ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 5, Issue 9, March 2016

Communication Network as a Non-Cooperative Game with Multiple Nash Equilibria

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Abstract*—In this paper, we propose a new mathematical model for communication network using game theory. In this model we consider a number of users who wish to send their throughput demand in the form of packets through one or more links with minimum cost or more efficiently (say in less time) than the other users. This results into a kind of non-cooperative situation in communication network. Two particular scenarios are studied: first when number of packets to be sent is less than the link capacity, and the second when number of packets exceeds the link capacity. Both these situations are modeled by multiple shot games. We discuss different type of cost functions and existence of Nash equilibrium. Our model is different from other transmission network models studied elsewhere in three aspects. These are regarding introduction of time (discrete) variable, a new idea of direct penalty to the users, and indirect penalty in the form of cost of transmission as an increasing function of time. This model of non-cooperative game leads to multiple Nash equilibrium points. To select one equilibrium point from these multiple points we use the concept of focal point and Pareto optimal point which lead to socially preferable solution.

***Index Terms*—Compact, Convex, Focal points, Nash equilibrium, Paretooptimality.**

I. INTRODUCTION

A. Orda, Rom and Shimkin (1993)[1]proposed a game theoretical model to deal with the routing problem in networking, contributing to the understanding of the dynamic of the modern network. Here dynamic means users change their behavior based on the state of the network. Each user knows its individual throughput demands, can measure the load on the network links and routing is selected by each user so as to optimize a certain selfish criterion such as cost. The performance of a user i is measured by the cost function which depends on the system flow configuration f. The system flow configuration f represents the flow of all users in a vector form. The authors defined cost function for user i as $J^i(\mathbf{f}): [0,\infty) \to [0,\infty]$ and the general family of the cost function is referred as type A which includes two special category type B and type C. Theorem related to the uniqueness of Nash equilibrium is also established for type A function. Elementary Stepwise System (ESS) shows simple convergence to the NEP for the network.

IsmetSahin and Marwan A. Simon (2006) [2] proposed a model for two node parallel link communication system with multiple competing users. The authors derived flow and routing control policy for each user to get the Nash equilibrium point. Maximizing throughput and minimizing delay are two main objectives which combine additively in a function known as a utility function for each user. Preference constants are also introduced for the two objectives and the links. Each user is given the flexibility of choosing its own objective between the two objectives and certain links over the other links. For the first time, users are given such a flexibility which not only to balance between throughput and delay but increase the usage of desirable links. The utility function for user i is defined in the form of "benefit – cost". It depends on the flow rate of user i on link m (λ^i_m) and link capacity c_m .

$$U^i\left(\hat{\lambda}^1, \hat{\lambda}^2, \dots\dots \hat{\lambda}^N\right) = \sum_{m\epsilon M} \alpha^i_m \lambda^i_m - \sum_{m\epsilon M} \frac{\beta^i_m \lambda^i_m}{c_m - \hat{\lambda}_m}$$

The parametersα^i_m and β^i_m are the preferences that user i assigns to each link for the benefit and the cost term of the utility function and their ratio $\gamma^i_m = \frac{\alpha^i_m}{\beta^i_m}$is the i^{th} trade off parameter for link m. The authors conclude that a unique NE exists that satisfies the following equation (for user$i$ and link$m$)

$$\lambda^{i^*}_m = (c_m - \bar{\lambda}^*_m)^2 \gamma^i_m - (c_m - \bar{\lambda}^*_m)$$

Where $\bar{\lambda}^*_m = c_m - \frac{(N-1)+\sqrt{(N-1)^2 + 4\bar{\gamma}_m c_m}}{2\,\bar{\gamma}_m}$. This NE is feasible if $\frac{1}{\gamma^i_m}\left(\frac{\bar{\gamma}_m}{\gamma^i_m} - (N-1)\right) \le c_m$ for all $i\epsilon N,\ m\epsilon M$

E. Altman, T. Basar , T. Jimenez and N. Shimkin [3] establish the conditions for the uniqueness of the Nash Equilibrium. The cost for the link l is the function of total load λ_{lt} on that link i.e .

$f_l(\lambda_{lt}) = a_l.(\lambda_{lt})^{p(l)} + b_l, \quad l\epsilon L$ where $a_l, b_l \; and \; p(l)$ are links specific positive parameters. The cost for player i is given by

$$J^i(\lambda) = J^i(\lambda^i, \lambda^{-i}) = \min_{\tilde{\lambda}^i} J^i(\tilde{\lambda}^i, \lambda^{-i}), \qquad i\epsilon N$$

Now the condition for uniqueness of NE is $p^* \coloneqq (3N-1)/(N-1)$ where $0 < p(l) < p^*, \;\; p^* > 3 \; for \; all \; N \ge 2$. If additionally $b_l = 0$ in the cost for the link and all users have the same source and destination, the resulting NE is globally optimal and the link flows of different users are proportional to their total traffic.

Altman and Wynter (2004)[4] highlighted a number of areas in which common features between transportation and telecommunication network model exist.

Bottleneck Routing Games were studied by R.Banner and A. Orda [5]. The author investigated that "Bottleneck" (worst) routing games appear in two main routing scenarios, namely when a user can split its traffic over more than one

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 5, Issue 9, March 2016

path (Splittable bottleneck game) and when it cannot (unsplitable bottleneck game). Also they have shown that a bottleneck game has always admitted a Nash equilibrium, moreover, best response dynamics in unsplitable games converge to a Nash equilibrium in finite time. This Nash equilibrium (both in splitable and unsplitable bottleneck games) can be very inefficient. In order to cope with this inefficiency, the authors investigated for each game “reasonable “conditions under which Nash equilibrium are socially optimal. The condition is

Given a Nash flow f is said to satisfy the efficiency condition if all users route their traffic along paths with a minimum number of bottlenecks i.e. for each $u \epsilon U$ and $p_1, p_2 \epsilon P^{(s_u, t_u)}$ with flow $f^u_{p_1} > 0$ it holds that $N_f(p_1) \leq N_f(p_2)$.

Massey [6] proposed a telecommunication model using queuing theory. In this model the impact of time varying behavior on communication system was studied.

II. PROPOSED MODEL “A DYNAMIC MODEL”

The work presented here deals with routing data packets in a communication network. The players/users come to the "game" with the knowledge of the number of packets they wish to send through one or more network link over the m chances/shots. As usual each link at a given chance/shot has a finite capacity to carry the packets. In other models, the users are dissuaded from sending the number of packets exceeding capacity of a link by making the cost infinity for such a situation and there is a transmission failure in the sense that no one packet is sent through the link.

In our model, the cost of transmission remains finite even if the sum of packets wished to be sent by the users through a given link in a given slot/shot exceeds the specified capacity of the link. However, we introduce a mechanism which will be followed to cope up with the above situation.

III. MATHEMATICAL MODELING

In the present communication network model, we consider two users sharing one link connecting a source node to a destination node. We assume that the link is available to the users over a discrete range of time known as time slots and there are m time slots in a single cycle. Users are rational and selfish for this competitive game. Each user n has throughput demand $D^{(n)}$ which he/she wants to ship from source to destination. A user sends its throughput demand in the form of data packets through the communication link and is able to decide at any time how the data packets will be transmitted and what fraction of throughput demand should be sent at that time through the link.

A. Mechanism for packet transmission

Let the first user plan to send $\{p_1, p_2, p_3, \dots . p_m\}$ packets and second user plan to send $\{q_1, q_2, q_3, \dots . q_m\}$ packets in m number of time slots. i.e. in first time slot first user and second user plan to send p_1 and q_1 packets respectively.

If $p_1 + q_1 \leq \lambda$ (Capacity of the link) then all packets of both users will be transmitted. Since the capacity λ of the link is fixed and this is a noncooperative game therefore it is impossible for the user to get the information of strategy of other user.

So the situation may occur when $p_1 + q_1 > \lambda$. In this case the user with minimum packets will able to transmit his /her data and other user cannot transmit his/her data packets completely in the first slot. For example if $p_1 > q_1$ then q_1 packets of user 2 will be transmitted but user 1 can transmit only $\lambda - q_1$ packets from p_1 packets. Remaining $p_1 - (\lambda - q_1)$ packets will be transmitted in the next time slot with p_2. Now modified p_2 is $p_2 + (p_1 - (\lambda - q_1))$.

To keep record of non-transmitted and planned packets we need two more strategy sets. Consider these sets are intermediate strategy sets and actual strategy sets. Assume that an intermediate strategy set for first user as $\{r_1, r_2, r_3, \dots . r_m\}$ and for second user as $\{s_1, s_2, s_3, \dots . s_m\}$. Now actual strategy sets (i.e. Data packets in a different time slot) may be different from the intermediate and planned strategy set for both users. Let $\{u_1, u_2, u_3, \dots . u_m\}$ and $\{v_1, v_2, v_3, \dots . v_m\}$ be the data packets transmitted by first and second user respectively in $\{1, 2, 3, \dots, m\}$ time slots. Initially

$$p_1 = r_1 \text{ and } q_1 = s_1$$

The situations discussed above can be described using these intermediate and actual strategy sets as follows:

Case I: When $p_1 + q_1 \leq \lambda$

i.e. number of packets wanted to be shipped by users on the link are less or equal to the link capacity. In this case all data packets will be transmitted. Mathematically this situation can be expressed as

$p_1 = r_1 = u_1$ and $q_1 = s_1 = v_1$

Case I : When $p_1 + q_1 > \lambda$

In this case number of packets wanted to be shipped by the users exceeds the link capacity. Now it is not possible to transmit all packets in the same time slot. Hence we need a mechanism to reduce these packets according to the capacity of the link.

(i) $p_1, q_1 \leq \lambda$, This inequality shows that both users want to use λ, the maximum capacity of the link

(ii) If $p_1 > q_1$ then all q_1 packets of user 2 will be transmitted to the link and from p_1 only $\lambda - q_1$ packets will transmit i.e. $u_1 = \lambda - q_1$ and $p_1 - u_1$ packets will be added to p_2 to get r_2. Mathematically
Step 1: $v_1 = q_1 = s_1, \; u_1 = \lambda - q_1$
Step 2: $r_2 = p_2 + (r_1 - u_1), \; and \; s_2 = q_2$

(iii) Similarly if $q_1 > p_1$ then all p_1 packets of user 1 will be transmitted and from q_1 only $\lambda - p_1$ packets will transmit to the link i.e. $v_1 = \lambda - p_1$ and $q_1 - v_1$ packets will be added to q_2 to get s_2. Mathematically

Step 1: $u_1 = p_1 = r_1, \; v_1 = \lambda - p_1$
Step 2: $s_2 = q_2 + (s_1 - v_1), \; and \; r_2 = p_2$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 5, Issue 9, March 2016

(iv) If $p_1 = q_1$ then $p_1/2$ packets of each user will be transmitted i.e. $u_1 = v_1 = p_1/2$ and $r_2 = p_2 + \frac{p_1}{2}$ and $s_2 = q_2 + \frac{q_1}{2}$.

Now in the next step, in spite of comparing p_2 and q_2, we will compare r_2 and s_2. Step (ii) or (iii) will be performed for each r_i and s_i where $2 \le i \le m$. Since r_i and s_i may be greater than p_i and q_i respectively therefore the cost may be greater than the cost of the planned strategy set.

B. Strategy sets and constraints

Now we will discuss the properties of element of all strategy sets.

P₁ : $p_i, q_i, r_i, s_i, u_i, v_i \ge 0 \quad \forall i = 1,2,3, \dots m$ (Non – negative constraint)

P₂ : $\sum_i p_i = D^{(1)}$ and $\sum_i q_i = D^{(2)}$ (Demand constraint for first and second user)

P₃ : $u_i \le p_i \le r_i$ and $v_i \le q_i \le s_i$ (Relation between three strategy set)

P₄ : $r_i = p_i + (r_{i-1} - u_{i-1})$ and $s_i = q_i + (s_{i-1} - v_{i-1})$ (Recurrence relations of r_i and s_i)

P₅ : $\lambda_i = u_i + v_i \le \lambda \quad \forall i = 1,2,3, \dots m$ (Capacity constraint for each i)

P₆ : $\sum_i u_i \le D^{(1)}$ and $\sum_i v_i \le D^{(2)}$

This property shows that actual data transmitted in all time slots are less or equal to the throughput demand of the user. In this paper, we consider only those cases in which user's throughput demands satisfied the conditions $\sum_i u_i = D^{(1)}$ and $\sum_i v_i = D^{(2)}$.

IV. ROUTING SCHEME AND COST FUNCTION

We describe routing scheme in a single link communication network for users. Users can route one or more packets on this link at time slot i

A. General Assumptions of Cost Function

The following general assumptions on the cost function $C^{(n)}$ of each user where $n = 1, 2$ will be imposed throughout the paper.

A₁ : $C^{(n)} = \sum_i C_i^{(n)}$ It is a sum of the cost of routed packets over the link in each time slot i by user n.

A₂ : Cost function is non- linear and non- negative function .

A₃ : $C_i^{(n)} : I \times I \to R^+$ is a continuous function for each time slot i.

A₄ : Cost function is strictly increasing with the number of packets in intermediate strategy set say x_i, total number of packets actually transmitted by the users λ_i and time slot i. We shall consider the cost of transmitting packets at time slot i to be of the type

$$C_i^{(n)} = f(i) \cdot \emptyset(x_i, \lambda_i) \quad \dots\dots\dots\dots (1)$$

Where $n = 1, 2. \quad i = 1,2,3, \dots m$

$x_i = r_i, s_i$ and $\lambda_i = u_i + v_i$

A₅: $C_i^{(n)}$ is continuously differentiable with respect to x_i and λ_i.

A₆: $C_i^{(n)}$ is zero when x_i is zero.

Additional assumptions concerning the time function $f(i)$ are

T₁ : Each user gets m number of instances called discrete time slots

T₂ : Flow of packets is continuous which implies that there is no congestion in the system.

T₃ : Users can transmit more than one packet on the link at the same time slot. They must obey the capacity constraint (P₅) and non-negative constraint (P₁).

Also we consider that the game will not be over until all the packets are transmitted and fixed number of the time slot is over. And hence the efficiency of user is measured by the term unit cost which depends on the total cost, number of packets and time slots. i.e.

$$\text{Unit cost for the user} = \frac{\text{Total Cost}}{(\text{No. of transmitted packets})}$$

Mathematical expression of the above formula is

$$\text{Unit cost for the user n} = \frac{C^{(n)}}{D^{(n)}} \dots\dots\dots\dots (2)$$

Efficient user = user 1 if $\frac{C^{(1)}}{D^{(1)}} < \frac{C^{(2)}}{D^{(2)}}$

and Efficient user = user 2 if $\frac{C^{(2)}}{D^{(2)}} < \frac{C^{(1)}}{D^{(1)}}$

Now

Unit cost for the Network = unit cost for the user 1 + unit cost for the user 2

$$= \frac{\sum_n C^{(n)}}{\sum_n D^{(n)}} \dots\dots\dots\dots (3)$$

B. Nash equilibrium point

According to [9] , " A Nash equilibrium is a profile of strategies such that each player's strategy is an optimal response to the other player's strategies". In other words, NEP is the stability point of the game in which no user finds it beneficial to change its strategies.

In this paper, we assume that user 1 and 2 will transmit their packets in different time slot such that the combination becomes optimal for the user. If $C^{(1)}$ and $C^{(2)}$ are the costs for user 1 and 2 respectively and the strategies of them are denoted by the collection $\{p_1, p_2, p_3, \dots p_m\}$ and $\{q_1, q_2, q_3, \dots q_m\}$, then the combination of strategies $\{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \dots \tilde{p}_m\}$ for user 1 and $\{\tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \dots \tilde{q}_m\}$ for user 2 is a Nash Equilibrium Point provided

$$C^{(1)}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \dots \tilde{p}_m, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \dots \tilde{q}_m) \le C^{(1)}(p_1, p_2, p_3, \dots p_m, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \dots \tilde{q}_m) \quad \dots\dots (4)$$

for all possible strategies $\{p_1, p_2, p_3, \dots p_m\}$ of user 1 and

$$C^{(2)}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \dots \tilde{p}_m, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \dots \tilde{q}_m) \le C^{(2)}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \dots \tilde{p}_m, q_1, q_2, q_3, \dots q_m) \quad \dots\dots (5)$$

for all possible strategies $\{q_1, q_2, q_3, \dots q_m\}$ of user 2.

We start our investigation with the conditions that needs to be satisfied by cost functions in order to guarantee the Nash equilibrium point.

C. Optimization and Existence of NEP

The Kuhn- Tucker conditions are simply the first order conditions for a constrained optimization problem.

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
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Let $f : R^n \to R$ and $G : R^n \to R^m$ be continuously differentiable functions and let $b \in R^m$. We want to characterize those vectors $\tilde{x} \in R^n$ that satisfy

(*) $\tilde{x}$ is a solution of the problem

Maximize $f(x)$ subject to $x \geq 0$ and $G(x) \leq b$,

i.e subject to $x_1, x_2, \ldots \ldots x_n \geq 0$ and

$G^i(x) \leq b_i$ for $i = 1, 2, \ldots \ldots m$

The Kuhn-Tucker conditions are the first order conditions that characterize the vectors $\tilde{x}$ that satisfy (*)$\exists\ \lambda_1, \lambda_2, \ldots \ldots \lambda_m \in R^+$ such that

(KT1) For $j = 1, 2 \ldots . n$: $\frac{\partial f}{\partial x_j} \leq \sum_{i=1}^{m} \lambda_i \frac{\partial G^i}{\partial x_j}$

$with\ equality\ if\ \tilde{x}_j > 0;$

(KT2) For $i = 1, 2 \ldots . m$: $G^i(\tilde{x}) \leq b_I$

$with\ equality\ if\ \lambda_i > 0;$

Where the partial derivatives are evaluated at $\tilde{x}$.

D. Implementation of Kuhn-Tucker Condition in the model

The cost function $C^{(n)}: I^m \times I^m \to R^+$ be continuously differentiable function in which n=1,2 and m total number of time slots.

$(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \ldots . \tilde{p}_m, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \ldots . \tilde{q}_m)$ is the solution of the problem

Minimize $\frac{\sum_n C^{(n)}}{\sum_n D^{(n)}}$

where $C^{(n)} = \sum_{i=1}^{m} C_i^{(n)} = \sum_{i=1}^{m} f(i) . \emptyset(x_i, \lambda_i)$ by(1)

If $\exists\ \delta_1, \delta_2, \ldots \ldots \delta_m, \delta_1', \delta_2', \ldots \ldots \delta_m' \in R^+$

$\tilde{p}_i, \tilde{q}_i > 0 \to x_i, \lambda_i > 0$

$$\Rightarrow \sum_{i=1}^{m} \frac{\partial C_i^{(n)}}{\partial x_i} = \sum_{i=1}^{m} \delta_i \ and \ \sum_{i=1}^{m} \frac{\partial C_i^{(n)}}{\partial \lambda_i} = \sum_{i=1}^{m} \delta_i' \ \ldots . (6)$$

Also if $\tilde{p}_i = 0\ or\ \tilde{q}_j = 0\ for\ some\ i, j\ then$

$$\sum_{i=1}^{m} \frac{\partial C_i^{(n)}}{\partial x_i} \geq \sum_{i=1}^{m} \delta_i \ and \ \sum_{i=1}^{m} \frac{\partial C_i^{(n)}}{\partial \lambda_i} \geq \sum_{i=1}^{m} \delta_i' \ \ldots \ldots \quad (7)$$

Inequality (7) satisfies because to satisfy the throughput demand of users, it is not possible to have all possible strategies $\tilde{p}_i, \ \tilde{q}_i = 0$ and $C^{(n)} = \sum_{i=1}^{m} C_i^{(n)}$.

The Kuhn-Tuckar conditions given by (6) and (7) constitute the necessary and sufficient condition for a feasible solution to be a Nash Equilibrium Point.

Another important theorem, which helps to prove the existence of Nash Equilibrium point, is the Kakutani's Fixed Ponit Theorem. This theorem states:

"Let $X \subset R^n$ be closed, bounded, and convex. For every $x \in X$ let $F(x)$ be a non-empty, convex subset of X. Assume that the graph of the set- valued function is closed in $X \times X$. Then there exists a point $x^* \in X$ such that $x^* \in F(x^*)$"

The existence of a Nash Equilibrium is equivalent to X having a fixed point.Kakutani's fixed point theorem guarantees the existence of a fixed point if the following four conditions are satisfied.

1. The domain is compact, convex, and nonempty.
2. F(x) is nonempty.
3. F(x) is convex.
4. F(x) is continuous.

For the existence of Nash equilibrium point, we define the cost function which is continuous, convex and its domain is bounded and closed.

V. CLASSES OF COST FUNCTION

The expected cost for the user n at i^{th} time slot depends on the number of packets x_i to be routed by user n and total number of packets λ_i by both users, which can be expressed as below

$C_i^{(n)} = f(i). \ \emptyset(x_i, \lambda_i)$

We analyze several classes of cost functions which satisfy equation (6) and (7) and general properties of cost function describe in section 4.1.

A. Exponential Cost Function

We consider the exponential cost function, which increases with time slot i and exponentially increases with x_i. For a fixed i, the cost function $C_i^{(n)}$ depends on $\emptyset(x_i, \lambda_i)$. Let the form of exponential cost function be

$$C_i^{(n)} = f(i). \ \emptyset(x_i, \lambda_i) = \frac{\sqrt{i}\,(e^{x_i} - 1)}{\lambda + 1 - (\lambda_i)}, \quad when\ \lambda_i \leq \lambda \ \ldots \ldots (8)$$

Here the time function $f(i)$ is the square root of time slot (or instance).

Hence the total cost of user n to send its data packets through the link is

$$C^{(n)} = \sum_i f(i) . \emptyset(x_i, \lambda_i) = \sum_i \frac{\sqrt{i}\,(e^{x_i} - 1)}{\lambda + 1 - (\lambda_i)}$$

B. Non- Linear Cost Function

We consider the Nonlinear cost function, which increases with time slot i and $(x_i)^a$.Where a is any positive integer such that $a \geq 1$. For a fixed i, the cost function $C_i^{(n)}$ depends on $\emptyset(x_i, \lambda_i)$. Let the form of the nonlinear cost function is

$$C_i^{(n)} = f(i). \ \emptyset(x_i, \lambda_i) = \frac{\sqrt{i}\,(x_i)^a}{\lambda + 1 - (\lambda_i)}, \quad when\ \lambda_i \leq \lambda \ and\ a \geq 1 \ \ldots \ldots \ldots \ (9)$$

Again we assume that the time function $f(i)$ is the square root of time slot (or instance).

Hence the total cost of user n to send its data packets through the link is

$$C^{(n)} = \sum_i f(i) . \emptyset(x_i, \lambda_i) = \sum_i \frac{\sqrt{i}\,(x_i)^a}{\lambda + 1 - (\lambda_i)}$$

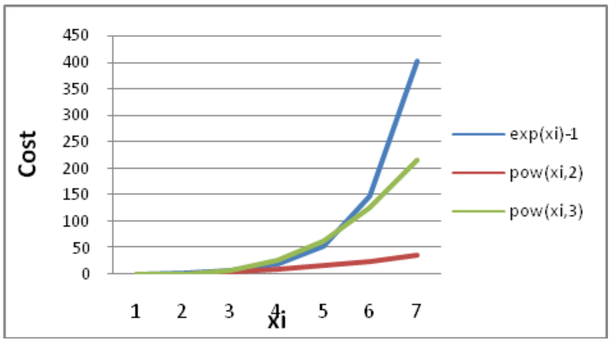

Fig 1: Curve of different cost functions

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 5, Issue 9, March 2016

VI. CONDITION OF CONVEXITY

In the preceding section IV the general assumptions of cost function define it as a continuous differentiable of type

$$C_i^{(n)} = f(i) . \emptyset(x_i,\lambda_i)$$

The following theorems establish the existence of Nash Equilibrium Point for the cost function.

Theorem 1: In a communication network cost function $C_i^{(n)}: I \times I \to R^+$ for each time slot, defined as (1) is convex.

Proof : To prove that cost function $C_i^{(n)}$ is convex we will use following theorem (by [7])

"A function $f(X)$ is convex if the Hessian matrix $H(X) = \left[\frac{\partial^2 f(X)}{\partial x_i \, \partial x_j}\right]$ is positive semi definite. If $H(X)$ is positive definite, the function $f(X)$ will be strictly convex."

By equation (1) cost function can be expressed as

$$C_i^{(n)} = f(i). \emptyset(x_i,\lambda_i)$$

For a fixed time slot, $f(i)$ is constant, therefore $C_i^{(n)}$ (for simplicity consider $C_i^{(n)} = C$) will be a function of two variables x_i and λ_i, therefore the Hessian Matrix for C is

$$H(C) = \begin{bmatrix} \frac{\partial^2 C}{\partial x_i^2} & \frac{\partial^2 C}{\partial x_i \partial \lambda_i} \\ \frac{\partial^2 C}{\partial x_i \partial \lambda_i} & \frac{\partial^2 C}{\partial \lambda_i^2} \end{bmatrix}$$

The following theorems establish the existence of Nash Equilibrium Point for the exponential cost function. (Similarly we can prove these theorems for Non-Linear cost function.)

Verification of condition: In a communication network cost function

$C_i^{(n)}: I \times I \to R^+$ for each time slot, defined as (8) is convex.

By equation (8) cost function can be expressed as

$$C_i^{(n)} = f(i). \emptyset(x_i,\lambda_i) = \frac{\sqrt{i}\,(e^{x_i} - 1)}{\lambda + 1 - (\lambda_i)}$$

For a fixed time slot, $f(i)$ is constant, therefore $C_i^{(n)}$ (for simplicity consider $C_i^{(n)} = C$) will be a function of two variables x_i and λ_i, therefore the Hessian Matrix for C is

$$H(C) = \begin{bmatrix} \frac{\partial^2 C}{\partial x_i^2} & \frac{\partial^2 C}{\partial x_i \partial \lambda_i} \\ \frac{\partial^2 C}{\partial x_i \partial \lambda_i} & \frac{\partial^2 C}{\partial \lambda_i^2} \end{bmatrix}$$

Case I: When $x_i > 0$

$$C = \frac{\sqrt{i}\,(e^{x_i} - 1)}{\lambda + 1 - (\lambda_i)}$$

Let $\sqrt{i} = K'$ and $\lambda + 1 - (\lambda_i) = M$

$$\frac{\partial C}{\partial x_i} = \frac{K' e^{x_i}}{M}$$

$$\frac{\partial^2 C}{\partial x_i^2} = \frac{K' e^{x_i}}{M}$$

$$\frac{\partial C}{\partial \lambda_i} = \frac{K'(e^{x_i} - 1)}{M^2} \text{ and } \frac{\partial^2 C}{\partial \lambda_i^2} = \frac{2K'\,(e^{x_i} - 1)}{M^3}$$

$$\frac{\partial^2 C}{\partial x_i \partial \lambda_i} = \frac{K' e^{x_i}}{M^2}$$

Now Hessian Matrix

$$H(C) = \begin{bmatrix} \frac{K' e^{x_i}}{M} & \frac{K' e^{x_i}}{M^2} \\ \frac{K' e^{x_i}}{M^2} & \frac{2K'\,(e^{x_i} - 1)}{M^3} \end{bmatrix}$$

$$|H(C)| = \begin{vmatrix} \frac{K' e^{x_i}}{M} & \frac{K' e^{x_i}}{M^2} \\ \frac{K' e^{x_i}}{M^2} & \frac{2K'\,(e^{x_i} - 1)}{M^3} \end{vmatrix}$$

$$= \frac{K'^2 e^{x_i}}{M^4}[e^{x_i} - 2]$$

$$> 0 \quad (\because e^1 = 2.71828)$$

Case II: When $x_i = 0$

$$C = \frac{\sqrt{i}\,(e^{x_i} - 1)}{\lambda + 1 - (\lambda_i)} = 0$$

$$\frac{\partial^2 C}{\partial x_i^2} = \frac{\partial^2 C}{\partial \lambda_i^2} = \frac{\partial^2 C}{\partial x_i \partial \lambda_i} = 0$$

Therefore $|H(C)| = 0$ which is non-negative.

Since $H(C)$ is positive definite in all cases therefore the function $C_i^{(n)}$ will be strictly convex for each time slot.

The cost function $C_i^{(n)}$ for each i and n is convex and compact. Therefore by Kakutani Fixed Point Theorem there exists a fixed point and such a point will be Nash equilibrium point.

VII. CONDITION FOR FEASIBLE AND UNFEASIBLE SOLUTIONS

We can obtain the general condition for feasible and unfeasible solutions, which will be applicable for all classes of cost functions.

If $\sum_{i=1}^{m} u_i = D^{(1)}$ and $\sum_{i=1}^{m} v_i = D^{(2)}$ I.e. both users transmit their throughput demand completely then the strategy sets $\{p_1, p_2, p_3, \dots . p_m\}$ *and* $\{q_1, q_2, q_3, \dots . q_m\}$ are called points for the feasible solution of cost function otherwise the solution will be non feasible.

A. *Condition for Nash Equilibrium Point*

If $p_i + q_i = \lambda \ \forall i$ then all data packets will be transmitted without any penalty and hence cost of data transmission will be less than any other combination of p_i and q_i i.e. condition (4) and (5) will satisfy. Mathematically this situation can be expressed as

$$p_i = r_i = u_i \text{ and } q_i = s_i = v_i \qquad \dots \dots \dots . . (10)$$

Now the cost function becomes

$$C^{(n)} = \sum_{i=1}^{m} C_i^{(n)} = \sum_{i=1}^{m} \sqrt{i}\,(e^{x_i} - 1)$$

$$where\ x_i = p_i, q_i \ and\ n = 1,2$$

(in case of Exponential cost function)

$$C^{(n)} = \sum_{i=1}^{m} C_i^{(n)} = \sum_{i=1}^{m} \sqrt{i}\,(x_i)^a$$

(in case of Non- linear cost function)

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 5, Issue 9, March 2016

VIII. MULTIPLE NASH EQUILIBRIUM POINT

Many games have more than one Nash equilibrium point. Two games the "Battle of the sexes" and the "Prisoner's Dilemma Game" are very popular games with multiple Nash Equilibrium points and they are discussed in [8] and [9]. In case of multiple Nash equilibrium point, we select a point which is better than all other NEP, called Pareto optimal point.

Pareto optimality is a state of game where resources are allocated in the most efficient manner. In other words, Pareto optimality is a set of conditions under which the state of economic efficiency (where no one can be made better off by making someone worse off) occurs.

The model describes in this paper presented a game with multiple Nash equilibrium points. By [8], the best situation is when a game has single Nash equilibrium. If there are multiple Nash equilibrium, then there is some hope that only one of them is admissible.

A. Focal Points and Pareto Optimality

How should the users behave, when there are multiple Nash equilibrium points? Which strategies are best for both users? How users can identify the strategy which gives Pareto Optimal solution? With the help of the concept of focal points some of these questions can be answered. The concept of the focal points and Pareto optimality is discussed in [9].

The theory of "Focal Points" was introduced by Schelling's (1960). This theory suggests that in some "real life" situations players may be able to coordinate on a particular equilibrium by using information that is abstracted away by the strategic form. The example described in [9] can easily explain the Focal point. In this example, two players are asked to name an exact time, with the promise of a reward if their choices match. Both players will try to match their choices. In order to this they will speak some common time like "12 noon", "1:00 p.m.", "2:30 p.m." etc. which are focal points for this game but "1:43 p.m.", "2:17 p.m." etc. are not. The "Focalness" of various strategies depends on the players/user's past experiences and information provided to him/her.

Now we will try to find out the answers of the following questions based on our network communication model.

1. What should be the focal point for the user?
2. Can we get a Pareto optimal solution using these focal points?
3. How can we express it mathematically?

From section 3 , We have $D^{(1)}$ and $D^{(2)}$ as throughput demands of user 1 and user 2 respectively. Both users want to transmit their throughput demand in m time slots over a link with capacity λ . Using equation (6), the optimal solution to the problem can be achieved when $p_i, q_i > 0 \quad \forall\, i = 1,2, \dots . m$. If some $p_i, q_i = 0$ and $D^{(1)}, D^{(2)} > 0$ then obviously $p_{i+1}, q_{i+1} > 0$. This shows that the user will transmit its data in the next time slot but cost function is directly proportional to the time slot and hence cost increases as i increases.

Both users want to minimize the cost of transmission, therefore they will prefer the strategy in which $p_i, q_i > 0 \quad \forall\, i = 1,2, \dots . m$. That means first they will focus on the strategy set having all nonzero elements. Now there may be number of strategy sets which satisfy this nonzero condition, so the next object is to find the focal points from these sets.

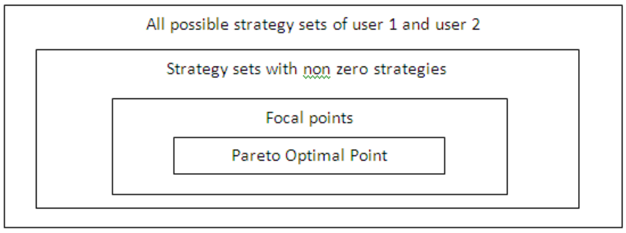

Fig 2: Relation between different strategies

From equation (1), we have assumed that cost is an increasing function of time slots and the number of packets in an intermediate strategy set which is directly related to the planned strategy set. This implies that the cost function increases with time slot and number of packets in the planned strategy set. Therefore the best option for the user is to plan to route same or nearly the same number of packets in each time slot. This statement can also verify with the special principle of Discrete Mathematics named as "Extended Pigeonhole Principle". The Extended pigeonhole principle states that :

"If n pigeons are assigned to m pigeonholes (The number of pigeons is very large than the number of pigeonholes), then at least one of the pigeon holes must contain $\lfloor (n-1)/m \rfloor + 1$ pigeons."

We can conclude from the above statement that almost all pigeonholes contain $\lfloor (n-1)/m \rfloor$ pigeons and some pigeonholes contain one more pigeon than other pigeonholes. It depends on the number of pigeonholes and number of pigeons.

Each user can use this principle to find focal points from all possible planned strategy set. In this case number of time slots m will be considered as pigeonholes and throughput demand will be considered as pigeons. Therefore in every slot, the user can transmit at least $\lfloor (D^{(n)}-1)/m \rfloor$ packets and in some time slot he/she can transmit $\lfloor (D^{(n)}-1)/m \rfloor + 1$ packets. Now we will try to find the number of time slots that contains

$$\lfloor (D^{(n)}-1)/m \rfloor + 1 \text{ packets.}$$

$D^{(n)}$ = Throughput demand of user n

m = Number of time slots

In a time slot at least $\lfloor (D^{(n)}-1)/m \rfloor$ packets can be transmitted.

Therefore in m time slots at least $\lfloor (D^{(n)}-1)/m \rfloor * m$ can be transmitted Remaining packets = $D^{(n)} - \left(\lfloor (D^{(n)}-1)/m \rfloor * m \right)$

Finally, the user n transmits $\lfloor (D^{(n)}-1)/m \rfloor + 1$ packets in $D^{(n)} - \left(\lfloor (D^{(n)}-1)/m \rfloor * m \right)$ time slots. This kind of distribution of packets will give the focal points to the user.

The user can get so many combinations from m time slot which contain $\lfloor (D^{(n)}-1)/m \rfloor + 1$ packets. All these

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 5, Issue 9, March 2016

combinations will be the focal points for the user or user can decide planned strategy set based on these focal points.

The user can get benefit to choose first $D^{(n)} - \left(\left\lfloor (D^{(n)} - 1)/m \right\rfloor * m\right)$ slots to transmit $\left\lfloor (D^{(n)} - 1)/m \right\rfloor + 1$ packets. In other words if the user transmits $\left\lfloor (D^{(n)} - 1)/m \right\rfloor + 1$ packets in the starting time slots then the cost will be minimum for that user.

Again we have two users in this game and the link capacity λ is fixed. If both users follow above stated rule to find minimum cost there may be a violation of capacity constraint and this combination will not produce a Nash equilibrium point in the game and hence it will not be the solution of the game.

Now we think about the social optimal point i.e. The "Pareto optimal point". In spite of minimizing cost of user 1 or user 2, let us try to minimize the cost of network or improve the performance of the network which is possible only at the Pareto optimal point.

B. Procedure to find the Pareto optimal point

Let throughput demand of the first user be greater than the second user i.e. $D^{(1)} > D^{(2)}$ and demands of both users can satisfy by the link with capacity λ in m time slot. Mathematically $D^{(1)} + D^{(2)} = m\lambda$. Also consider that

$$\alpha_1 = D^{(1)} - \left(\left\lfloor \frac{D^{(1)} - 1}{m} \right\rfloor * m\right) \ and \ \beta_1 = \left\lfloor \frac{D^{(1)} - 1}{m} \right\rfloor$$

$$\alpha_2 = D^{(2)} - \left(\left\lfloor \frac{D^{(2)} - 1}{m} \right\rfloor * m\right) \ and \ \beta_2 = \left\lfloor \frac{D^{(2)} - 1}{m} \right\rfloor$$

For the Pareto optimal point the first user should transmit $\beta_1 + 1$ packets in 1 to α_1 time slots and β_1 packets in $\alpha_1 + 1$ to m time slots. Similarly the second user should transmit $\beta_2$packets in 1 to α_2 time slots and $\beta_2 + 1$ packets in $\alpha_2 + 1$ to m time slots. Mathematically above scheme can be expressed as

$$C^{(1)} = \sum_{i=1}^{m} C_i^{(1)} = \sum_{i=1}^{\alpha_1} C_i^{(1)} + \sum_{i=\alpha_1+1}^{m} C_i^{(1)}$$

$$= \emptyset(\beta_1 + 1, \lambda) \sum_{i=1}^{\alpha_1} f(i) + \emptyset(\beta_1, \lambda) \sum_{i=\alpha_1+1}^{m} f(i)$$

Similarly for second user

$$C^{(2)} = \emptyset(\beta_2, \lambda) \sum_{i=1}^{\alpha_2} f(i) + \emptyset(\beta_1 + 1, \lambda) \sum_{i=\alpha_2+1}^{m} f(i)$$

Illustrative Examples

We present three examples to illustrate the solution approach described in previous sections.

<u>Example 1:</u> **Unequal and Even number of demands**

Consider a communication network with two nodes connected with a single link and two users. Let the link capacity is $\lambda = 6$ and time slots $m = 5$ i.e. in these 5 time slots both users can transmit 30 packets together. Assume that throughput demand of user 1 ($D^{(1)}$) is 18 and throughput demand of user 2 ($D^{(2)}$) is 12. These demands satisfied the condition $D^{(1)} + D^{(2)} = m\lambda$

Now we will try to find the number of ways by which user 1 and user 2 can transmit their 18 and 12 packets respectively in 5 different time slots over the link with capacity 6.

For user 1, we have 1190 ways to transmit 18 packets in 5 time slots over the link. Similarly there are 1190 ways for user 2 to transmit 12 packets in 5 time slots over the link. If both users transmit their packets simultaneously then total number of possible combinations is

$1190 \times 1190 = 14{,}16{,}100$

i.e all possible strategy set contains 14,16,100 elements, but all these combinations are not feasible. Now the question may arise that "How many combinations are feasible from this set?"

To answer this question we can use the concept of NEP and equation (10) discussed in the previous section. In which we have

$$p_i = r_i = u_i \text{ and } q_i = s_i = v_i$$

This show that if planned strategy set become an actual strategy set for both users then all packets will be transmitted smoothly in each time slot.

For every point in all possible strategy set of user 1, user 2 has a strategy which satisfies the equation (8) and vice versa. Hence we get 1190 Nash equilibrium points. This can be explained in the following table:

S. No.	Planned strategy set of User1 for 18 packets $\{p_1, p_2, p_3, p_4, p_5\}$	Planned strategy set of User2 for 12 packets $\{q_1, q_2, q_3, q_4, q_5\}$	Packets transmitted in each time slot
1	{6,6,6,0,0}	{0,0,0,6,6}	{6,6,6,6,6}
2	{5,6,3,4,0}	{1,0,3,2,6}	{6,6,6,6,6}
3	{4,4,3,2,5}	{2,2,3,4,1}	{6,6,6,6,6}
⋮	⋮	⋮	⋮

Table 1: different combination of strategies

Next step is to find focal points for both users from 1190 Nash equilibrium points. We will use the Extended Pigeonhole principle described in the previous section.

$D^{(1)} = 18 \ and \ D^{(2)} = 12$

For user 1

$$\left\lfloor (D^{(1)} - 1)/m \right\rfloor = \lfloor (18 - 1)/5 \rfloor = 3$$

And $D^{(1)} - \left(\left\lfloor (D^{(1)} - 1)/m \right\rfloor * m\right) = 18 - (3 * 5) = 3$

This implies that user 1 should transmit 3 packets in 2 time slots and 4 packets in 3 time slots. Similarly we can calculate condition to obtain focal points for user 2 also. For user 2, it will be 2 packets in 3 time slots and 3 packets in 2 time slots. There will be 10 permutations which satisfy the condition stated above. These permutations, respective cost of the individual user and the cost of the network are shown below:

S. No	**Planned strategy of user1 and user2**	**Cost for user 1 $C^{(1)}$**	**Cost for user 2 $C^{(2)}$**	**Total Cost of the network**
1	User1= {4,4,3,3,4} User 2= {2,2,3,3,2}	320.47	100.94	421.41
2	User1= {4,3,4,4,3} User 2= {2,3,2,2,3}	323.30	99.90	423.20
3	User1= {4, 3,3,4,4}	340.69	93.50	434.19

	User 2= {2, 3,3,2,2}			
4	User1= {3,4,4,4,3} User 2= {3,2,2,2,3}	337.59	94.64	432.23
5	User1= {3,4,3,4,4} User 2= {3,2,3,2,2}	354.99	88.24	443.23
6	**User1= {4,4,4,3,3} User 2= {2,2,2,3,3}**	**303.08**	**107.34**	**410.42**
7	User1= {4,4,3,4,3} User 2= {2,2,3,2,3}	312.33	103.94	416.27
8	User1= {4,3,4,3,4} User 2= {2,3,2,3,2}	331.44	96.90	428.34
9	User1= {3,4,4,3,4} User 2= {3,2,2,3,2}	345.74	91.64	437.38
10	User1= {3,3,4,4,4} User 2= {3,3,2,2,2}	365.96	84.20	450.16

Table 2: Strategies of user 1& 2 and their cost

All above 10 combinations are focal points for both users. Using these focal points we can find the Pareto optimal point in this game. The minimum cost of the network can be obtained using 6th strategy set which will be the Pareto optimal point.

Note that this strategy set does not generate a minimum cost for the user with less throughput demand, though it generates minimum cost of the complete network.

IX. CONCLUSION

In this work, we attempted to present mathematical modeling of transmission in a communication network, using game theoretical concept with multiple chances available to users. The cost function involves the number of packets to be routed and time variables, in a non-linear fashion. Penalty to the users are also introduced in the cost function. We have proved theorems which show the existence of Nash equilibrium point in this non- cooperative game. We have obtained sufficient condition for the Nash equilibrium point. According to these conditions game have multiple Nash equilibrium points. We developed a procedure to find Pareto optimal point from these multiple Nash equilibrium points. The examples also demonstrated based on that procedure.

Despite the results accomplished so far, there is space for more detailed investigation for multiuser; complex network with non-symmetrical links (i.e. links with different speed). Furthermore, different demands and different source and destination seem to play a critical role in this packet transmission that has not been investigated in detail yet.

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