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The Mathematical Safe Problem Solving On The Matrices with a Prime Number of States

Yaghoub Aghaei Agh Ghamish (Iran) Kiev Polytechnic Institute, Kiev, Ukraine

***Abstract**- This paper considers the problem of mathematical safes on matrices. The safes with the same type locks with a plain number of states are studied. The solution of the problem, illustrated by examples, is proposed. Index terms- the mathematical safe. the locks of safe. a set of states locks, incidence matrix of locks, matrix of a system of linear equations, module of system equations, inverse matrix of system equations, matrix of solutions of system equations.*

I. INTRODUCTION

We consider the problem in some computer games. A matrix of zeros and ones is given. One move consists in selecting any element. Then all the zeros in row and column defining this element are converted into ones and ones are converted into zeros. It is required to find a sequence of moves that would lead to the formation of a matrix consisting of zeros (or ones) only. There exists a real interpretation of the problem: identical locks keyholes are arranged in form on the safe door. There are two states of each lock - it is open or closed. If a key is inserted into one of the keyholes and one turn is made, then the same turn will be made in all the locks of the same row and the same column. It is necessary to find a sequence of keyholes to open the safe turning their keys. This will happen when all the locks are open.

II. THE PROBLEM FORMULATING

The global mathematical safe problem is formulated in [1].

The problem.

System S(Z, b, $\langle \mathbf{Z} \rangle$) consisting of a set of locks Z = {z_1, z_2, . . . , z_N } , safe states vector $\mathrm{b} = (b_1, b_2, \ldots, b_N)$, where $b_i \in \{0,1,\ldots,k_i - 1\}$ is a state of i-th lock , and a set $\langle Z \rangle = \{Z_1, Z_2, \ldots, Z_N\}$, $z_l \notin Z_l$, $Z_l \in 2^Z (1 \le i, l \le N)$ is called the mathematical safe.

As a result of a key turning clockwise in the lock z_l all the locks $z_j \in Z_l$ move from state b_j into state $(b_j + 1)(\bmod k_j)$. The safe is considered open if its state is $b = (0,0,\ldots,0) = b_{fin}$. It is necessary to find for each lock z_j number x_i of the key turns to open the safe.

We call vector $\mathbf{X} = (x_1, x_2, \ldots, x_N)$ the safe problem solution. The set $\langle \mathbf{Z} \rangle$ is called the incidence set. It can be written in the form of the incidence matrix $\mathbf{A_0} = a_{ij}^0$ of size $N \times N$ with zeros on the main diagonal and $a_{ij}^0 = 1$ if z_j belongs to the set $Z_i (1 \le i, j \le N)$ and zero otherwise. It is obvious that x key clockwise turns in lock z_l are equivalent to $k_l - x$ counterclockwise turns. Matrix $\mathbf{A_0}$ can be associated with a directed graph $\mathbf{G}(\mathbf{Z})$, where z_i vertex is connected with z_j vertex when $a_{ij}^0 = 1$. Different mathematical safe problems arise depending on the complexity of this matrix

We denote $\mathbf{A} = \mathbf{A_0} + \mathbf{E}_N$ where $\mathbf{E}_N$ identity matrix is. In its column corresponding to j-th lock ones are situated in front of the locks affecting j-th lock state. Taking into account number of all turns in these locks and number of turns x_j in j-th lock, we obtain total number of turns in j-th lock. Adding it to the initial state of j-th lock we have to get 0 (mod k_j). Then the global safe problem is reduced to solving of linear comparisons

$$\mathbf{X}\bar{\mathbf{a}}_\mathbf{i} + b_i \equiv 0(\bmod k_i), (1 \le i \le N),$$

where $\bar{\mathbf{a}}_\mathbf{i}$ is i-th column of matrix A We suppose that the initial state of safe **B** is known or at least easy calculated. If $k_i = K = const$ for all $1 \le i \le N$ the locks are called of one type. It follows from the problem formulation that the solution realization does not depend on the order of key turns in locks. So depending on the problem type we give examples in which the key turn sequence will be different. In [2] and [3] safes defined on oriented or non-oriented graphs was studied. It follows from the global definition of the mathematical safe that any of them can be set using adjacency matrix of some graph. However, there exist mathematical safes with the specificity that allows setting them using matrices.

Consider generally safe as defined above, in which all locks are arranged in a rectangular table of size $m \times n$. For it $N = mn$, $l = n(i-1) + j$ *(i = 1, 2,..., M; j = 1, 2, .., n)*. Denote Z_l the set of locks combining i-th row and j-th column locks, and let all the locks have arbitrary number of states, and belong to the same type, that is $k_l = K$. To any initial state $\vec{b}$ the safe matrix $B = (b_{ij})_{m,n}$ where $b_{ij} \in \{0,1,\ldots,K-1\}$ corresponds. It is necessary to find such a sequence of locks and the corresponding number of turns in them to "open the safe",

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that is to go into the safe state $B_{fin} = (b_{ij} = 0)_{m,n}$. Let $X = (x_{ij})_{m,n}$ be a solution to the problem, where x_{ij} is the number of turns of the key in the lock Z_l. Then the condition of transforming an element b_{ij} of the matrix X into zero is represented by the ratio

$$\sum_{k=1}^{n} x_{ik} + \sum_{\substack{k=1 \\ k\neq i}}^{m} x_{kj} + b_{ij} \equiv 0 \,(\mathrm{mod}\, K)\ , \qquad (1)$$

where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

III. THE PROBLEM SOLUTION

Let us denote $\vec{x} = (x_{11}, x_{12}, \ldots\ldots, x_{1n}, x_{21}, x_{22}, \ldots\ldots, x_{2n}, \ldots\ldots, x_{m,n-1}, x_{mn})$ column vector obtained from the matrix X by sequentially recording her lines.

Similarly, from the matrix B we obtain the column vector $\vec{b}$. In addition, let $\Im_n$ be a matrix of size $n \times n$, consisting of ones, E_n be the identity matrix of the same size, and I_n be a row vector of n ones. The condition of transformation (1) for the whole matrix B can be written as a system of equations

$$A\vec{x} + \vec{b} \equiv 0 (\mathrm{mod}\, K) \qquad (2)$$

Where matrix A of size $mn \times mn$ consists of m^2 sells:

$$A = \begin{pmatrix} \Im_n & E_n & E_n & \ldots & E_n \\ E_n & \Im_n & E_n & \ldots & E_n \\ E_n & E_n & \Im_n & \ldots & E_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ E_n & E_n & E_n & \ldots & \Im_n \end{pmatrix} \qquad (3)$$

This problem can easily be reduced to the solving of a system of linear Diophantine equations, if you add the to the right-hand sides of (2) the terms Ky_i, (i = 1, 2, ..., mn), using known methods described in[4]. However, the specifics of the problem allows to find its solution directly, since the matrix A is standard and does not depend on the values of B. Its rank and determinant depend only on the values of m and n. In [5] matrices of zeros and ones were studied.

Since the matrix A is symmetric, hereinafter all the arguments concerning the rows of the matrix hold for the columns of the same name, and vice versa.

If the matrix A rank is mn, then system (2) is

$$\vec{x} = -A^{-1}\vec{b} \,(\mathrm{mod}\, K) \qquad (4)$$

Thus, the problem is reduced to finding the inverse matrix A^{-1}. In the general case for arbitrary m, n and K, it may not exist. Then the system (2) may have a solution if the initial state satisfies certain constraints. Therefore, in such cases, the problem arises to correct the initial state so that the problem solution exists.

Consider the symmetric square matrix of order **n**, depending on two parameters $H_n(\alpha,\beta) = (\alpha - \beta)E_n + \beta\Im_n$.

$$H_n(\alpha,\beta) = \begin{pmatrix} \alpha & \beta & \beta & \ldots & \beta \\ \beta & \alpha & \beta & \ldots & \beta \\ \beta & \beta & \alpha & \ldots & \beta \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta & \beta & \beta & \ldots & \alpha \end{pmatrix} \qquad (5)$$

We use it to build a square matrix of order mn depending on four parameters and consisting of m^2 submatrices

$$T_{m,n}(\alpha,\beta,\gamma,\delta) = \begin{pmatrix} H_n(\alpha,\beta) & H_n(\gamma,\delta) & \ldots & H_n(\gamma,\delta) \\ H_n(\gamma,\delta) & H_n(\alpha,\beta) & \ldots & H_n(\gamma,\delta) \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ H_n(\gamma,\delta) & H_n(\gamma,\delta) & \ldots & H_n(\alpha,\beta) \end{pmatrix} \qquad (6)$$

Matrices of this type will be called T-matrices. In this notation, the identity matrix and the matrix A are also T-matrices, namely, $E_{mn} = T_{m,n}(1,0,0,0)$, and $A = T_{mn}(1,1,1,0)$. We will calculate the inverse matrix A^{-1} of system (2) in the form of T-matrix $A^{-1} = T_{m,n}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. In [6] values for α_i are given.

$$\left.\begin{aligned} \alpha_1 &\equiv \frac{1}{m-1} + \frac{1}{n-1} - 1 + \alpha_4 \\ \alpha_2 &\equiv \frac{1}{n-1} + \alpha_4 \\ \alpha_3 &\equiv \frac{1}{m-1} + \alpha_4 \\ \alpha_4 &\equiv -\left(\frac{1}{n-1} + \frac{1}{m-1}\right)\frac{1}{m+n-1} \end{aligned}\right\} (\mathrm{mod}\, K) \qquad . \qquad (7)$$

Consider the inverse matrix A^{-1}, provided its parameters are given in (7), and find an explicit expression for the solution of system (2) in the form (4) for a prime K. As mentioned above, the vector $\vec{b}$ (and $\vec{x}$ also) can be expressed as an element of the matrix B' (respectively X) as $b_l = b_{n\lambda+\eta}$, where

$$\lambda = 0,1,\ldots,m-1; \qquad \eta = 1,2,\ldots,n. \qquad (8)$$

The multiplication of the inverse matrix A^{-1} by the vector $\vec{b}$ can be represented as the parameters $\alpha_i (i = 1,2,3,4)$ multiplication by the elements of the matrix B according

$$\begin{aligned} &\lambda = i; \quad \eta = j \quad \sim\alpha_1; \\ &\lambda = i; \quad \eta \neq j \quad \sim\alpha_2; \qquad (9) \\ &\lambda \neq i; \quad \eta = j \quad \sim\alpha_3; \end{aligned}$$

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$\lambda \neq i; \quad \eta \neq j \quad \sim \alpha_4.$

It can be displayed in a matrix

$$\lambda = i \begin{pmatrix} \alpha_4 & \ldots & \alpha_4 & \overset{\eta=j}{\alpha_3} & \alpha_4 & \ldots & \alpha_4 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_4 & \ldots & \alpha_4 & \alpha_3 & \alpha_4 & \ldots & \alpha_4 \\ \alpha_2 & \ldots & \alpha_2 & \alpha_1 & \alpha_2 & \ldots & \alpha_2 \\ \alpha_4 & \ldots & \alpha_4 & \alpha_3 & \alpha_4 & \ldots & \alpha_4 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_4 & \ldots & \alpha_4 & \alpha_3 & \alpha_4 & \ldots & \alpha_4 \end{pmatrix} \quad (10)$$

Then system (2) solution can be written

$$x_{ij} \equiv b_{ij}\left(1 - \alpha_4 - \frac{1}{m-1} - \frac{1}{n-1}\right) + \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{m} b_{\lambda j}\left(-\alpha_4 - \frac{1}{m-1}\right) \quad (11)$$

$$+ \sum_{\substack{\eta=1 \\ \eta \neq j}}^{n} b_{i\eta}\left(-\alpha_4 - \frac{1}{n-1}\right) + \sum_{\substack{\lambda=1 \\ \lambda \neq i}}^{m} \sum_{\substack{\eta=1 \\ \eta \neq j}}^{n} b_{\lambda\eta}(-\alpha_4)$$

It is easy to convert to the form

$$x_{ij} = b_{ij} - \frac{1}{m-1} \cdot \sum_{\lambda=1}^{m} b_{\lambda j} - \frac{1}{n-1} \cdot \sum_{\eta=1}^{n} b_{i\eta} + \sum_{\lambda=1}^{m} \sum_{\eta=1}^{n} b_{\lambda\eta}(-\alpha_4).$$

Denote

$$\sum_{\lambda=1}^{m} \sum_{\eta=1}^{n} b_{\lambda\eta} = \sum(b); \sum_{\lambda=1}^{m} b_{\lambda j} = \lambda_j; \sum_{\eta=1}^{n} b_{i\eta} = \sigma_i.$$

Then

$$x_{ij} \equiv \left[b_{ij} + \frac{1}{m-1}\left(\frac{\sum(b)}{m+n-1} - \lambda_j\right) + \frac{1}{n-1}\left(\frac{\sum b}{m+n-1} - \sigma_i\right)\right](\bmod \mathbf{K}) \quad (12)$$

If none of the conditions (7) is satisfied, $\det S \equiv 0(\bmod \mathrm{K})$ and the system (2) may not have a solution. Depending on the parameters settings four principal cases appear.

(V_1)

$$m + n \equiv 1(\bmod \mathrm{K}), m \neq 1(\bmod \mathrm{K}), n \neq 1(\bmod \mathrm{K}).$$

In this case, all fractional expressions with $m+n-1$ in the denominator turn in uncertainty. In order to this uncertainty could be an integer, it is necessary to satisfy the condition

$$\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} \equiv 0(\bmod \mathrm{K}) \quad (13)$$

The rest of fractions, as before, correspond to some integers. Under the condition (13) the uncertainty may be expressed as

$$\frac{\sum(b)}{m+n-1} = \rho. \quad (14)$$

Substitute this value in (12), and then (12) substitute in (1)

$$\sum_{k=1}^{n} b_{ik} + \frac{1}{m-1}\left[n\rho - \sum(b)\right] + \frac{n}{n-1}(\rho - \sigma_i) + \left(\sum_{\substack{k=1 \\ k \neq i}}^{m} b_{kj} + b_{ij}\right)$$

$$+ \rho - \lambda_j + \frac{1}{n-1}\left[(m-1)\rho - \sum(b) + \sigma_i\right]$$

After the simplification we obtain

$$\rho(m+n-1) - \sum(b) \equiv 0(\bmod \mathrm{K}),$$

as this is equivalent to (14). This relationship holds for any ρ, so the system (2) has K solutions

$$x_{ij} \equiv \left[b_{ij} + \frac{1}{m-1}(\rho - \lambda_j) + \frac{1}{n-1}(\rho - \sigma_i)\right](\bmod \mathrm{K}), \quad (15)$$

where $\rho \in \{0, 1, \ldots, \mathrm{K}-1\}$.

IV. INVESTIGATES

Example 1. Let K = 7, m = n = 4 and matrix B is given. As the condition (13) is not satisfied in matrix B, the framed element should be replaced, which will result in the matrix B', allowing the problem solution.

$$B = \begin{pmatrix} 3 & 5 & 4 & 3 \\ 4 & 3 & 0 & 5 \\ 3 & 4 & 3 & 2 \\ 6 & 0 & 5 & 4 \end{pmatrix}, \quad B' = \begin{pmatrix} 3 & 5 & 4 & 3 \\ 4 & 3 & 0 & 5 \\ 3 & 4 & 3 & 2 \\ 6 & 0 & 5 & 6 \end{pmatrix}.$$

Calculate values $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 5$, $\lambda_4 = 2$; $\sigma_1 = 1$, $\sigma_2 = \sigma_3 = 5$;

$$\frac{1}{m-1} = \frac{1}{n-1} \equiv -2(\bmod 7); \rho \in \{0, 1, \ldots, 6\}.$$

Then $\quad x_{ij} \equiv (b_{ij} + 2\lambda_j + 2\sigma_i + 3\rho)(\bmod 7).$

If we put $\rho \equiv 0$ and $\rho \equiv 4$, we obtain two solutions:

$$X^{(1)} = \begin{pmatrix} 2 & 3 & 2 & 2 \\ 4 & 2 & 6 & 5 \\ 3 & 3 & 2 & 2 \\ 2 & 2 & 0 & 2 \end{pmatrix}, \quad X^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 4 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{pmatrix}.$$

Obviously, the solution $X^{(2)}$ is preferable, since in it you must make 17 turns of the key (using only 7 locks) to open the safe, in contrast to the first solution, in which 42 key turns using 15 (almost all) locks must be done. Let us verify the second solution.

$$B' = \overset{+1}{\begin{pmatrix} 3 & 5 & 4 & 3 \\ 4 & 3 & 0 & 5 \\ 3 & 4 & 3 & 2 \\ 6 & 0 & 5 & 6 \end{pmatrix}} \to \overset{+2}{\begin{pmatrix} 4 & 6 & 5 & 4 \\ 4 & 4 & 0 & 5 \\ 3 & 5 & 3 & 2 \\ 6 & 1 & 5 & 6 \end{pmatrix}} \to \overset{+4}{\begin{pmatrix} 6 & 6 & 5 & 4 \\ 6 & 6 & 2 & 0 \\ 5 & 5 & 3 & 2 \\ 1 & 1 & 5 & 6 \end{pmatrix}} \to \overset{+3}{\begin{pmatrix} 6 & 6 & 2 & 4 \\ 3 & 3 & 6 & 4 \\ 5 & 5 & 0 & 2 \\ 1 & 1 & 2 & 6 \end{pmatrix}} \to$$

$$\to \overset{+1}{\begin{pmatrix} 6 & 6 & 2 & 0 \\ 6 & 6 & 2 & 0 \\ 5 & 5 & 0 & 5 \\ 1 & 1 & 2 & 2 \end{pmatrix}} \to \overset{+1}{\begin{pmatrix} 0 & 6 & 2 & 0 \\ 0 & 6 & 2 & 0 \\ 6 & 6 & 1 & 6 \\ 2 & 1 & 2 & 2 \end{pmatrix}} \to \overset{+5}{\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 2 & 2 \end{pmatrix}} \to \quad b_{fin}.$$

Selection of the optimal solution is not difficult. If in the formula for x_{ij} 3ρ we replace by ρ (this is equivalent for

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simple K), the second solution can be obtained from the first, if we put ρ = -2, i.e., each element of $X^{(1)}$ subtract 2. Taking into account these considerations represent (15) in the form

$$x_{ij} \equiv \left(b_{ij} - \frac{\lambda_j}{m-1} - \frac{\sigma_i}{n-1} + \rho\right)(\operatorname{mod} K), \text{где } \rho \in \{0,1,\ldots,K-1\}. \quad (16)$$

Now search for the optimal solution is reduced to the following steps: assume ρ = 0 and find the solution $X^{(1)}$ from (16); find the most common element $x_{ij}^* \in \mathbf{X}^{(1)}$; the optimal solution is $\mathbf{X}^* = \mathbf{X}^{(1)} - x_{ij}^* \cdot \tau$, where τ is a matrix of size $m \times n$, all the elements of which are equal to 1.

$$n \equiv 1(\operatorname{mod} K), m \neq 1(\operatorname{mod} K), m+n \neq 1(\operatorname{mod} K). \quad (V_2)$$

Under these conditions, the third term in (12) can not always be converted to an integer. This uncertainty is solvable only if

$$\frac{\sum(b)}{m+n-1} \equiv \sigma_i (\operatorname{mod} K), i = 1,2,\ldots,m. \quad (17)$$

Since the left-hand side of this expression is a constant, then all σ_i should be equal to each other, which can be expressed by the relations

$$\sum_{j=1}^{n} b_{1j} - \sum_{j=1}^{n-1} b_{ij} \equiv b_{in} (\operatorname{mod} K), (i = 2,3,\ldots,m). \quad (18)$$

Let us denote

$$\frac{1}{n-1}\left[\frac{\sum(b)}{m+n-1} - \sigma_i\right](\operatorname{mod} K) = p_i; i = 1,2,\ldots,m. \quad (19)$$

Summing up all the values of p_i, we obtain

$$\frac{1}{n-1}\left[\frac{m\sum(b)}{m+n-1} - \sum b\right] \equiv \sum_{i=1}^{m} p_i (\operatorname{mod} K).$$

If p_i take arbitrary values, it follows only limitation for them

$$\sum_{i=1}^{m} p_i + \frac{\sum b}{m+n-1} \equiv 0(\operatorname{mod} K). \quad (20)$$

This makes it possible to obtain K^{m-1} solutions of the form

$$x_{ij} \equiv \left[b_{ij} + \frac{1}{m-1}\left(\frac{\sum(b)}{m+n-1} - \lambda_i\right) + p_i\right](\operatorname{mod} K),$$
$$\sum_{i=1}^{m} p_i + \frac{\sum b}{m+n-1} \equiv 0(\operatorname{mod} K), \quad (21)$$
where $p_i \in \{0,1,\ldots,K-1\}, i = 1,2,\ldots,m.$

Example 2. Пусть K = 3, m = 5, n = 4 и задана матрица B, где элементы в рамке не удовлетворяют условию (18). Поэтому трансформируем её в матрицу B′.

$$B = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}.$$

Let us calculate the parameters:

$$\lambda_1 = 0; \lambda_2 = \lambda_3 = -1; \lambda_4 = 1; \sigma_i = 1, (i = 1,2,\ldots,5);$$
$$\frac{1}{m-1} = \frac{1}{4} \equiv 1(\operatorname{mod} 3); \frac{1}{m+n-1} = \frac{1}{8} \equiv 2(\operatorname{mod} 3); \sum(b) \equiv 2(\operatorname{mod} 3);$$
$$\frac{\sum(b)}{m+n-1} \equiv 1(\operatorname{mod} 3); \sum_{i=1}^{5} p_i = 2(\operatorname{mod} 3).$$

With these parameters, the solution of (21) will acquire the specific form.

$$x_{ij} \equiv \left[b_{ij} + (1-\lambda_j) + p_i\right](\operatorname{mod} 3),$$
$$\sum_{i=1}^{5} p_i \equiv 2(\operatorname{mod} 3); \quad (22)$$
$$i = 1,2,3,4,5; \ j = 1,2,3,4.$$

Consider the first solution when $p_1 = p_2 = p_3 = p_4 \equiv 0(\operatorname{mod} 3), \quad p_5 = \sum(b) = 2$.

The result is matrix $X^{(1)}$.

$$X^{(1)} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix}, \quad X^* = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

As it is known, the total number of solutions here is equal to 3^4=81. As in Example 2, we assume that the optimal solution is one in which the matrix X contains the most of zero elements. Let us denote it X*, and H(y) - matrix of size $m \times n$, in which i-th row is composed of elements y_i.

To calculate X*, it is necessary to solve the following problem: Find values y_i (i=1,2, …,m) such that

1) $\sum_{i=1}^{m} y_i \equiv 0(\operatorname{mod} K)$;

2) $X^{(1)}+H(\vec{\mathbf{y}})$ contains minimal number of zeros.

В данном примере $\vec{y} = (-1,1,1,0,-1)$, что приводит к матрице X*, указанной выше. Проверим это решение.

In this example $\vec{y} = (-1,1,1,0,-1)$, which results in a matrix X*, specified above. Check out this solution

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$$B' = \overset{-1}{\begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}} \to \overset{+1}{\begin{pmatrix} -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}} \to \overset{-1}{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}} \to$$

$$\overset{+1}{\begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 0 & -1 \end{pmatrix}} \to \overset{+1}{\begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & -1 \end{pmatrix}} \to \overset{-1}{\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}} \to$$

$$\overset{+1}{\begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix}} \to \overset{+1}{\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{pmatrix}} \to \qquad b_{fin}.$$

Case $m \equiv 1(\operatorname{mod} K)$, $n \neq 1(\operatorname{mod} K)$ and $m+n \neq 1(\operatorname{mod} K)$ is reduced to this case as well, if m and n are interchanged. $m+n \equiv 1(\operatorname{mod} K)$, $n \equiv 1(\operatorname{mod} K)$.

Hence $m \equiv 0(\operatorname{mod} K)$. (V_3)

Under these conditions, the solution of (12) is possible if the initial state corresponds to the safe restrictions:

$$\sum(b) \equiv 0(\operatorname{mod} K), \qquad \frac{\sum(b)}{m+n-1} \equiv \sigma_i(\operatorname{mod} K). \tag{23}$$

This case combines cases (V_1) and (V_2). Denote like in (V_1)

$$\frac{\sum(b)}{m+n-1} = \rho; \frac{1}{n-1}\left(\frac{\sum(b)}{m+n-1} - \sigma_i\right) = p_i, i = 1,2,\ldots,m.$$

Then for the matrix B the conditions (18) must be carried out, and the solution of the problem will be similar to (21):

$$x_{ij} \equiv \left[b_{ij} + \frac{1}{m-1}(\rho - \lambda_i) + p_i\right](\operatorname{mod} K), \qquad \sum_{i=1}^{m} p_i + \rho \equiv 0(\operatorname{mod} K), \tag{24}$$

where $p_i \in \{0,1,\ldots,K-1\}$.

As ρ is given, in this case the problem has K^{m-1} solutions as well.

Example 3. Let K = 3, m =3, n = 4, and matrix B, not satisfying the conditions (23) is given. After changing framed element we obtain the required matrix B'.

$$B = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 \end{pmatrix}, \quad B' = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix}.$$

Let us calculate the parameters: $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = 1$, $\lambda_4 = 0$; $\sigma_1 = \sigma_2 = \sigma_3 = 1$;

$$\sum(b) = 0; \frac{1}{m-1} = \frac{1}{2} \equiv 2(\operatorname{mod} 3); \rho = 1; \sum_{i=1}^{3} p_i \equiv -1(\operatorname{mod} 3).$$

With these parameters, the solution (24) takes on a specific type:

$$x_{ij} \equiv [b_{ij} + \lambda_i + p_i - 1](\operatorname{mod} 3)$$

Consider first the solution when $p_i \equiv 0(\text{mod}3)$. The result is a matrix $X^{(1)}$. It is clear now that the optimal solution is obtained when p1 = 0, p2 and p3 = -1 = 0, which gives the matrix $X^{(2)}$.

$$X^{(1)} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} - 1 \ \to \ X^{(2)} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

Let us check the solution $X^{(2)}$.

$$B = \overset{+1}{\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix}} \to \overset{-1}{\begin{pmatrix} -1 & 0 & -1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}} \to \overset{-1}{\begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}} \to$$

$$\to \overset{+1}{\begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \to \overset{-1}{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}} \to \qquad b_{fin}.$$

Case m ≡ 1(mod3) and m + n ≡ 1(mod3), reduced to (V_3), is solved similarly, if m and n are interchanged.

m ≡ n ≡ 1(mod K). Then m + n – 1 ≡ 1(mod K). (V_4)

Reasoning by analogy, we conclude that the problem has a solution with the following restrictions on the matrix B:

$$\sum(b) \equiv \sigma_i(\operatorname{mod} K), \ i = 1,2,\ldots,m; \qquad \sum(b) \equiv \lambda_j(\operatorname{mod} K), \ j = 1,2,\ldots,n. \tag{25}$$

It gives a specific dependence of its elements

$$\lambda_1 - \sum_{j=1}^{n-1} b_{ij} \equiv b_{in}(\operatorname{mod} K), \ i = 1,2,\ldots,m; \qquad \lambda_1 - \sum_{i=1}^{m-1} b_{ij} \equiv b_{mj}(\operatorname{mod} K), \ j = 1,2,\ldots,n. \tag{26}$$

Denote

$$\frac{1}{m-1}\left(\frac{\sum(b)}{m+n-1} - \lambda_j\right) = q_j; \frac{1}{n-1} = \left(\frac{\sum(b)}{m+n-1} - \sigma_i\right) = p_i.$$

For these parameters

$$\sum_{i=1}^{m} p_i \equiv \sum_{j=1}^{n} q_j \equiv -\sum(b)(\operatorname{mod} K) \tag{27}$$

takes place.

Then the problem has K^{m+n-2} solutions of the type:

$$x_{ij} \equiv \lfloor b_{ij} + p_i + q_j \rfloor (\operatorname{mod} K),$$

$$\sum_{i=1}^{m} p_i \equiv \sum_{j=1}^{n} q_j \equiv -\sum (b) (\operatorname{mod} K), \qquad (28)$$

where $p_i, q_j \in \{0,1,\ldots,K-1\}$.

Example 4. Let K = 3, m = n = 4, and matrix B is proposed below. Elements in the frame should be adjusted to the conditions (26). Find $\lambda_1 \equiv 1 (\operatorname{mod} 3)$. Then the framed elements are uniquely determined, as reflected in the matrix B'.

$$B = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \quad B' = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

As $\sum (b) \equiv 1 (\operatorname{mod} 3)$, $\sum_{i=1}^{m} p_i \equiv \sum_{j=1}^{n} q_j \equiv -1 (\operatorname{mod} 3)$.

We put first in (28) $p_i \equiv q_j \equiv 0 (\operatorname{mod} 3)$. We obtain the matrix B. If we substitute then $p_1 = p_4 = q_3 = q_4 \equiv 1 (\operatorname{mod} 3)$, and rest of p_i и q_j are equal to **0**, we get solution $X^{(1)}$.

$$X^{(1)} = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \quad X^{(2)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

However, the optimal solution is obtained by putting q1 = q3 ≡ 1 (mod 3), p1 ≡ -1 (mod 3), and the rest are 0. Then we obtain the matrix $X^{(2)}$. Check out this solution.

$$B' = \begin{pmatrix} \overset{-1}{-1} & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \overset{-1}{1} & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & \overset{+1}{0} & -1 \\ -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 0 & \overset{+1}{0} & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & \overset{-1}{1} & -1 \\ 0 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & \overset{-1}{1} & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 0 & 0 & 0 & \overset{+1}{-1} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \rightarrow b_{\text{fin}}.$$

V. CONCLUSIONS

We formulate a new problem in the matrix, which is called the problem of mathematical safe. It is given by a system of linear equations in the class of residues modulo end. We prove the existence of an inverse matrix for this system. This matrix is obtained using combinatorial methods. Checking solutions to the problem is shown in the examples

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AUTHOR'S PROFILE

Yaghoub Aghaei Agh Ghamish (Iran)- master of sciences., post-graduate student, National Polytechnic Institute 03056, Kiev-56, Ukraine. e-mail: yaghoubaghaei@yahoo.com