of these modified sub-matrices are obtained and added. This is as if all the 2×2 leader determinants had been calculated and they add up to the c_{n-2} coefficient of the characteristic polynomial in expression (1). Hence, an efficient method has been achieved in order to calculate all the remaining determinants corresponding to each characteristic polynomial coefficient of the original matrix. Finally, the weak hawkins-simon condition was verified by a Mexican's algorithm through multi-core cpu-gpu equipment.

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International Journal of Engineering and Innovative Technology (IJEIT)
Volume 6, Issue 5, November 2016

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Volume 6, Issue 5, November 2016

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International Journal of Engineering and Innovative Technology (IJEIT)
Volume 6, Issue 5, November 2016

APPENDIX A.

A FRAMEWORK OF THE SUCCESSIVE TRANSFORMATIONS OF THE REDUCTION STEP IN EACH OF THE SUBMATRICES BY APPLYING SUCH TRANSFORMATIONS TO $n \times n$ MATRIX

$(-1)^n$	A	$\mathbf{M}''$	$\mathbf{M}''\mathbf{A} = \mathbf{A}'$	P	$\mathbf{M}''$	A′′	P	…	
	$\mathbf{A}_{n,n}$	$\mathbf{M}''_n$	$\mathbf{M}''_n \mathbf{A}_{n,n} = {}^{n}_{n}\mathbf{A}'_{n-1,n-1}$	${}^{n}_{n}\mathbf{A}'_{n-1,n-1}$	${}^{n}\mathbf{M}''_{n-1}$	${}^{n}\mathbf{M}_{n-1}\,{}^{n}_{n}\mathbf{A}'_{n-1,n-1} = {}^{n-1}_{n}\mathbf{A}''_{n-2,n-2}$	${}^{n-1}_{n}\mathbf{A}''_{n-2,n-2}$	…	${}^{n}\mathbf{M}_{2}\,{}^{3}_{n}\mathbf{A}^{(n-2)}_{2,2} = {}^{2}_{n}$ [cut off]
							${}^{n-2}_{n}\mathbf{A}''_{n-3,n-3}$	…	
							⋮		
							${}^{4}_{n}\mathbf{A}''_{3,3}$	…	
							${}^{3}_{n}\mathbf{A}''_{2,2}$	…	
				${}^{n-1}_{n}\mathbf{A}'_{n-2,n-2}$	${}^{n}\mathbf{M}''_{n-2}$	${}^{n}\mathbf{M}''_{n-2}\,{}^{n-1}_{n}\mathbf{A}'_{n-2,n-2} = {}^{n-2}_{n}\mathbf{A}''_{n-3,n-3}$	${}^{n-2}_{n}\mathbf{A}''_{n-3,n-3}$	…	
				⋮			⋮		
				${}^{4}_{n}\mathbf{A}'_{3,3}$	${}^{n}\mathbf{M}''_{3}$	${}^{n}\mathbf{M}''_{3}\,{}^{4}_{n}\mathbf{A}'_{3,3} = {}^{3}_{n}\mathbf{A}''_{2,2}$			
				${}^{3}_{n}\mathbf{A}'_{2,2}$	${}^{n}\mathbf{M}''_{2}$	${}^{n}\mathbf{M}''_{2}\,{}^{3}_{n}\mathbf{A}'_{2,2} = {}^{2}_{n}\mathbf{A}''_{1,1}$			
	$\mathbf{A}_{n-1,n-1}$	$\mathbf{M}''_{n-1}$	$\mathbf{M}''_{n-1}\mathbf{A}_{n-1,n-1} = {}^{n-1}_{n-1}\mathbf{A}'_{n-2,n-2}$	${}^{n-1}_{n-1}\mathbf{A}'_{n-2,n-2}$	${}^{n-1}\mathbf{M}''_{n-2}$	${}^{n-1}\mathbf{M}''_{n-2}\,{}^{n-1}_{n-1}\mathbf{A}'_{n-2,n-2} = {}^{n-2}_{n-1}\mathbf{A}''_{n-3,n-3}$			
				${}^{n-2}_{n-1}\mathbf{A}'_{n-3,n-3}$	${}^{n-1}\mathbf{M}''_{n-3}$	${}^{n-1}\mathbf{M}''_{n-3}\,{}^{n-2}_{n-1}\mathbf{A}'_{n-3,n-3} = {}^{n-3}_{n-1}\mathbf{A}''_{n-4,n-4}$			
				⋮			⋮		
				${}^{4}_{n-1}\mathbf{A}'_{3,3}$	${}^{n-1}\mathbf{M}''_{3}$	${}^{n-1}\mathbf{M}''_{3}\,{}^{4}_{n-1}\mathbf{A}'_{3,3} = {}^{3}_{n-1}\mathbf{A}''_{2,2}$	${}^{3}_{n-1}\mathbf{A}''_{2,2}$	…	
				${}^{3}_{n-1}\mathbf{A}'_{2,2}$	${}^{n-1}\mathbf{M}''_{2}$	${}^{n-1}\mathbf{M}''_{2}\,{}^{3}_{n-1}\mathbf{A}'_{2,2} = {}^{2}_{n-1}\mathbf{A}''_{1,1}$			
	$\mathbf{A}_{3,3}$	$\mathbf{M}''_3$	$\mathbf{M}''_3\mathbf{A}_{3,3} = {}^{3}_{3}\mathbf{A}'_{2,2}$	${}^{3}_{3}\mathbf{A}'_{2,2}$	${}^{3}\mathbf{M}''_{2}$	${}^{3}\mathbf{M}''_{2}\,{}^{3}_{3}\mathbf{A}'_{2,2} = {}^{2}_{3}\mathbf{A}''_{1,1}$			
	$\mathbf{A}_{2,2}$	$\mathbf{M}''_2$	$\mathbf{M}''_2\mathbf{A}_{2,2} = {}^{2}_{2}\mathbf{A}'_{1,1}$						
$c_n = (-1)^n$	$c_{n-1} = Tr(\mathbf{A}_{n,n})$		$c_{n-2} = Tr({}^{n}_{n}\mathbf{A}'_{n-1,n-1}) + Tr({}^{n-1}_{n-1}\mathbf{A}'_{n-2,n-2}) + \dots + Tr({}^{3}_{3}\mathbf{A}'_{2,2}) + Tr({}^{2}_{2}\mathbf{A}'_{1,1})$			$c_{n-3} = Tr({}^{n-1}_{n}\mathbf{A}''_{n-2,n-2}) + Tr({}^{n-2}_{n}\mathbf{A}''_{n-3,n-3}) + \dots + Tr({}^{3}_{n}\mathbf{A}''_{2,2}) + Tr({}^{2}_{n}\mathbf{A}''_{1,1}) + Tr({}^{n-2}_{n-1}\mathbf{A}''_{n-3,n-3}) + Tr({}^{n-3}_{n-1}\mathbf{A}''_{n-4,n-4}) + \dots + Tr({}^{3}_{n-1}\mathbf{A}''_{2,2}) + Tr({}^{2}_{n-1}\mathbf{A}''_{1,1}) + \dots + Tr({}^{2}_{3}\mathbf{A}''_{1,1})$		…	$c_0 = Tr({}^{2}_{n}\mathbf{A}^{(n-}_{1,1}$ [cut off]

Fig 1. The Tree Structure of the Algorithm for an input matrix of rank *n=5*. The decomposition process which is applied to each new obtained sub-matrix, is composed by the partition process *S* (left edges) and by the transformation process *M* (right edges). The original matrix is at the top of the tree.

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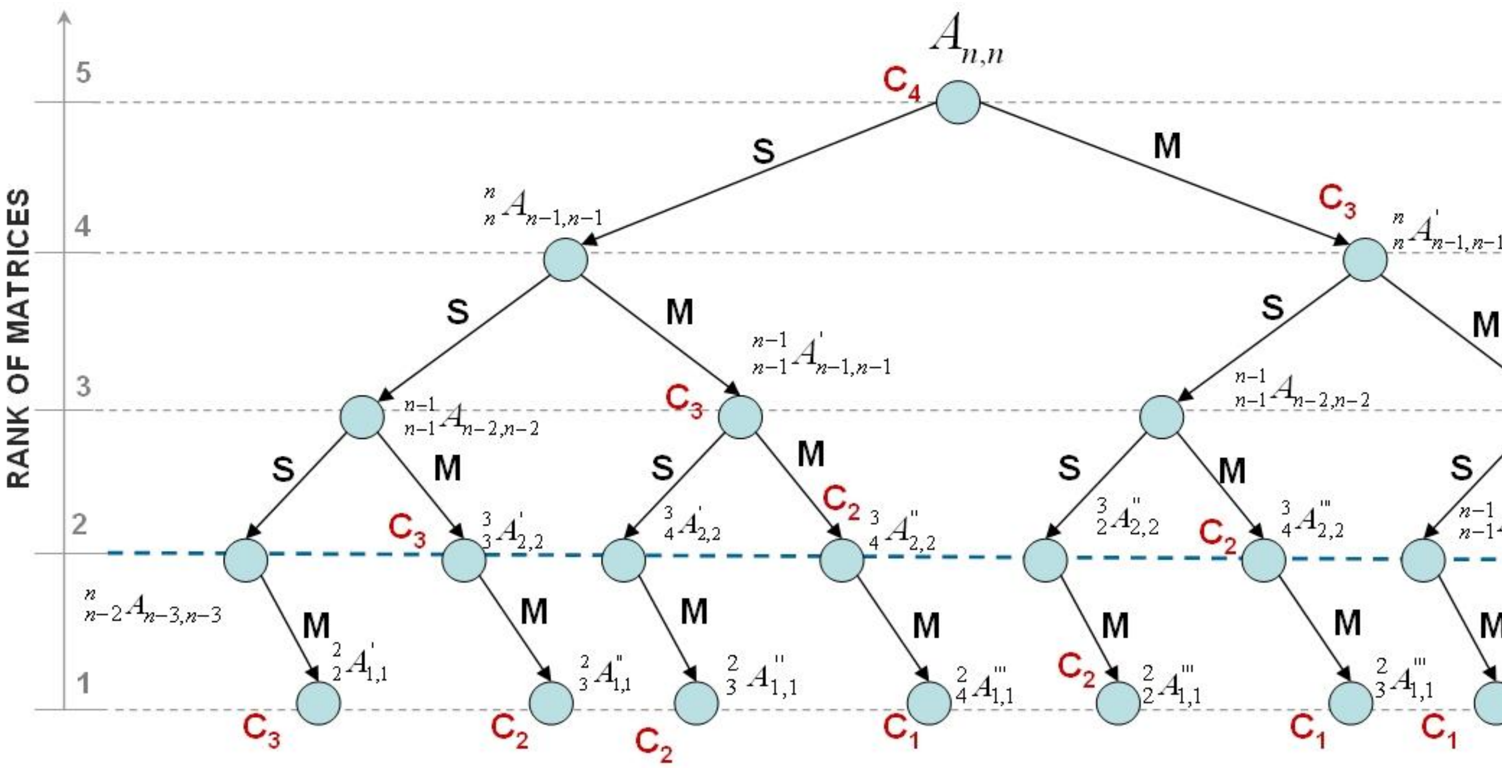
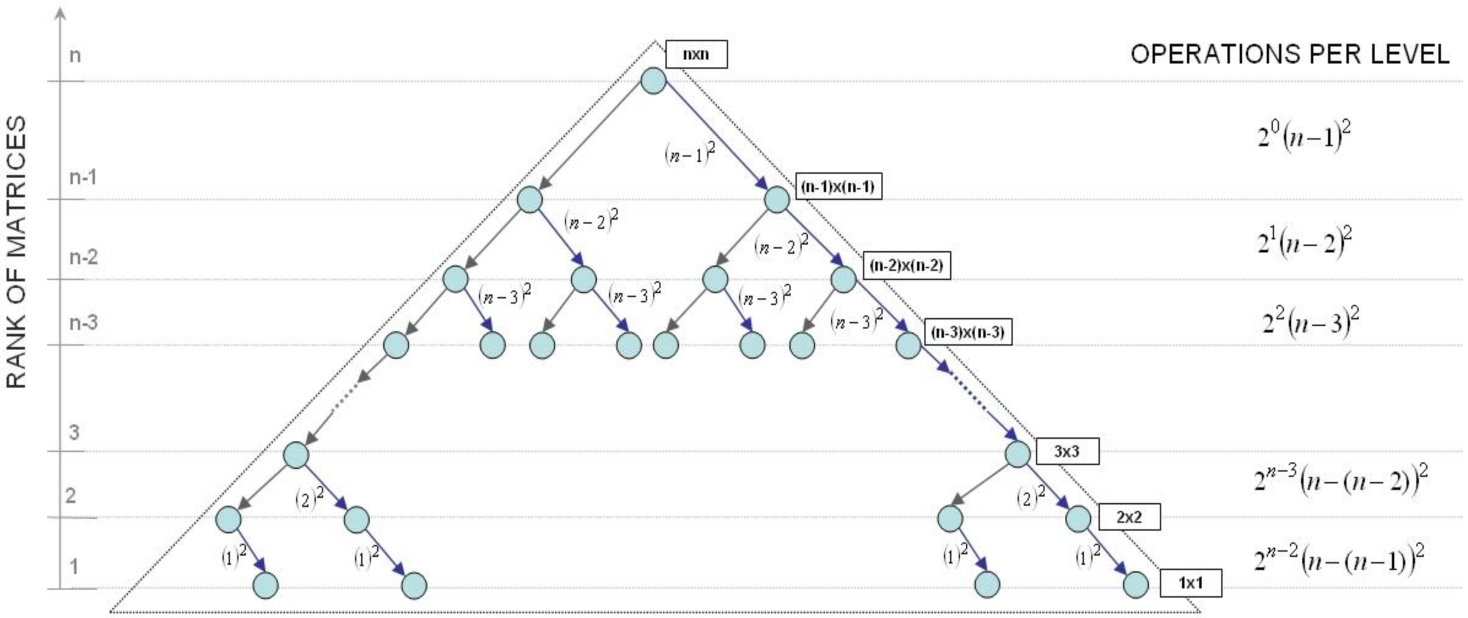
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 6, Issue 5, November 2016

(a)





Therefore, the total number of required operations for the algorithm is defined by the following series where n is the dimension of the original matrix. The former expression could be rewritten as:

$$\sum_{k=1}^{n-1} 2^{k-1}(n-k)^2$$

which is reduced by induction to

$$3 \cdot 2^n - (n^2 + 2n + 3)$$

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ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 6, Issue 5, November 2016

which is the total number of field operations required when the original matrix has a dimension of n. Hence, the complexity of the algorithm is governed by the term $3 \cdot 2^n$, finally $O(2^n)$.

APPENDIX B. AN EXAMPLE OF THE NEW METHOD. Below, we show the successive transformations of the reduction step in each of the sub-matrices by applying the new transformations to the same matrix used in appendix A of Abdeljaoued's paper [18]

$-1)^5$	A	M´´	M´´A=A´	P	M´´	A´´	P	M´´	A´´´	M´´	A´´´´
	$\begin{pmatrix} (-3) & 5 & -4 & 2 & 1 \\ 2 & (-1) & 3 & 0 & 2 \\ 5 & 3 & (1) & -3 & 0 \\ 1 & 2 & 4 & (-1) & -5 \\ 2 & 1 & -3 & 0 & (2) \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 & 0 \\ -5 & 0 & -3 & 0 & 0 \\ -1 & 0 & 0 & -3 & 0 \\ -2 & 0 & 0 & 0 & -3 \end{pmatrix}$	$\begin{pmatrix} (-7) & -1 & -4 & -8 \\ -34 & (17) & -1 & -5 \\ -11 & -8 & (11) & 14 \\ -13 & 17 & -4 & (-8) \end{pmatrix}$	$\begin{pmatrix} (-7) & -1 & -4 & -8 \\ -34 & (17) & -1 & -5 \\ -11 & -8 & (8) & 14 \\ -13 & 17 & -4 & (-8) \end{pmatrix}$	$\frac{1}{-3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 34 & -7 & 0 & 0 \\ 11 & 0 & -7 & 0 \\ 13 & 0 & 0 & -7 \end{pmatrix}$	$\begin{pmatrix} (51) & 43 & 79 \\ -15 & (17) & 62 \\ 44 & 8 & (16) \end{pmatrix}$	$\begin{pmatrix} (51) & 43 & 79 \\ -15 & (17) & 62 \\ 44 & 8 & (16) \end{pmatrix}$	$\frac{1}{-7}\begin{pmatrix} 0 & 0 & 0 \\ 15 & 51 & 0 \\ -44 & 0 & 51 \end{pmatrix}$	$\begin{pmatrix} (-216) & -621 \\ 212 & (380) \end{pmatrix}$	$\frac{1}{51}\begin{pmatrix} 0 & 0 \\ -212 & -216 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (972) \end{pmatrix}$
							$\begin{pmatrix} (17) & 62 \\ 8 & (16) \end{pmatrix}$	$\frac{1}{-7}\begin{pmatrix} 0 & 0 \\ -8 & 17 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (32) \end{pmatrix}$		
				$\begin{pmatrix} (17) & -1 & -5 \\ -8 & (1) & 14 \\ 17 & -4 & (-8) \end{pmatrix}$	$\frac{1}{-3}\begin{pmatrix} 0 & 0 & 0 \\ 8 & 17 & 0 \\ -17 & 0 & 17 \end{pmatrix}$	$\begin{pmatrix} (-3) & -66 \\ 17 & (17) \end{pmatrix}$	$\begin{pmatrix} (-3) & -66 \\ 17 & (17) \end{pmatrix}$	$\frac{1}{17}\begin{pmatrix} 0 & 0 \\ -17 & -3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (63) \end{pmatrix}$		
				$\begin{pmatrix} (1) & 14 \\ -4 & (-8) \end{pmatrix}$	$\frac{1}{-3}\begin{pmatrix} 0 & 0 \\ 4 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (-16) \end{pmatrix}$					
	$\begin{pmatrix} -1 & 3 & 0 & 2 \\ 3 & 1 & -3 & 0 \\ 2 & 4 & -1 & -5 \\ 1 & -3 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} (-10) & 3 & -6 \\ -10 & (1) & 1 \\ 0 & 0 & (-4) \end{pmatrix}$	$\begin{pmatrix} (-10) & 3 & -6 \\ -10 & (1) & 1 \\ 0 & 0 & (-4) \end{pmatrix}$	$\frac{1}{-1}\begin{pmatrix} 0 & 0 & 0 \\ 10 & -10 & 0 \\ 0 & 0 & -10 \end{pmatrix}$	$\begin{pmatrix} (-20) & 70 \\ 0 & (-40) \end{pmatrix}$	$\begin{pmatrix} (-20) & 70 \\ 0 & (-40) \end{pmatrix}$	$\frac{1}{-10}\begin{pmatrix} 0 & 0 \\ 0 & -20 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (-80) \end{pmatrix}$		
				$\begin{pmatrix} (1) & 1 \\ 0 & (-4) \end{pmatrix}$	$\frac{1}{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (4) \end{pmatrix}$					
	$\begin{pmatrix} 1 & -3 & 0 \\ 4 & -1 & -5 \\ -3 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} (11) & -5 \\ -9 & (2) \end{pmatrix}$	$\begin{pmatrix} (11) & -5 \\ -9 & (2) \end{pmatrix}$	$\frac{1}{1}\begin{pmatrix} 0 & 0 \\ 9 & 11 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (-23) \end{pmatrix}$					
	$\begin{pmatrix} -1 & -5 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (-2) \end{pmatrix}$								
$= -1$	$c_4 = -3 - 1 + 1 - -1 + 2 = -2$		$c_3 = -7 + 17 + 11 - 8 - -10 + 1 - 4 + 11 + +2 - 2 = 1$			$c_2 = 51 + 17 + 16 - -3 + 17 - 16 - -20 - 40 + 4 - -23 = 3$			$c_1 = -216 + 380 + +32 + 63 - 80 = =179$		$c_0 = 972$

As we have the first negative coefficient, the following ones are simply alternated whereby

$$(-1)x^5 + (-2)x^4 - (1)x^3 + (3)x^2 - (179)x + 972 = 0$$

Finally $x^5 + 2x^4 + x^3 - 3x^2 + 179x - 972 = 0$

APPENDIX C. AN EXAMPLE OF THE NEW METHOD WITH NULL PIVOT. Below, we show the successive transformations of the reduction step in each of the sub-matrices by applying the new transformation to an arbitrary matrix.

$(-1)^5$	A	M´´	M´´A=A´	P	M´´	A´´	P	M´´
	$\begin{pmatrix} (1) & 2 & 1 & -1 & 2 \\ 2 & (2) & 2 & -1 & -1 \\ -1 & -1 & (-1) & 2 & -2 \\ 1 & -1 & -1 & (-1) & 2 \\ 1 & 1 & -1 & -1 & (-1) \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} (-2) & 0 & 1 & -5 \\ 1 & (0) & 1 & 0 \\ -3 & -2 & (0) & 0 \\ -1 & -2 & 0 & (-3) \end{pmatrix}$	$\begin{pmatrix} (-2) & 0 & 1 & -5 \\ 1 & (0) & 1 & 0 \\ -3 & -2 & (0) & 0 \\ -1 & -2 & 0 & (-3) \end{pmatrix}$	$\frac{1}{1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 3 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$	$\begin{pmatrix} (0) & -3 & 5 \\ 4 & (3) & -15 \\ 4 & 1 & (1) \end{pmatrix}$	By permuting the diagonal elements, we have:	
	$\begin{pmatrix} (-1) & 2 & -2 & -1 & -1 \\ -1 & (-1) & 2 & -1 & 1 \\ -1 & -1 & (-1) & 1 & 1 \\ 2 & -1 & -1 & (2) & 2 \\ 1 & -1 & 2 & 2 & (1) \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ -2 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} (3) & -4 & 0 & -2 \\ 3 & (-1) & -2 & -2 \\ -3 & 5 & (0) & 0 \\ -1 & 0 & -1 & (0) \end{pmatrix}$	$\begin{pmatrix} (3) & -4 & 0 & -2 \\ 3 & (-1) & -2 & -2 \\ -3 & 5 & (0) & 0 \\ -1 & 0 & -1 & (0) \end{pmatrix}$	$\frac{1}{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$	$\begin{pmatrix} (-9) & 6 & 0 \\ -3 & (0) & 6 \\ 4 & 3 & (2) \end{pmatrix}$	$\begin{pmatrix} (-9) & 6 & 0 \\ -3 & (0) & 6 \\ 4 & 3 & (2) \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ 3 & -9 & 0 \\ -4 & 0 & -9 \end{pmatrix}$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 6, Issue 5, November 2016

							$\begin{pmatrix} (0) & 6 \\ 3 & (2) \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 0 & 0 \\ -3 & 0 \end{pmatrix}$
				$\begin{pmatrix} (-1) & -2 & -2 \\ 5 & (0) & 0 \\ 0 & -1 & (0) \end{pmatrix}$	$\frac{1}{-1}\begin{pmatrix} 0 & 0 & 0 \\ -5 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} (-10) & -10 \\ -1 & (0) \end{pmatrix}$	$\begin{pmatrix} (-10) & -10 \\ -1 & (0) \end{pmatrix}$	$\frac{1}{-1}\begin{pmatrix} 0 & 0 \\ 1 & -10 \end{pmatrix}$
				$\begin{pmatrix} 0 & 0 \\ -1 & (0) \end{pmatrix}$	$\frac{1}{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (0) \end{pmatrix}$		
	$\begin{pmatrix} -1 & 2 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 2 & 2 \\ -1 & 2 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} (3) & -2 & 0 \\ 3 & (-3) & -1 \\ 0 & -3 & (0) \end{pmatrix}$	$\begin{pmatrix} (3) & -2 & 0 \\ 3 & (-3) & -1 \\ 0 & -3 & (0) \end{pmatrix}$	$\frac{1}{-1}\begin{pmatrix} 0 & 0 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	$\begin{pmatrix} (3) & 3 \\ 9 & (0) \end{pmatrix}$	$\begin{pmatrix} (3) & 3 \\ 9 & (0) \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 0 & 0 \\ -9 & 3 \end{pmatrix}$
				$\begin{pmatrix} (-3) & -1 \\ -3 & (0) \end{pmatrix}$	$\frac{1}{-1}\begin{pmatrix} 0 & 0 \\ 3 & -3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (3) \end{pmatrix}$		
	$\begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} (-1) & -1 \\ -4 & (-3) \end{pmatrix}$	$\begin{pmatrix} (-1) & -1 \\ -4 & (-3) \end{pmatrix}$	$\frac{1}{-1}\begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (1) \end{pmatrix}$		
	$\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & (-2) \end{pmatrix}$					
$c_5 = -1$	$c_4 = -1-1-1+ +2+1=0$		$c_3 = 3-1+0+0+ +3-3+0-1- -3-2=-4$			$c_2 = -9+0+2- -10+0+0+ +3+0+3+ +1=-10$		

As we have the first negative coefficient, the following ones are simply alternated whereby

$$(-1)x^5 + (0)x^4 - (-4)x^3 + (-10)x^2 - (-5)x + 38 = 0$$

Finally

$$x^5 - 4x^3 + 10x^2 - 5x - 38 = 0$$

APPENDIX D. DOMESTIC SYMMETRIC MATRIX A OF TECHNICAL COEFFICIENTS 2012 (MEXICAN ECONOMY MATRIX-INEGI)*

SECTOR	Agricultura, cría y explotación de animales, aprovechamiento forestal, pesca y caza	Minería	…	Otros servicios excepto actividades gubernamentales	A gu ju in
Agricultura, cría y explotación de animales, aprovechamiento forestal, pesca y caza	0.08847293	0.00000000	…	0.00001388	
Minería	0.00105086	0.02084388	…	0.00000000	
Generación, transmisión y distribución de energía eléctrica, suministro de agua y de gas por ductos al consumidor final	0.01483633	0.00637391	…	0.01951979	
Construcción	0.00000260	0.00381412	…	0.00122540	
Industrias Manufactureras	0.14414032	0.04927820	…	0.06344915	
Comercio	0.03712817	0.01474015	…	0.02049364	
Transportes, correos y almacenamiento	0.00773213	0.00385789	…	0.00685727	
Información en medios masivos	0.00016660	0.00093975	…	0.00803073	
Servicios financieros y de seguros	0.00525449	0.00547710	…	0.00150495	
Servicios inmobiliarios y de alquiler de bienes muebles e intangibles	0.00363795	0.00488991	…	0.04217268	
Servicios profesionales, científicos y técnicos	0.00115497	0.01110308	…	0.01860827	
Corporativos	0.00007714	0.00864235	…	0.00163119	
Servicios de apoyo a los negocios y manejo de desechos y servicios de remediación	0.00095955	0.00775232	…	0.02633665	
Servicios educativos	0.00000176	0.00000021	…	0.00000173	
Servicios de salud y de asistencia social	0.00000000	0.00000000	…	0.00000000	
Servicios de esparcimiento culturales y deportivos, y otros servicios recreativos	0.00000000	0.00000000	…	0.00000000	
Servicios de alojamiento temporal y de preparación de alimentos y bebidas	0.00028229	0.00139655	…	0.00254327	
Otros servicios excepto actividades gubernamentales	0.00093491	0.00091919	…	0.01284199	
Actividades legislativas, gubernamentales, de	0.00000000	0.00000000	…	0.00000000	

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 6, Issue 5, November 2016

impartición de justicia y de organismos internacionales y extraterritoriales					

- http://www3.inegi.org.mx/sistemas/tabuladosbasicos/tabniveles.aspx?c=33683

APPENDIX E. INVERSE LEOTIENF MATRIX $(I - \mathbf{A})^{-1}$, DOMESTIC SYMMETRIC MATRIX OF TECHNICAL COEFFICIENTS 2012 (MEXICAN ECONOMY MATRIX-INEGI)

SECTOR	Agricultura, cría y explotación de animales, aprovechamiento forestal, pesca y caza	Minería	…	Otros servicios excepto actividades gubernamentales	Activida gubernar justicia y internaci
Agricultura, cría y explotación de animales, aprovechamiento forestal, pesca y caza	1.106505353	0.003168391	…	0.00433753	
Minería	0.018134656	1.027038442	…	0.007958122	
Generación, transmisión y distribución de energía eléctrica, suministro de agua y de gas por ductos al consumidor final	0.020107983	0.008290962	…	0.022898589	
Construcción	0.000366222	0.004336243	…	0.001586772	
Industrias Manufactureras	0.199960566	0.067052306	…	0.091478241	
Comercio	0.055403467	0.020645465	…	0.029388824	
Transportes, correos y almacenamiento	0.013236828	0.005957503	…	0.010111178	
Información en medios masivos	0.001739706	0.002276633	…	0.010024069	
Servicios financieros y de seguros	0.008109791	0.007353906	…	0.003860876	
Servicios inmobiliarios y de alquiler de bienes muebles e intangibles	0.007561392	0.007114868	…	0.04624308	
Servicios profesionales, científicos y técnicos	0.00536251	0.014688379	…	0.024021114	
Corporativos	0.001395982	0.010128893	…	0.002767666	
Servicios de apoyo a los negocios y manejo de desechos y servicios de remediación	0.008658906	0.012230142	…	0.033288381	
Servicios educativos	3.58104E-05	2.90444E-05	…	4.45982E-05	
Servicios de salud y de asistencia social	0	0	…	0	
Servicios de esparcimiento culturales y deportivos, y otros servicios recreativos	2.75328E-06	3.35017E-06	…	1.3556E-05	
Servicios de alojamiento temporal y de preparación de alimentos y bebidas	0.000897494	0.001793443	…	0.003190694	
Otros servicios excepto actividades gubernamentales	0.001842812	0.001417089	…	1.013722436	
Actividades legislativas, gubernamentales, de impartición de justicia y de organismos internacionales y extraterritoriales	5.98634E-06	2.2328E-06	…	3.19163E-06	

APPENDIX F. CHARACTERISTIC POLYNOMIAL OF THE INVERSE LEOTIENF MATRIX $(I - \mathbf{A})^{-1}$, THE ROOTS OF THE CHARACTERISTIC POLYNOMIAL ARE THE EIGENVALUE OF THE DOMESTIC SYMMETRIC MATRIX OF TECHNICAL COEFFICIENTS 2012 (MEXICAN ECONOMY MATRIX-INEGI). ALL PRINCIPAL MINORS ARE POSITIVE AND AT LEAST ONE ROOT IT IS REAL.

ALL PRINCIPAL MINORS (1)	VALUES	ALL PRINCIPAL MINORS (2)	VALUES	ALL PRINCIPAL MINORS (3)	VALUES	ALL PRINCIPAL MINORS (4)
m[0] =	1.11	m[29] =	1.19	m[36] =	1.11	m[65] =
m[1] =	1.03	m[30] =	1.15	m[37] =	1.15	m[66] =
m[2] =	1.01	m[31] =	1.11	m[38] =	1.22	m[67] =
m[3] =	1.08	m[32] =	1.11	m[39] =	1.34	m[68] =
m[4] =	1.19	m[33] =	1.11	m[40] =	1.16	m[69] =

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 6, Issue 5, November 2016

m[5] =	1.02	m[34] =	1.11	m[41] =	1.17	m[70] =
m[6] =	1.03	m[35] =	1.12	m[42] =	1.19	m[71] =
m[7] =	1.05	m[36] =	1.11	m[43] =	1.26	m[72] =
m[8] =	1.11	m[37] =	1.15	m[44] =	1.15	m[73] =
m[9] =	1.02	m[38] =	1.22	m[45] =	1.22	m[74] =
m[10] =	1.07	m[39] =	1.34	m[46] =	1.22	m[75] =
m[11] =	1.07	m[40] =	1.16	m[47] =	1.18	m[76] =
m[12] =	1.04	m[41] =	1.17	m[48] =	1.14	m[77] =
m[13] =	1.00	m[42] =	1.19	m[49] =	1.14	m[78] =
m[14] =	1.00	m[43] =	1.26	m[50] =	1.14	m[79] =
m[15] =	1.00	m[44] =	1.15	m[51] =	1.14	m[80] =
m[16] =	1.00	m[45] =	1.22	m[52] =	1.15	m[81] =
m[17] =	1.01	m[46] =	1.22	m[53] =	1.14	m[82] =
m[18] =	1.00	m[47] =	1.18	m[54] =	1.24	m[83] =
m[19] =	1.14	m[48] =	1.14	m[55] =	1.35	m[84] =
m[20] =	1.12	m[49] =	1.14	m[56] =	1.17	m[85] =
m[21] =	1.19	m[50] =	1.14	m[57] =	1.19	m[86] =
m[22] =	1.31	m[29] =	1.19	m[58] =	1.21	m[87] =
m[23] =	1.13	m[30] =	1.15	m[59] =	1.27	m[88] =
m[24] =	1.14	m[31] =	1.11	m[60] =	1.17	m[89] =
m[25] =	1.16	m[32] =	1.11	m[61] =	1.23	m[90] =
m[26] =	1.22	m[33] =	1.11	m[62] =	1.23	m[91] =
m[27] =	1.12	m[34] =	1.11	m[63] =	1.19	m[92] =
m[28] =	1.19	m[35] =	1.12	m[64] =	1.15	m[93] =
ALL PRINCIPAL MINORS (18076)	**VALUES**	**ALL PRINCIPAL MINORS (18077)**	**VALUES**	**ALL PRINCIPAL MINORS (18078)**	**VALUES**	**ALL PRINCIPAL MINORS (18079)**
m[524186] =	1.04	m[524215] =	1.05	m[524244] =	1.02	m[524273] =
m[524187] =	1.05	m[524216] =	1.04	m[524245] =	1.01	m[524274] =
m[524188] =	1.05	m[524217] =	1.05	m[524246] =	1.02	m[524275] =
m[524189] =	1.05	m[524218] =	1.05	m[524247] =	1.00	m[524276] =
m[524190] =	1.04	m[524219] =	1.04	m[524248] =	1.02	m[524277] =
m[524191] =	1.05	m[524220] =	1.05	m[524249] =	1.01	m[524278] =
m[524192] =	1.05	m[524221] =	1.04	m[524250] =	1.00	m[524279] =
m[524193] =	1.04	m[524222] =	1.05	m[524251] =	1.01	m[524280] =
m[524194] =	1.05	m[524223] =	1.04	m[524252] =	1.00	m[524281] =
m[524195] =	1.04	m[524224] =	1.05	m[524253] =	1.02	m[524282] =
m[524196] =	1.05	m[524225] =	1.05	m[524254] =	1.00	m[524283] =
m[524197] =	1.04	m[524226] =	1.05	m[524255] =	1.02	m[524284] =
m[524198] =	1.05	m[524227] =	1.04	m[524256] =	1.01	m[524285] =

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 6, Issue 5, November 2016

m[524199] =	1.05	m[524228] =	1.05	m[524257] =	1.02	m[524286] =	
m[524200] =	1.05	m[524229] =	1.05	m[524258] =	1.00	m[524273] =	
m[524201] =	1.04	m[524230] =	1.00	m[524259] =	1.02	m[524274] =	
m[524202] =	1.05	m[524231] =	1.00	m[524260] =	1.01	m[524275] =	
m[524203] =	1.05	m[524232] =	1.00	m[524261] =	1.00	m[524276] =	
m[524204] =	1.04	m[524233] =	1.01	m[524262] =	1.00	m[524277] =	
m[524205] =	1.04	m[524234] =	1.00	m[524263] =	1.01	m[524278] =	
m[524206] =	1.05	m[524235] =	1.00	m[524264] =	1.00	m[524279] =	
m[524207] =	1.04	m[524236] =	1.00	m[524265] =	1.00	m[524280] =	
m[524208] =	1.04	m[524237] =	1.01	m[524266] =	1.01	m[524281] =	
m[524209] =	1.05	m[524238] =	1.00	m[524267] =	1.00	m[524282] =	
m[524210] =	1.04	m[524239] =	1.00	m[524268] =	1.02	m[524283] =	
m[524211] =	1.05	m[524240] =	1.01	m[524269] =	1.00	m[524284] =	
m[524212] =	1.04	m[524241] =	1.00	m[524270] =	1.02	m[524285] =	
m[524213] =	1.05	m[524242] =	1.02	m[524271] =	1.01	m[524286] =	
m[524214] =	1.05	m[524243] =	1.00	m[524272] =	1.02		

$$x^{19} - 19.83x^{18} + 186.23x^{17} - 1100.64x^{16} + 4589.73x^{15} - 14348.91x^{14} + 34877.1x^{13} - 67447.89x^{12} + 105313.62x^{11} - 133938.64x^{10} + 139324.4x^{9} -$$

$$+ 82147.28x^{7} - 45966.45x^{6} + 20465.44x5 - 7084.74x^{4} + 1838.9x^{3} - 336.82x^{2} + 38.83x - 2.12 = 0$$