Fatigue Failure of an Annular Plate Under the Action of Pulsating Moment and Pressure along the internal contour

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Abstract: - We solve a problem on fatigue failure of an annular elastic plastic plate in the case when internal contour is loaded with pulsating moment and pressure. The damage condition that determines the amount of loadings before appearance of first damages and the cyclic strength condition determining the number of loadings to fatigue failure are used. Intensity of residual deformations was accepted as a determining parameter of fatigue failure. The relations of plastic flow theory were used at basic deformation. Possibility of appearance of secondary plastic deformations was taken into account at unloading.

Keywords: annular plate, pulsating moments and pressures; ideal plasticity; secondary plastic deformations; damage time; failure time.

I. INTRODUCTION

We consider an annular plate loaded by a pulsating moment and pressure along the internal contour. It is assumed that at the deformation a part of a plate goes into the plastic state. The material's properties are described by the equations of plastic flow theory [1], the material's strengthening is excluded. After some number of loading cycles in the plane there happens fatigue failure (disturbance of continuity) the goal of the paper is to determine the number of pulsating acting loadings preceding the appearance in the plate the first damages and the number of loadings to failure. The intensity of residual deformations is accepted as a determining parameter of fatigue failure. The unloading process is studied with regard to appearance of secondary plastic deformations [3]. In order to determine the intensity of residual deformations the solution of the elastico-plastic problem is used at basic loading of the plate from the natural state that was obtained by Nordgren and Naghdy [2].

II. STATEMENT AND SOLUTION OF THE PROBLEM

Within the plastic flow theory [1] we give the statement of a mathematical problem on deformation from the natural (unreformed) state of an annular plate with internal radius a, the external radius b at loading along the internal contour by the pressure \( p(t) \) and distributed moment \( M(t) \), where \( t \) is the time :

\[ 0 \leq t \leq t_\ast \] is the time to fatigue failure.

According to [1]. The equation of plastic flow theory accept in the form

\[ \dot{\varepsilon}_{ij}^p = \lambda \frac{\partial F}{\partial \sigma_{ij}}, \quad (1) \]

Here \( \dot{\varepsilon}_{ij}^p \) are velocities of plastic deformations \( \varepsilon_{ij}^p \) as the loading parameter: \( \sigma_{ij} \) are current stresses, \( F \) is the loading function. We assume that the plate's material has no strengthening. Therefore, in the plastic area we have \( F = 0, \quad \lambda \geq 0 \). To relations (1) we should add the relations for elastic components of deformations

\[ \varepsilon_{ij}^e = \frac{1+v}{E} \left( \sigma_{ij} - \frac{3v}{1+v} \sigma \delta_{ij} \right), \quad (2) \]

where \( \sigma \) is the Possion ratio, \( E \) is the modulus of longitudinal elasticity; \( \delta_{ij} \) are the Kronecker symbols,

\[ \sigma = \sigma_{ij} \delta_{ij}/3 \] is the mean stress.

We shall use the cylindrical system of coordinates \((r, \theta, z)\), whose origin coincides with the center of the annular plate.

Therewith \( a \leq r \leq b; \quad 0 \leq \theta \leq 2\pi \).

The stress components \( \sigma_r, \sigma_\theta, \sigma_{r\theta} \) should satisfy the following equilibrium equations

\[ \frac{\partial \sigma_r}{\partial r} = \frac{\sigma_\theta - \sigma_r}{r} \quad (3) \]

\[ \frac{\partial \sigma_{r\theta}}{\partial r} = -\frac{2}{r} \sigma_{r\theta} = 0 \quad (4) \]

Between the components of displacements \( u_r, u_\theta \) and \( \varepsilon_r, \varepsilon_\theta, \varepsilon_{r\theta} \) deformations it holds the Cauchy kinematic relation

\[ \varepsilon_r = \frac{\partial u_r}{\partial r}; \quad \varepsilon_\theta = \frac{u_r}{r}; \quad \varepsilon_{r\theta} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right). \quad (5) \]

The following boundary conditions should be satisfied:
\[ \sigma_r \big|_{r=a} = -p(t); \sigma_r \big|_{r=b} = 0, \quad u_\theta \big|_{r=b} = 0 \quad \text{(6)} \]

\[ \int_0^{2\pi} \sigma_r \theta r^2 d\theta = M(t). \quad \text{(7)} \]

Furthermore, these hold the conjugation conditions \( \mathbf{U}_r \) and \( \sigma_r \), for \( r = r_s \), where \( r_s \) is the radius of a circle separating elastic and plastic areas. Elasto-plastic problem (1) - (7) was solved in the paper of Nordgren and Naghdi [2]. Write this solution:

The loading function \( F \) chosen in the form:

\[ F = \frac{1}{4} \left( \sigma_r - \sigma_\theta \right)^2 + \sigma_r^2 - 2\tau_s^2, \]

where \( \tau_s \) is the yield point at pure shear.

\[ \sigma_r = \frac{M(t)}{2\pi r^2}, \quad \text{for } a \leq r \leq b \quad \text{(9)} \]

In the elastic domain \( r_s \leq r \leq b \) we have:

\[ \sigma_\theta = \frac{1}{r} \left[ \left( \frac{M(t)}{2\pi r_s^2} \right)^{2-1/2} \left( 1 + \frac{r_s^2}{r^2} \right) \tau_s, \right. \quad \text{(10)} \]

\[ \varepsilon_r = \frac{1}{E} \left( \frac{M(t)}{2\pi r_s^2} \right)^{2-1/2} \left( \frac{1 + 2\nu}{2} \right) r_s, \quad \text{(11)} \]

\[ \varepsilon_r \theta = \frac{1 + \nu}{E} \left( \frac{M(t)}{2\pi r_s^2} \right)^{2-1/2} \tau_s, \quad \text{(12)} \]

\[ u_r = \frac{r_s}{E} \left[ \left( \frac{M(t)}{2\pi r_s^2} \right)^{2-1/2} \left( \frac{1 + 2\nu}{2} \right) r_s + \frac{1 + 2\nu}{2} \right], \quad \text{(13)} \]

\[ u_\theta = -\frac{(1 + \nu)M(t)}{2\pi E} \left( \frac{1}{r} - \frac{r}{b^2} \right) \quad \text{(14)} \]

In the plastic domain \( a \leq r \leq r_s \) we have

\[ \sigma_r = \tau_s \left\{ \ln \left[ \frac{r_s^2 + \left( \frac{M(t)}{2\pi r_s^2} \right)}{r_s^2} \right]^{1/2} + \left[ \frac{\left( \frac{M(t)}{2\pi r_s^2} \right)}{r_s^2} \right]^{1/2} \right\} - p(t), \quad \text{(15)} \]

\[ \sigma_\theta = \sigma_r + \frac{2\pi}{r^2} \left[ r^4 - \left( \frac{M(t)}{2\pi r_s^2} \right)^{2} \right]^{1/2}, \quad \text{(16)} \]

\[ \varepsilon_r = \frac{1}{E} \left[ \frac{2\tau_s}{2\pi r_s^2} \left( \frac{M(t)}{2\pi r_s^2} \right)^{1/2} \right] \left( \frac{1 + \nu}{2} \right) \left[ \frac{r_s^4 - \left( \frac{M(t)}{2\pi r_s^2} \right)^{2}}{r_s^2} \right]^{1/2}, \quad \text{(17)} \]

\[ \varepsilon_\theta = \frac{1}{E} \frac{2\tau_s}{2\pi r_s^2} \left[ r_s^4 - \left( \frac{M(t)}{2\pi r_s^2} \right)^{2} \right]^{1/2}, \quad \text{(18)} \]

\[ u_r = \frac{1}{E} \left( \frac{M(t)}{2\pi r_s^2} \right)^{1/2} \left( \frac{1}{r} - \frac{r}{b^2} \right) - \frac{1}{\pi} \left( \frac{1}{b} - \frac{1}{r} \right) \int_0^t \dot{M}(t) dt, \quad \text{(19)} \]

\[ u_\theta = \frac{\bar{\sigma}}{E} \left( \frac{1 + \nu}{2} \right) \left( \frac{1}{r} - \frac{r}{b^2} \right) - \frac{1}{\pi} \left( \frac{1}{b} - \frac{1}{r} \right) \int_0^t \dot{M}(t) dt. \quad \text{(20)} \]

The variable \( \sigma_r \) contained in (16) (18) is represented by formula (15). The loading parameter \( \dot{M} \) is determined by the formula:

\[ \dot{M} = \frac{2}{E} \left( \frac{M(t)/(4\pi r_s^2)}{r^4 - \left( \frac{M(t)}{2\pi r_s^2} \right)^{2}} \right)^{1/2} + \frac{2\tau_s^2}{2\pi r_s^2} \left( \frac{M(t)}{2\pi r_s^2} \right)^{1/2} \right\}. \quad \text{(22)} \]

The unknown radius \( r_s \) is the solution of the following equation:

\[ \frac{p(t)}{r_s} = \ln \frac{r_s^2 + \left( \frac{M(t)}{2\pi r_s^2} \right)^{2}^{1/2}}{\frac{1}{4} + \left[ \frac{\left( \frac{M(t)}{2\pi r_s^2} \right)}{r_s^2} \right]^{2}^{1/2}} - \frac{r_s^4 - \left( \frac{M(t)}{2\pi r_s^2} \right)^{2}}{b^2} \right\}. \quad \text{(23)} \]

The represented solution holds for

\[ \frac{M(t)}{2\pi r_s^2} \leq \alpha^2, \quad \text{(16)} \]

\[ p(t) > 1 - \left( \frac{M(t)}{2\pi r_s^2} \right)^{2}^{1/2}. \]

The mentioned conditions are connected with the use of determining equations of plastic flow theory.
The first plastic deformations will appear on the internal contour of the plate. Therewith, \( r_s^2 = a \) and from (23) it follows the relation that holds between the internal pressure \( P(t) \) and distributed moments \( M(t) \): 

If \( M(t) = 0 \), we have \( p_s = \frac{a^2}{b^2 \tau_s} \) but if \( P(t) = 0 \), then \( M_s = 2\pi a^2 \tau_s \).

Suppose that external forces \( P(t) \) and \( M(t) \) are monotonically increasing in time interval \( 0 \leq t \leq t_1 \).

Let beginning with time \( t = \frac{t_1}{2} \) these forces begin to decrease and for time \( t = t_1 \) become zero: \( P(t_1) = 0 \), \( M(t_1) = 0 \). The unloading in the plate happens at time interval \( \frac{t_1}{2} \leq t \leq t_1 \). We shall assume that the unloading process is accompanied by appearance of secondary plastic deformations. Using the above represented solution, define the residual stresses, strains and displacements that remain the annular disk after removing the internal pressure \( P(t) \) and distributed moment \( M(t) \). And we'll assume that the forces \( P(t) \) and \( M(t) \) at loading from the natural state obtained such a value that in the unloading process necessarily the secondary plastic deformations well appear. Now we must determine the conditions of appearance of secondary plastic deformations, and also find the residual stresses, strains and displacements. We'll use V.V. Moskvitin's theorem on secondary plastic deformations [3]. According to this theorem, the residual stresses \( \sigma_r^0, \sigma_\theta^0, \sigma_{r\theta}^0 \), residual strains \( \varepsilon_r^0, \varepsilon_\theta^0, \varepsilon_{r\theta}^0 \), residual displacements \( u_r^0, u_\theta^0 \) may be determined by the following formulas:

\[
\begin{align*}
\sigma_r^0 &= \sigma_r - \sigma_r^*; & \sigma_\theta^0 &= \sigma_\theta - \sigma_\theta^*; & \sigma_{r\theta}^0 &= \sigma_{r\theta} - \sigma_{r\theta}^*; \\
\varepsilon_r^0 &= \varepsilon_r - \varepsilon_r^*; & \varepsilon_\theta^0 &= \varepsilon_\theta - \varepsilon_\theta^*; & \varepsilon_{r\theta}^0 &= \varepsilon_{r\theta} - \varepsilon_{r\theta}^*; \\
u_r^0 &= u_r - u_r^*; & u_\theta^0 &= u_\theta - u_\theta^*.
\end{align*}
\]

(24)

Here \( \sigma_r, \sigma_\theta, \sigma_{r\theta}, \varepsilon_r, \varepsilon_\theta, \varepsilon_{r\theta}, u_r, u_\theta \) are stresses, strains and displacements before the beginning of unloading and are determined by the formulas (10)-(14) with regard to (22). Therewith, in these formulas, \( p(t) \) and \( M(t) \) should be replaced by \( p(t_1/2)M(t_1/2) \). Unlike the considered plate, the yield point at the shear of the fictitious plate material is \( 2\sigma_s \). The secondary plastic deformations will appear in the zone adjoining to the interior contour of the plate. Let \( r_0 \) be a radius of a circle that separates the area of secondary plastic deformations: \( a \leq r \leq r_0 \). For composing the equation plastic determining \( r_0 \), we should use equation (23) having substituted in it \( 2\sigma_s \) for \( \sigma_s \):

\[
\frac{P(t_1/2)}{2\tau_s} = \frac{\gamma_0^2}{a^2 + \left[ \frac{\gamma_0}{4\pi \tau_s} \right]^{1/2}} + \frac{\alpha^2}{b^2 \tau_s} - \frac{\alpha^2}{4\pi \tau_s} \left[ 1 - \frac{\gamma_0}{4\pi \tau_s \alpha^2} \right]^{1/2}.
\]

(25)

According to (24)

\[
\left( \frac{M(t_1/2)}{4\pi \tau_s} \right) \leq \alpha^2, \quad \frac{P(t_1/2)}{2\tau_s} < 1 + \left[ 1 - \frac{\gamma_0}{4\pi \tau_s \alpha^2} \right]^{1/2},
\]

(26)

will be a condition of appearance of secondary plastic deformations.

If \( M(t) \equiv 0 \), there should be \( P > \frac{2(b^2 - a^2)}{a^2 b^2} \tau_s \). But if \( P(t) \equiv 0 \) then \( M > 4\pi a^2 \tau_s \).

In the area of secondary plastic deformations \( a \leq r \leq r_0 \), the residual stresses, strains and displacements will be:

\[
\sigma_r^0, \sigma_\theta^0, \sigma_{r\theta}^0, \varepsilon_r^0, \varepsilon_\theta^0, \varepsilon_{r\theta}^0, u_r^0, u_\theta^0 \]

\[
= \tau_s \left[ \frac{\gamma_0^2}{a^2 + \left[ \frac{\gamma_0}{4\pi \tau_s} \right]^{1/2}} \right]^{1/2} \left[ \frac{\gamma_0}{4\pi \tau_s \alpha^2} \right]^{1/2}.
\]

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The quantity \( \sigma^0_{r\theta} \) that is contained in formulas (28), (30), (31) and (33) is determined by formula (27). The formula for the quantity \( \dot{\lambda}_0 \) contained in (32) and (34) is written similar to formula (22) having substituted at the last one \( 2 \tau_s \) for \( \tau_{sr} \) and \( \tau_{r} \) for \( t \).

The residual stresses, strains and displacements in the area \( r_0 \leq r \leq r_s \), where the unloading process happens elastically, may be found also by formulas (24). Therewith the formulas

\[ \sigma_r, \sigma_\theta, \sigma_{r\theta}, \varepsilon_r, \varepsilon_\theta, \varepsilon_{r\theta}, u_r, u_\theta \]

are determined as in the previous case of loading. The quantities

\[ \sigma^*_r, \sigma^*_\theta, \sigma^*_{r\theta}, \varepsilon^*_r, \varepsilon^*_\theta, \varepsilon^*_{r\theta}, u^*_r, u^*_\theta \]

are the stresses, strains and displacements that appear in the fictitious annular plate at its elastic deformation by the forces \( (t_{1/2}), M(t_{1/2}) \). For finding them, we’ll use independent solutions of Lame’s elastic problem on action of the internal pressure \( P(t_{1/2}) \) in the plate under consideration and the distributed moment \( M(t_{1/2}) \) along the internal contour of this plate. Proceeding from this, the quantities

\[ \sigma^*_r, \sigma^*_\theta, \sigma^*_{r\theta}, \varepsilon^*_r, \varepsilon^*_\theta, \varepsilon^*_{r\theta}, u^*_r, u^*_\theta \]

are determined by the following formulas

\[ \sigma^*_r \]

\[ \sigma^*_\theta \]

\[ \varepsilon^*_r \]

\[ \varepsilon^*_\theta \]

\[ u^*_r \]

\[ u^*_\theta \]

Consequently, the residual stresses, strains and displacements in the area \( r_0 \leq r \leq r_s \) are determined as the differences of appropriate formulas (15)-(21) subject to with regard to (22) and substitution of \( t_{1/2} \) for \( t \). It remains to define the residual sought-for quantities in the area \( r_s \leq r \leq b \). These quantities are also found on the base of formula (24). Therewith the quantities

\[ \sigma_\theta, \sigma_r, \sigma_{r\theta}, \varepsilon_r, \varepsilon_\theta, \varepsilon_{r\theta}, u_r, u_\theta \]

are expressed by the formulas (15)-(21), where \( r_s \) is determined by formula (23).
The quantities \( \sigma^*_e, \sigma^*_r, \sigma^*_q, \varepsilon^*_e, \varepsilon^*_r, \varepsilon^*_q, u^*_r, u^*_\theta \) are represented by formulas (35)-(40). Now consider an issue on fatigue failure of the plate under consideration. Let the plate under consideration after total unloading be again loaded by monotonically increasing forces \( P(t) \) and \( M(t) \) and let the loading process continue to time \( t = \frac{3}{2} t_1 \) and in the sequel the total loading process happen to time \( \left( \frac{3}{2} t_1, 2 t_1 \right) \). Suppose that such loading and unloading process of the plate continues up to fatigue failure. Therewith the time of each loading-unloading cycle equals \( t_f = t_r / t_1 \). The number of cycles to fatigue failure \( N_0 \) will be \( N_0 = t_f / t_1 \). The number of the cycles before appearance of first damages \( N_f \) is expressed by the formula: \( N_f = t_f / t_1 \), where \( t_f \) is the time to appearance of first damages. From physical reasoning it is clear that the first damages in the plate will appear on the internal contour. The plate will fail just from this contour. Determine the number of loading cycles before the appearance of first damages and the number of the cycles to failure of the plate along the radius \( r \). To determine them we use the following known formulas [3]:

\[
N_0 = A_0 \left( \frac{\varepsilon^0}{\varepsilon_S} \right)^{\alpha_0}; N_f = A_1 \left( \frac{\varepsilon^0}{\varepsilon_S} \right)^{\alpha_1},
\]

(41)

where \( A_0, \alpha_0, A_1, \alpha_1 \) are universal constants of the material, \( \varepsilon^0 \) is the intensity of residual strains, \( \varepsilon_S \) is some reduction strain.

In conformity to the problem under consideration the intensity of residual strains are determined by the formula:

\[
\varepsilon^0 = \frac{2}{3} \left[ \left( \varepsilon^0_r \right)^2 + \left( \varepsilon^0_q \right)^2 - \varepsilon^0_r \varepsilon^0_q + 3 \left( \varepsilon^0_r \varepsilon^0_\theta + \varepsilon^0_q \varepsilon^0_\theta \right)^2 \right]^{\frac{1}{2}}.
\]

(42)

Since the plates material is cyclic-ideal, then the residual stresses and displacement in any k-th loading cycle will be the same as in the first unloading. We take into account this fact in calculating the intensity of residual strains, and in defining the numbers \( N_0 \) and \( N_f \).

Processing of experimental data with regard to cyclic durability for the steel of the mark EH-826, that are contained in the paper [4], for \( \varepsilon_S = 0.01 \) gave the following results:

\[
\alpha_0 \approx \alpha_1 \approx 1.6; \quad \frac{A_1}{A_0} \approx 0.56; \quad A_0 \approx 1.8 \cdot 10^5
\]
cycles. By using these data and after calculating the intensity of residual strains by formula (42) allowing for relations (27),(30)-(32), the number of loading cycles to appearance of first damages and to fatigue failure of the internal contour of the plate were found. Therewith formula (41) was used. The graphs of dependences \( N_0 \sim M(t_1/2), N_f \sim M(t_1/2) \) on the interval \( \frac{N_0}{2 \pi \alpha^2 \tau_s} \) are depicted in fig.1.

Besides the abovementioned digital data, the following values of parameters were used:

\[
\begin{align*}
\alpha &= 0.3; & E &= 2 \cdot 10^4 \text{ MPa}; & \\
\nu &= 0.3; & \frac{b}{a} &= 4;
\end{align*}
\]

Kinetics of plate failure along the radius \( r \) for \( M(t_1/2)/(2 \pi \alpha^2 \tau_s) = 0.4 \) was determined (fig 2). After onset of failure of the internal contour, the amplified fatigue failure of the remaining part the plate is observed.

III. CONCLUSIONS

1. The conditions of appearance of secondary plastic deformations at total unloading after prior elastic-plastic deformation of an annular plate under the action of pressure and distributed moment along the internal contour were determined.
2. The number of loadings to fatigue failure of an annular plate at its cyclic elastico-plastic deformation under
pressure and moment pulsating acting along the internal contour was determined.

REFERENCES


