

Transient thermo elastic problem of a semi-infinite circular beam

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Abstract- This paper is concerned with transient thermo elastic problem in which we need to determine the temperature distribution, displacement function and thermal stresses of a semi-infinite circular beam when the boundary conditions are known. Integral transform techniques are used to obtain the solution of the problem.

Key Words: Semi-infinite circular beam, transient problem, Integral transform, internal heat source.

I. INTRODUCTION

Khobragade et al. [2-16] studied the various problems on circular plate, rectangular plate and hollow cylinder.

In this paper, we analyzed inverse thermo elastic problem of temperature and thermal stresses of thick, semi-infinite circular beam due to heat generation. The governing heat conduction equation has been solved by using Marchi-Zgrablich and Fourier Cosine transform techniques. The result presented here will be more useful in engineering applications.

II. STATEMENT OF THE PROBLEM

Consider a thick circular beam occupying the space D: $a \leq r \leq b$, $0 \leq z < \infty$. The material is homogeneous and isotropic. The differential equation governing the displacement potential function $\phi(r, z, t)$ as Noda et al. [1] is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \left[\frac{1+\nu}{1-\nu} \right] \alpha_t T \quad (1)$$

Where, ν and α_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate and T is temperature of the plate satisfying the differential equation as Noda et al. [1] is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \chi(r, z, t) = \frac{1}{k} \frac{\partial T}{\partial t} \quad (2)$$

Subject to initial condition:

$$T(r, z, 0) = f(r, z) \quad (3)$$

and boundary conditions are

$$\left[T(r, z, t) + k_1 \frac{\partial T(r, z, t)}{\partial r} \right]_{r=a} = g_1(z, t) \quad (4)$$

$$\left[T(r, z, t) + k_2 \frac{\partial T(r, z, t)}{\partial r} \right]_{r=b} = g_2(z, t) \text{ (Known)} \quad (5)$$

$$[T(r, z, t)]_{r=b} = G(z, t) \text{ (Unknown)} \quad (6)$$

$$\left[\frac{\partial T(r, z, t)}{\partial z} \right]_{z=0} = f_1(r, t) \quad (7)$$

$$\left[\frac{\partial T(r, z, t)}{\partial z} \right]_{z=\infty} = f_2(r, t), \quad 0 \leq r \leq a, \quad t > 0 \quad (8)$$

Where k is the thermal diffusivity of the material of the plate. The displacement function in the cylindrical co-ordinate system are represented by the Goodier thermo elastic function ϕ and Love's function L as Noda et al. [1] are

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 L}{\partial r \partial z} \quad (9)$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu) \nabla^2 L - \frac{\partial^2 L}{\partial z^2} \quad (10)$$

in which Goodier thermo elastic potential must satisfy the equation as Noda et al. [1] is

$$\nabla^2 \phi = \left(\frac{1+\nu}{1-\nu} \right) \alpha_t T \quad (11)$$

The Love's function must satisfy

$$\nabla^2 (\nabla^2 L) = 0 \quad (12)$$

where,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The component of stresses are represented by the use of the potential ϕ and Love's function L as Noda et al. [1] are

$$\sigma_{rr} = 2G \left\{ \left[\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi \right] + \frac{\partial}{\partial z} \left[\nu \nabla^2 L - \frac{\partial^2 L}{\partial r^2} \right] \right\} \quad (13)$$

$$\sigma_{\theta\theta} = 2G \left\{ \left[\frac{1}{r} \frac{\partial \phi}{\partial r} - \nabla^2 \phi \right] + \frac{\partial}{\partial z} \left[\nu \nabla^2 L - \frac{1}{r} \frac{\partial^2 L}{\partial r^2} \right] \right\} \quad (14)$$

$$\sigma_{zz} = 2G \left\{ \left[\frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi \right] + \frac{\partial}{\partial z} \left[(z-\nu) \nabla^2 L - \frac{\partial^2 L}{\partial z^2} \right] \right\} \quad (15)$$

$$\sigma_{rz} = 2G \left\{ \frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left[(1-\nu) \nabla^2 L - \frac{\partial^2 L}{\partial z^2} \right] \right\} \quad (16)$$

Equations (1) to (16) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE PROBLEM

Applying finite **Marchi-Zgrablich transform** defined in [3] to the equations (2) and using equations (4), (5) one obtains

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \chi(r, z, t) = \frac{1}{k} \frac{\partial T}{\partial t} \quad (17)$$

By using the operational property of finite Marchi-Zgrablich transform, we get

$$\frac{\partial^2 \bar{T}}{\partial z^2} - \mu_n^2 \bar{T} + \bar{\chi} = \frac{1}{k} \frac{\partial \bar{T}}{\partial t} + g(z, t) \quad (18)$$

Again, applying Fourier cosine transform to the equation (18), we get

$$\frac{d \bar{T}_c^*}{dt} + kp^2 \bar{T}_c^* = \bar{\phi}_1^* + \bar{\chi}_1^* \quad (19)$$

where

$$\bar{\chi}_1^* = k \bar{\chi}_c^* \quad \text{and} \quad \bar{\phi}_1^* = k\mu - k\mu_n^2 \bar{T}_c^* - k g_c^*$$

Equation (19) is a linear equation whose solution is given by

$$\bar{T}^* (n, z, t) = e^{-kp^2 t} \int_0^t (\bar{\phi}_1^* + \bar{\chi}_1^*) e^{-kp^2 t'} dt' + C e^{-kp^2 t} \quad (20)$$

Using (3), we get

$$C = F^* (m, n)$$

Thus, we have,

$$\bar{T}^* (n, z, t) = e^{-kp^2 t} \left[\int_0^t (\bar{\phi}_1^* + \bar{\chi}_1^*) e^{-kp^2 t'} dt' + \bar{F}^* (m, n) \right] \quad (21)$$

Applying inversion of Fourier cosine transform and Marchi-Zgrablich transform to the equation (21), one obtains

$$T(r, z, t) = \sum_{n=1}^{\infty} \frac{1}{C_n} \left\{ e^{-kp^2 t} \left[\int_0^t (\bar{\phi}_1^* + \bar{\chi}_1^*) e^{-kp^2 t'} dt' + \bar{F}^* (m, n) \right] \right\} \times S_0(k_1, k_2, \mu_n r) \quad (22)$$

$$G(z, t) = \sum_{n=1}^{\infty} \frac{1}{C_n} \left\{ e^{-kp^2 t} \left[\int_0^t (\bar{\phi}_1^* + \bar{\chi}_1^*) e^{-kp^2 t'} dt' + \bar{F}^* (m, n) \right] \right\} \times S_0(k_1, k_2, \mu_n b) \quad (23)$$

These are the desired solutions of the given problem. Let us assume Love's function L, which satisfy condition (11) as

$$L(r, z) = \sum_{n=1}^{\infty} \frac{1}{C_n} \psi S_0(k_1, k_2, \mu_n r) \quad (24)$$

Where,

$$\psi = e^{-kp^2 t} \left[\int_0^t (\bar{\phi}_1^* + \bar{\chi}_1^*) e^{-kp^2 t'} dt' + \bar{F}^* (m, n) \right]$$

The displacement potential is given by

$$\phi = A \sum_{n=1}^{\infty} \frac{1}{C_n} \psi S_0(k_1, k_2, \mu_n r) [\psi + B(t)] \quad (25)$$

$$\text{where, } A = \left(\frac{1+\nu}{1-\nu} \right) \alpha_t$$

$$B(t) = e^{-kp^2 t} \left[\int_0^t (\bar{\phi}_1^* + \bar{\chi}_1^*) e^{-kp^2 t'} dt' + \bar{F}^* (m, n) \right] dt$$

IV. DETERMINATION OF DISPLACEMENT FUNCTION

Substituting the equations (24) and (25) in the equation (8) one obtains

$$u_r = A \sum_{n=1}^{\infty} \frac{\mu_n}{C_n} S_0'(k_1, k_2, \mu_n r) [\psi + B(t)] - \sum_{n=1}^{\infty} \frac{\mu_n}{C_n} \psi S_0'(k_1, k_2, \mu_n r) \quad (26)$$

$$u_z = 2(1-\nu)$$

$$\left[\sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} \psi S_0'(k_1, k_2, \mu_n r) + \frac{1}{r} \sum_{n=1}^{\infty} \frac{\mu_n}{C_n} \psi S_0'(k_1, k_2, \mu_n r) \right] \quad (27)$$

V. DETERMINATION OF STRESS FUNCTIONS

Substituting the values from the equation (24) and (25) in the equation (10) to (13) we get

$$\sigma_{rr} = 2G \left\{ \begin{aligned} & \left[\begin{aligned} & A \sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} S_0''(k_1, k_2, \mu_n r) [\psi + B(t)] \\ & - A \sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} S_0''(k_1, k_2, \mu_n r) [\psi + B(t)] - \\ & \frac{A}{r} \sum_{n=1}^{\infty} \frac{\mu_n}{C_n} S_0'(k_1, k_2, \mu_n r) [\psi + B(t)] \end{aligned} \right] \\ & + \frac{\partial}{\partial z} \left[\begin{aligned} & \left[\begin{aligned} & A \sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} S_0''(k_1, k_2, \mu_n r) \\ & + \frac{A}{r} \sum_{n=1}^{\infty} \frac{\mu_n}{C_n} \psi S_0'(k_1, k_2, \mu_n r) \end{aligned} \right] \\ & - \sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} S_0''(k_1, k_2, \mu_n r) \end{aligned} \right] \end{aligned} \right\} \quad (28)$$

$$\sigma_{\theta\theta} = 2G \left\{ \begin{aligned} & \left[\begin{aligned} & \frac{A}{r} \sum_{n=1}^{\infty} \frac{\mu_n}{C_n} S_0'(k_1, k_2, \mu_n r) [\psi + B(t)] \\ & \left[\begin{aligned} & A \sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} S_0''(k_1, k_2, \mu_n r) [\psi + B(t)] + \\ & \frac{A}{r} \sum_{n=1}^{\infty} \frac{\mu_n}{C_n} S_0'(k_1, k_2, \mu_n r) [\psi + B(t)] \end{aligned} \right] \\ & + \frac{\partial}{\partial z} \left[\begin{aligned} & \left[\begin{aligned} & A \sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} \psi S_0''(k_1, k_2, \mu_n r) \\ & + \frac{1}{r} \sum_{n=1}^{\infty} \frac{\mu_n}{C_n} \psi S_0'(k_1, k_2, \mu_n r) \end{aligned} \right] \\ & - \frac{1}{r} \sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} \psi S_0''(k_1, k_2, \mu_n r) \end{aligned} \right] \end{aligned} \right\} \quad (29)$$

$$\sigma_{zz} = 2G \left\{ \begin{aligned} & \left[\begin{aligned} & A \sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} S_0''(k_1, k_2, \mu_n r) [\psi + B(t)] \\ & + \frac{A}{r} \sum_{n=1}^{\infty} \frac{\mu_n}{C_n} S_0'(k_1, k_2, \mu_n r) [\psi + B(t)] \end{aligned} \right] \\ & + (1-\nu) \sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} \psi S_0''(k_1, k_2, \mu_n r) \\ & + \frac{(1-\nu)}{r} \sum_{n=1}^{\infty} \frac{\mu_n}{C_n} \psi S_0'(k_1, k_2, \mu_n r) \end{aligned} \right\} \quad (30)$$

$$\sigma_{rz} = 2G \left[(1-\nu) \sum_{n=1}^{\infty} \frac{\mu_n^3}{C_n} \psi S_0'''(k_1, k_2, \mu_n r) - \frac{(1-\nu)}{r^2} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu_n^2}{C_n} \psi S_0''(k_1, k_2, \mu_n r) \right] \quad (31)$$

Where,

$$\begin{aligned} A &= \left(\frac{1+\nu}{1-\nu} \right) \alpha_t \text{ and } \psi \\ &= e^{-kp^2 t} \left[\int_0^t (\phi_1^* + \bar{\chi}_1^*) e^{kp^2 t'} dt' + \bar{F}^*(m, n) \right] \end{aligned}$$

$$B(t) = \int \psi dt$$

VI. SPECIAL CASE

$$\text{Set } F(r, z) = \delta(r - r_0)(z - e^{-z}) \quad (32)$$

Applying finite transform defined in Marchi Zgrablich [35] to the equation (32) one obtains

$$\bar{F}(n, z) = r_0 (z - e^{-z}) S_0(k_1, k_2, \mu_n r_0) \quad (33)$$

Substituting the value of (33) in the equations (22) to (23) one obtains

$$\begin{aligned} T(r, z, t) &= \sum_{n=1}^{\infty} \frac{1}{C_n} \left\{ e^{-kp^2 t} \left[\int_0^t (\phi_1^* + \bar{\chi}_1^*) e^{-kp^2 t'} dt' + \bar{F}^*(m, n) \right] \right\} \\ &\times S_0(k_1, k_2, \mu_n r) \end{aligned} \quad (34)$$

Where

$$\bar{F}^*(n, m) = r_0 S_0(k_1, k_2, \mu_n r_0) \int_0^{\infty} (z - e^{-z}) \cos \alpha z dz$$

$$G(z,t) = \sum_{n=1}^{\infty} \frac{1}{C_n} \left\{ e^{-kp^2t} \left[\int_0^t (\phi_1^* + \chi_1^*) e^{-kp^2t'} dt' + \bar{F}^*(m,n) \right] \right\} \times S_0(k_1, k_2, \mu_n b) \tag{35}$$

VII. NUMERICAL RESULTS

Put $a = 2, \xi = 2.3, b = 2.5, t = 1\text{sec}$ in equations (34) to (35) one obtains

$$T(r,z,t) = \sum_{n=1}^{\infty} \frac{1}{C_n} \left\{ e^{-kp^2t} \left[\int_0^1 (\phi_1^* + \chi_1^*) e^{-kp^2t'} dt' + \bar{F}^*(m,n) \right] \right\} \times S_0(k_1, k_2, \mu_n r) \tag{36}$$

$$G(z,t) = \sum_{n=1}^{\infty} \frac{1}{C_n} \left\{ e^{-kp^2t} \left[\int_0^1 (\phi_1^* + \chi_1^*) e^{-kp^2t'} dt' + \bar{F}^*(m,n) \right] \right\} \times S_0(k_1, k_2, \mu_n (2.5)) \tag{37}$$

VIII. MATERIAL PROPERTIES

The numerical calculation has been carried out for an Aluminum (pure) circular plate with the material properties as

Density $\rho = 169 \text{ lb/ft}^3$

Specific heat = 0.208 Btu/lbOF

Thermal conductivity $K = 15.9 \times 10^6 \text{ Btu/(hr. ftOF)}$

Thermal diffusivity $\alpha = 3.33 \text{ ft}^2/\text{hr.}$

Poisson ratio $\nu = 0.35$

Coefficient of linear thermal expansion

$\alpha_t = 12.84 \times 10^{-6} / \text{F}$

Lame constant $\mu = 26.67$

Young's modulus of elasticity $E = 70 \text{ G Pa}$

IX. DIMENSIONS

The constants associated with the numerical calculation are taken as

Radius of the disk $a = 2\text{ft}$

Radius of the disk $b = 2.5 \text{ ft}$

X. CONCLUSION

In this paper, we develop the analysis for the temperature field by introducing the methods of the Marchi- Zgrablich and Fourier cosine transform techniques and determined the expression for temperature distribution, displacement and thermal stresses of a semi-infinite, thick circular beam with known boundary conditions which is useful to design of structure or machines in engineering applications.

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