Thermoelastic Problem of a Thin Rectangular Plate with Moving Heat Source

Rajesh B. Pardhi, M. S. Warbhe and N. W. Khobragade
Department of Mathematics, Dr. Babasaheb College of Engineering & Research, Nagpur, India.
Sarvodaya Mahavidyalaya Sindewahi, Dist: Chandrapur, India.
Post graduate teaching department of Mathematics, RTM Nagpur University, Nagpur 440 033, India.

ABSTRACT- This paper is concern with the thermoelastic solution of a thin rectangular plate in which an attempt has been made to determine the temperature distribution and thermal stresses with the help of integral transform technique. The results are obtained in term of Bessel’s function in the form of infinite series.

KEY WORDS: Moving heat source, Marchi-Fasulo transform, Fourier Cosine Transform.

I. INTRODUCTION

Several researchers have considered the heat conduction equation with internal heat generation and find the solution by various methods. N.W.Khobragade and P.C.Wankhede [6] studied an inverse unsteady-state thermoelastic problem of a thin rectangular plate. Recently D.T.Solank and M.H.Durge [18] has been considered the heat conduction equation with internal moving heat source for a Neumann’s thin rectangular plate and find the thermal stresses by using Green’s theorem. Also D.T.Solank and M.H.Durge [19] has been determined the temperature distribution and thermal stresses in thin rectangular plate with moving line heat source taking second type boundary condition by using integral transform technique and Green’s theorem. Kidawa-Kukla J [9] has studied the temperature distribution in a Rectangular Plate heated by a Moving Heat Source.

In present paper author considered thermoelastic problem with second and third kind boundary condition in a thin rectangular plate occupying the region 

\( D: -a \leq x \leq a, -b \leq y \leq b, 0 \leq z \leq h \).

The solution of the problem is obtained by using finite Marchi-Fasulo transform and finite Fourier cosine transform techniques in the form of infinite series.

II. STATEMENT OF THE PROBLEM

Consider a thin rectangular plate occupying the region 

\( D: -a \leq x \leq a, -b \leq y \leq b, 0 \leq z \leq h \).

The plate is subjected to the motion of moving point heat source at the point \((0, y', z')\) which move its place along \(X, Y, Z\) axes with constant velocity vector 

\[ \vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k} \]

where \(v_1, v_2, v_3\) are component of velocity vector along \(X, Y, Z\) axes respectively. The temperature distribution of the rectangular plate is given by

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g(x, y, z, t)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (1) \]

Where \(k\) is the thermal conductivity and \(\alpha\) is thermal diffusivity of the material of the plate. Consider an instantaneous moving point heat source at point \((0, y', z')\) and releasing its heat spontaneously at time \(t'\) Such volumetric moving heat source in rectangular coordinates is given by

\[ g(x, y, z, t) = g_0 \delta(x) \delta(y - y') \delta(z - z') \delta(t - t') \]

Hence equation (1) becomes

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g_0 \delta(x) \delta(y - y') \delta(z - z') \delta(t - t')}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (2) \]

Where \(y' = v_2 t\) and \(z' = v_3 t\), with initial condition

\[ T(x, y, z, 0) = T_0 \quad (3) \]

And the boundary conditions are given by

\[ T(x, y, z, t) + k_1 \left( \frac{\partial T(x, y, z, t)}{\partial x} \right) _{x=a} = G_1(y, z, t) \quad (4) \]

\[ T(x, y, z, t) + k_2 \left( \frac{\partial T(x, y, z, t)}{\partial y} \right) _{y=b} = G_2(y, z, t) \quad (5) \]

\[ T(x, y, z, t) + k_3 \left( \frac{\partial T(x, y, z, t)}{\partial z} \right) _{z=h} = G_3(x, z, t) \quad (6) \]

\[ T(x, y, z, t) + k_4 \left( \frac{\partial T(x, y, z, t)}{\partial y} \right) _{y=0} = G_4(x, z, t) \quad (7) \]

\[ \left( \frac{\partial T(x, y, z, t)}{\partial x} \right) _{x=0} = f_1(x, y, t) \quad (8) \]

\[ \left( \frac{\partial T(x, y, z, t)}{\partial x} \right) _{x=a} = f_2(x, y, t) \quad (9) \]
Let us introduce a thermal stress function $\chi$ related to component of stress in the rectangular coordinates system as in [6] is

$$\chi = \chi_c + \chi_p$$

(10)

where $\chi_c$ is the complementary solution and $\chi_p$ is particular solution. $\chi_c$ and $\chi_p$ are governed by equations:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 \chi_c = 0$$

(11)

and

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \chi_p = -\alpha E \Gamma .$$

(12)

Since plate is thin $z$ is negligible and where

$$\Gamma = T - T_0, \ T_0 \text{ is initial temperature.}$$

Also component of stress functions are given by

$$\sigma_{xx} = \frac{\partial^2 \chi}{\partial y^2}$$

(13)

$$\sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2}$$

(14)

$$\sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y}$$

(15)

The boundary conditions are

$$\sigma_{yy} = 0, \ \sigma_{xy} = 0 \ \text{ at } y = b .$$

Equations (1) to (15) constitute the mathematical formulation of the problem under consideration.

### III. SOLUTION OF THE PROBLEM

Applying finite Marchi-Fasulo transform two times and then finite Fourier cosine transform to equation (2), also using given boundary conditions we get

$$-\alpha_s^2 \pi^2 = -b_m^2 \pi^2 - b_m^2 \pi^2 + G$$

$$+ \frac{n \pi}{h} \left( (-1)^{n+1} f_2 + f_1 \right) + \frac{1}{k} g_0 a_i a_m$$

$$\cos(a_i a) \cos(b_m b) \delta(t-t') \cos(d_{n,y}) \cos \left( \frac{n \pi'}{h} \right) = \frac{1}{d'} t' \cos \left( \frac{n \pi'}{h} \right)$$

(16)

Solving above equation and using initial condition we get,

$$T = e^{-\alpha_s^2 \pi^2 t} - b_m^2 \pi^2 + G$$

$$+ \frac{n \pi}{h} \left( (-1)^{n+1} f_2 + f_1 \right) + \frac{1}{k} g_0 a_i a_m$$

$$\cos(a_i a) \cos(b_m b) \delta(t-t') \cos(d_{n,y}) \cos \left( \frac{n \pi'}{h} \right)$$

(17)

Where

$$\Omega = \left[ \left[ \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right]^2 \right] \right] \times \left[ \left[ \left[ \frac{\partial}{\partial x} \right] \right] \right] \times \left[ \left[ \left[ \frac{\partial}{\partial y} \right] \right] \right]$$

(18)

And $G = G (G_1, G_2, G_3, G_4)$

Taking inverse finite Fourier cosine transform, and Marchi-Fasulo transform two times we get

$$\Gamma = \frac{2}{h} \sum_{l,m,n=1}^{\infty} \frac{p_0 (m) R_0 (m)}{\lambda_m} \mu_m$$

$$\times \left[ \left[ \left[ \frac{\alpha^2 + b_m^2 + \frac{n \pi^2}{h^2}}{h} \right] \right] \right] \times \left[ \left[ \left[ \frac{1}{t'} \cos \left( \frac{n \pi'}{h} \right) \right] \right] \right]$$

(19)

And

$$\Gamma = \frac{2}{h} \sum_{l,m,n=1}^{\infty} \frac{p_0 (m) R_0 (m)}{\lambda_m} \mu_m$$

$$\times \left[ \left[ \left[ \frac{1}{t'} \cos \left( \frac{n \pi'}{h} \right) \right] \right] \right]$$

(19)

**IV. DETERMINATION OF STRESS FUNCTION**

Let the suitable form of $\chi_c$ satisfying (11) is given by
\[ \chi_c = \sum_{l,m,n=1}^{\infty} \left\{ y^2 \left[ c_n e^{a} + c_{-n} e^{-a} \right] \sin \left( \frac{n \pi x}{h} \right) + y^2 \left[ c_n e^{a} + c_{-n} e^{-a} \right] \cos \left( \frac{n \pi y}{h} \right) \right\} \]

Let the suitable form of \( \chi_p \) satisfying (12) is given by

\[ \chi_p = \frac{2 \alpha E a^2 b^2}{h(a^2 + b^2)} \sum_{l,m,n=1}^{\infty} \frac{P_1(x) R_m(y)}{\lambda_l \mu_m} \times \left\{ \int \exp \left( a_i^2 + b_i^2 + \frac{n^2 \pi^2}{h^2} \right) (t-t') \left[ G + \frac{n \pi}{h} (-1)^{n+1} f_2 + f_1 \right] + \frac{a}{k} g_0 4a_j m_m \cos(a_j a) \cos(b_j b) \cos(a_n y') \delta(t-t') \sin \left( \frac{n \pi y'}{h} \right) \right\} - T_0 \]

Substituting equation (19) and (20) in (10), one obtains

\[ \chi = \sum_{l,m,n=1}^{\infty} \left\{ y^2 \left[ c_n e^{a} + c_{-n} e^{-a} \right] \sin \left( \frac{n \pi x}{h} \right) + y^2 \left[ c_n e^{a} + c_{-n} e^{-a} \right] \cos \left( \frac{n \pi y}{h} \right) \right\} + \frac{2 \alpha E a^2 b^2}{h(a^2 + b^2)} \sum_{l,m,n=1}^{\infty} \frac{P_1(x) R_m(y)}{\lambda_l \mu_m} \times \left\{ \int \exp \left( a_i^2 + b_i^2 + \frac{n^2 \pi^2}{h^2} \right) (t-t') \left[ G + \frac{n \pi}{h} (-1)^{n+1} f_2 + f_1 \right] + \frac{a}{k} g_0 4a_j m_m \cos(a_j a) \cos(b_j b) \cos(a_n y') \delta(t-t') \sin \left( \frac{n \pi y'}{h} \right) \right\} - T_0 \]

Using (21) in (13), (14), (15) we get

\[ \sigma_{xx} = \sum_{l,m,n=1}^{\infty} \left\{ 2 \left[ c_n e^{a} + c_{-n} e^{-a} \right] \sin \left( \frac{n \pi x}{h} \right) + 2 \left[ c_n e^{a} + c_{-n} e^{-a} \right] \cos \left( \frac{n \pi x}{h} \right) \right\} \]

\[ \sigma_{yy} = \sum_{l,m,n=1}^{\infty} \left\{ 2 \left[ c_n e^{a} + c_{-n} e^{-a} \right] \sin \left( \frac{n \pi y}{h} \right) + 2 \left[ c_n e^{a} + c_{-n} e^{-a} \right] \cos \left( \frac{n \pi y}{h} \right) \right\} \]

\[ \sigma_{xy} = \sum_{l,m,n=1}^{\infty} \left\{ 2 \left[ c_n e^{a} + c_{-n} e^{-a} \right] \sin \left( \frac{n \pi x}{h} \right) + 2 \left[ c_n e^{a} + c_{-n} e^{-a} \right] \cos \left( \frac{n \pi y}{h} \right) \right\} \]
\[ \sin \left( \frac{n\pi x}{h} \right) \int_t dt \cos \left( \frac{n\pi z}{h} \right) \] \quad (24)

Using the boundary conditions \( \sigma_y = 0, \sigma_{yy} = 0 \), at \( y = b \) and equation (22) and (23) we get
\[ C_1 = -\frac{1}{2} \left[ \frac{2a\gamma_1}{bn^2\pi^2} + \frac{2a\gamma_2}{2bn\pi} \right] \]
\[ C_2 = -\frac{1}{2} \left[ \frac{2a\gamma_1}{bn^2\pi^2} + \frac{2a\gamma_2}{2bn\pi} \right] e^{\frac{2n\pi x}{a}} + \frac{2a\gamma_2}{2bn\pi} e^{\frac{n\pi x}{a}} \]
And \( C_3 = C_4 = 0 \)

Where
\[ \gamma_1 = \frac{2\alpha E_2 a^2 b^2}{h(a^2 + b^2)} \left\{ \sum_{l,m,n=1}^{\infty} \frac{P_l(x)}{\lambda_l} \frac{R_m(y)}{\mu_m} \times \right. \]
\[ \left[ \exp\left( a^2 + b^2 + \frac{n^2\pi^2}{h^2} \right) (t-t') \right] \left( G + \frac{n\pi}{h} \left[ -1 \right]^{l+1} f_2 + f_1 \right) \]
\[ + \frac{a}{k} g_0 4a_i b_m \cos(a_i a) \cos(b_m b) \cos(a_n y') \delta(t-t') \]
\[ \sin \left( \frac{n\pi x}{h} \right) \int_t dt \cos \left( \frac{n\pi z}{h} \right) \] \quad (25)

\[ \gamma_2 = \frac{2\alpha E_2 a^2 b^2}{h(a^2 + b^2)} \left\{ \sum_{l,m,n=1}^{\infty} \frac{P_l(x)}{\lambda_l} \frac{R_m(y)}{\mu_m} \times \right. \]
\[ \left[ \exp\left( a^2 + b^2 + \frac{n^2\pi^2}{h^2} \right) (t-t') \right] \left( G + \frac{n\pi}{h} \left[ -1 \right]^{l+1} f_2 + f_1 \right) \]
\[ + \frac{a}{k} g_0 4a_i b_m \cos(a_i a) \cos(b_m b) \cos(a_n y') \delta(t-t') \]
\[ \sin \left( \frac{n\pi x}{h} \right) \int_t dt \cos \left( \frac{n\pi z}{h} \right) \] \quad (26)

Substituting the above values in (22) to (24), it gets
\[ \sigma_{xx} = \sum_{l,m,n=1}^{\infty} \left\{ \left[ \exp\left( a^2 + b^2 + \frac{n^2\pi^2}{h^2} \right) (t-t') \right] \left( G + \frac{n\pi}{h} \left[ -1 \right]^{l+1} f_2 + f_1 \right) \] + \frac{a}{k} g_0 4a_i b_m \cos(a_i a) \cos(b_m b) \cos(a_n y') \delta(t-t') \] \[ \sin \left( \frac{n\pi x}{h} \right) \int_t dt \cos \left( \frac{n\pi z}{h} \right) \] \quad (29)
V. SPECIAL CASE AND NUMERICAL RESULTS

Set

\[ f_1(x, y, t) = (x-a)^2 (x+a)^2 (y-b)^2 (y+b)^2 \bigg( T_0 e^{-t} \bigg) \]
\[ f_2(x, y, t) = (x-a)^2 (x+a)^2 (y-b)^2 (y+b)^2 \bigg( e^{h T_0 e^{-t}} \bigg) \]

\[ G_1 = G_2 = G_3 = G_4 = 0. \]

\[ a = 2, b = 4, h = 2, t = 1 \text{ sec in equation (18), we get} \]
\[ T = \sum_{l,m,n=1}^{\infty} \frac{p_i(x) R_m(y)}{\lambda_l} \frac{1}{\mu_m} \]
\[ \times \left\{ \int \exp\left[ a_l^2 + b_m^2 + \frac{n \pi^2}{4} \right] (1-t') G + \frac{n \pi}{2} \right\} (-1)^{n+1} f_2 + f_1 \]
\[ + \frac{a}{k} g o \cos(2a_1) \cos(4b_m) \cos(d_m y') \delta(t - t') \]
\[ \cos\left( \frac{n \pi c}{2} \right) dt \cos\left( \frac{n \pi e}{2} \right) \} \quad (30) \]

VI. CONCLUSION

In this paper, the temperature distribution and thermal stresses have been found by using the finite Marchi-Fasulo transform and the finite Fourier cosine transform with moving heat point source. The results are obtained in the form of infinite series.

REFERENCES


AUTHOR BIOGRAPHY

Dr. N.W. Khobragade for being M.Sc in statistics and Maths, he attained Ph.D in both subjects. He has been teaching since 1986 for 28 years at PGTD of Maths, RTM Nagpur University, Nagpur and successfully handled different capacities.

At present he is working as Professor. Achieved excellent experiences in Research for 18 years in the area of Boundary value problems (Thermoelasticity in particular) and Operations Research. Published more than 220 research papers in reputed journals. Eighteen students awarded Ph.D Degree and six students submitted their thesis in University for award of Ph.D Degree under their guidance.

Mr. Rajesh B. Pardhi: M.Sc in maths. He has been teaching since 1999 for 17 years. Currently working as Head & Asst. Professor (Dept. of Mathematics) at Dr. Babasaheb Ambedkar College of Engineering, Nagpur.