A Study of Fuzzy Modules through a Functor

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Abstract: In this paper first we construct a homotopy type invariant functor, then we study the fuzzy modules through this functor.

In this paper we show that:-

1. ‘Hom$_{F}$’: ‘CM’ $\rightarrow$ ‘CF’ is a Homotopy type invariant functor; where CM denotes the category of R-modules and R-homeomorphisms, R be a ring and CF denotes the category of fuzzy left R-module and fuzzy R-map;

2. All homotopy type invariant functors form a function space, this is denoted by CF$^{CM}$ or simply F$^{CM}$.

3. A fuzzy R-map $\tilde{f} \in \text{Hom}_{R}(\lambda_{M},\eta_{N})$ is called fuzzy split iff $\tilde{f}$ is an isomorphism.

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I. INTRODUCTION

The concept of fuzzy modules was introduced by Nagoita and Ralescu[1] and the category of fuzzy sets was introduced by Goguen[2] in 1967. In this paper we construct the homotopy type invariant functor and then we study the fuzzy modules. To do this we recall the following definitions and statements.

Definition 1.1
Let R be a ring and M be left or right R-module. (M, $\lambda$) is called a fuzzy left R-module iff there is a map $\lambda : M \rightarrow [0,1]$ satisfying the following conditions:

1. $\lambda(a+b) \geq \min\{\lambda(a),\lambda(b)\}, \forall a,b \in M$
2. $\lambda(-a) = \lambda(a), \forall a \in M$
3. $\lambda(0) = 1$
4. $\lambda(ra) = \lambda(a) (\forall a \in M, r \in R)$

We write (M, $\lambda$) by $\lambda_{M}$

Definition 1.2
Let $\lambda_{M}$ and $\eta_{N}$ be arbitrary fuzzy left R-modules. A fuzzy R-map

$\tilde{f} : \lambda_{M} \rightarrow \eta_{N}$ should satisfy the following conditions.

1. $f : M \rightarrow N$ is an R-map,
2. $\eta(f(a)) \geq \lambda(a), \forall a \in M$

Definition 1.3
Let $f : M \rightarrow N$ and $\mu$ be a fuzzy subset of N. The fuzzy subset $f^{-1}(\mu)$ of $\mu$ defined as follows;

for all $x \in M$, $f^{-1}(\mu)(x) = \mu(f(x))$ is called fuzzy preimage of $\mu$ under $f$.

Definition 1.4
A fuzzy sub modules of M is a fuzzy subset of M such that

1. $\mu(0) = 1$
2. $\mu(rx) \geq \mu(x), \forall r \in R \text{ and } \forall x \in M$
3. $\mu(x+y) \geq \min(\mu(x),\mu(y)), \forall x, y \in M$

Definition 1.5
A fuzzy R-map $\tilde{f} \in \text{Hom}_{R}(\lambda_{M},\eta_{N})$ is called fuzzy split iff there exists some

$\tilde{g} \in \text{Hom}_{R}(\eta_{N},\lambda_{M})$ such that

$\tilde{f}\tilde{g} = \tilde{I}_{M}$ and $\tilde{g}\tilde{f} = \tilde{I}_{N}$.

Definition 1.6
A category C consists of

(a) a class of objects X, Y, Z,.....denoted by Ob(C);
(b) For each ordered pair of objects X, Y a set of morphisms with domain X and range Y denoted by C(X,Y);
(c) For each order triple of objects X, Y and Z and a pair of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ their composite is denoted by $g \circ f : X \rightarrow Z$.

Definition 1.7
Let $\tilde{T} : C \rightarrow D$ be categories. A covariant functor $T$ from C to D consists of

1. an object function which assigns to every object X of C an object $T(X)$ of D; and
2. a morphism function which assigns to every morphism $f : X \rightarrow Y$ in C a morphism

$T(f) : T(X) \rightarrow T(Y)$ in D such that

a) $T(I_{X}) = I_{T(X)}$

b) $T(fg) = T(g)T(f)$, for $g : X \rightarrow Y$ in D.

Definition 1.8
Let $A$ and $B$ be topologically enriched categories. A functor $F : A \rightarrow B$ is called continuous if

$F : A(A,B) \rightarrow B(FA,FB)$ is continuous for all A and B in A.

A natural transformation $\alpha : F \Rightarrow G$ of continuous functors $F,G : A \rightarrow B$ is called continuous if is a
Lemma 1.9
A base point preserving continuous map $f: M \to N$ is a homotopy equivalence if there is a base point preserving continuous map $g: N \to M$ with $g \circ f \simeq 1_M$ and $f \circ g \simeq 1_N$, where $M$ and $N$ are two $R$-modules.

Lemma 1.10
Let $\text{Hom}_R(\lambda_M, \eta_N)$ denotes the set of all fuzzy $R$-maps from $\lambda_M$ to $\eta_N$, then $\text{Hom}_R(\lambda_M, \eta_N)$ is an additive group. Moreover, if $R$ is a commutative ring, then $\text{Hom}_R(\lambda_M, \eta_N)$ is a left $R$-modules.

Proposition 2.1
Let $R$ be a ring and $M$ be left or right $R$-module. Then $R$-modules and $R$-homormorphisms form a category. This category is denoted by ‘$CM’$.

Proposition 2.2
Let $R$ be a ring and $M$ be left or right $R$-module. Then fuzzy left $R$-module and fuzzy $R$-map forms a category. This category is denoted by ‘$CF’$.
Proposition 2.4
Let CF denotes the category of fuzzy R-modules and fuzzy R-maps and CM denotes the category of R-modules and R-homomorphisms, then there exists a covariant functor 
\[ \alpha : CM \rightarrow CF \]

Proof:
Define \( \alpha : CM \rightarrow CF \) by
\[ \lambda(M) = (M, \lambda) = \lambda_M, \] which is the object of CF
Let M, N are two R-modules in CM and f: M \rightarrow N be R-homomorphisms in CM, then
\[ \lambda(f): \lambda(M) \rightarrow \lambda(N) \] in CF.

\[ \lambda(f)(\alpha) = \alpha \cdot f^1, \forall \alpha in \lambda(M) \]

\[ \alpha_1(f^1)(\alpha) = \alpha_1(\mu(f)(x)) \] 
\[ \Rightarrow \alpha_2(f^1)(\mu(x)) = \alpha_2(\mu(f)(x)) \]
Thus \( \alpha_1 = \alpha_2 \Rightarrow \alpha_1 \cdot f^1 = \alpha_2 \cdot f^1 \)
\[ \Rightarrow \lambda(f)(\alpha_1) = \lambda(f)(\alpha_2) \] 
Let f: M \rightarrow N and g: N \rightarrow P are in CM, then \( \lambda(f) = \lambda(M) \rightarrow \lambda(N) \).

\[ \lambda(g) = \lambda(N) \rightarrow \lambda(P) \] and gof : M \rightarrow P are in CM.

Proposition 2.6
Let R be a ring and M be a fixed R-module, the R-homomorphism f:N \rightarrow P induces

i) a fuzzy R-homomorphism \( f : \text{Hom}_R(M,N) \rightarrow \text{Hom}_R(M,P) \) and

ii) an fuzzy R-homomorphism \( f' : \text{Hom}_R(P,N) \rightarrow \text{Hom}_R(P,M) \)

Proof: Using Definition 1.1 and Lemma 1.12, it follows.
Corollary 2.7
For any fixed R-module M, the fuzzy R module \( \text{Hom}_R(M,N) \) and their fuzzy R-homomorphisms forms category, for any R-module N; this category is denoted by ‘CF’
Proof
Using Proposition 2.6 and Lemma 1.12(a), it follows

Proposition 2.8
‘\( \text{Hom}_R \)’ is an invariant functor in the sense that it is both a covariant and a contravariant functor
\( \text{Hom}_R : CM \rightarrow CM \) is an invariant functor, for any fixed R-module M.

Proof:
Define \( \text{Hom}_R : CM \rightarrow CF \) by
\( \text{Hom}_R(N) = \text{Hom}_R(M,N) \), for any fixed R-module M, which is the object of CF.

Let N, P are two R-modules in CM and f:N \rightarrow P be R-homomorphisms in CM, then
\( \text{Hom}_R(f) = f : \text{Hom}_R(M,N) \rightarrow \text{Hom}_R(M,P) \) in CF and \( f' : \text{Hom}_R(P,M) \rightarrow \text{Hom}_R(P,N) \) are well defined mapping and so by Definition 1.1, Lemma 1.12 and Theorems 2.6, theorem follows.

Proposition 2.9
‘\( \text{Hom}_R \)’ is a Homotopy type functor in the sense that if f is a homotopy equivalence for any two R-modules M and N, then \( \text{Hom}_R(f) \) is a isomorphism
Proof:
Since f is a homotopy equivalence for any two R-modules M and N, there exists \( f:M \rightarrow N \) and \( g: N \rightarrow M \) such that \( g \circ f \cong \text{Id}_M \) and \( f \circ g \cong \text{Id}_N \), then \( \text{Hom}_R(f) : \text{Hom}_R(P,M) \rightarrow \text{Hom}_R(P,N) \) and
\( \text{Hom}_R(f) : \text{Hom}_R(N,P) \rightarrow \text{Hom}_R(N,M) \) are fuzzy R-homomorphisms, then \( \text{Hom}_R \) satisfies the following conditions:

i) \( f \cong g \Rightarrow \text{Hom}_R(f) = \text{Hom}_R(g) \)

ii) \( g \circ f \cong \text{Id}_M \Rightarrow \text{Hom}_R(g \circ f) = \text{Id} \Rightarrow \text{Hom}_R(g) \cdot \text{Hom}_R(f) = \text{Id} \)

iii) \( f \circ g \cong \text{Id}_N \Rightarrow \text{Hom}_R(f \circ g) = \text{Id} \Rightarrow \text{Hom}_R(f) \cdot \text{Hom}_R(g) = \text{Id} \)

Thus \( \text{Hom}_R(f) \) is isomorphic to \( \text{Hom}_R(g) \)

Corollary 2.10.
\( \text{Hom}_R \) is also a Homotopy type invariant functor.
Proposition 2.11

A fuzzy R-map $\tilde{f} \in \text{Hom}_R(\lambda_M, \eta_P)$ is called fuzzy split iff $\tilde{f}$ is an isomorphism.

Proof: By the definition of a fuzzy split, it follows that there exists $\tilde{g} \in \text{Hom}(\eta_P, \lambda_M)$ such that $\tilde{f} \circ \tilde{g} = \tilde{I}_P$ and $\tilde{g} \circ \tilde{f} = \tilde{I}_M$.

$\Rightarrow \tilde{f}$ is an epimorphism and $\tilde{f}$ is also a monomorphism.

$\Rightarrow \tilde{f}$ is an isomorphism.

Conversely suppose that a fuzzy R-map $\tilde{f}$ is an isomorphism, then $\tilde{f}$ is an epimorphism and monomorphism, i.e., there exists $\tilde{g} \in \text{Hom}(\eta_P, \lambda_M)$ such that $\tilde{f} \circ \tilde{g} = \tilde{I}_P$ and $\tilde{g} \circ \tilde{f} = \tilde{I}_M$.

$\Rightarrow \tilde{f}$ is fuzzy split.

Proposition 2.12

All homotopy type invariant functors form a function spaces from category CM to the category CF, it is denoted by $F^M$.

Proof: Using the Definition 1.8, Theorems 2.8, Theorem 2.9 and Theorem 2.10, it follows.

REFERENCES


AUTHOR BIOGRAPHY

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