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A Study of Fuzzy Modules through a Functor

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Abstract: ***In this paper first we construct a homotopy type invariant functor, then we study the fuzzy modules through this functor.***
In this paper we show that:-

i) 'Hom_R': 'CM' → 'CF' is a Homotopy type invariant functor; where CM denotes the category of R-modules and R-homeomorphisms ,R be a ring and CF denotes the category of fuzzy left R-module and fuzzy R-map;

ii) All homotopy type invariant functors form a function space, this is denoted by CF^{CM} or simply F^M.

iii) A fuzzy R-map $\tilde{f} \in Hom_R(\lambda_M,\eta_P)$ is called fuzzy split iff $\tilde{f}$ is an isomorphism

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Key words **Fuzzy modules, function spaces, fuzzy split, Category, invariant functor.**

I. INTRODUCTION

The concept of fuzzy modules was introduced by Nogoita and Ralescu[1] and the category of fuzzy sets was introduced by Goguen[2] in 1967. In this paper we construct the homotopy type invariant functor and then we study the fuzzy modules. To do this we recall the following definitions and statements.

Definition1.1

Let R be a ring and M be left or right R-module. (M,λ) is called a **fuzzy left R-module** iff there is a map $\lambda : M \to [0,1]$ satisfying the following conditions:

i) $\lambda(a+b) \geq \min\{\lambda(a), \lambda(b)\}, \forall a,b \in M$
ii) $\lambda(-a) = \lambda(a)$, $\forall$ $a \in M$
iii) $\lambda(0) = 1$
iv) $\lambda(ra) = \lambda(a)$ ($\forall$ $a \in M$, $r \in R$)

We write (M,λ) by λ_M

Definition1.2

Let λ_M and η_N be arbitrary fuzzy left R-modules. A **fuzzy R-map**
$\tilde{f} : \lambda_M \to \eta_N$ should satisfy the following conditions.

i) $f : M \to N$ is an R-map,
ii) $\eta(f(a)) \geq \lambda(a)$, $\forall$ $a \in M$

Definition 1.3

Let $f:M \to N$ and μ be a fuzzy subset of N . The fuzzy subset $f^{-1}(\mu)$ of N defined as follows;
for all $x \in M$, $f^{-1}(\mu)(x) = \mu(f(x))$ is called fuzzy preimage of μ under f.

Definition 1.4

A fuzzy sub modules of M is a fuzzy subset of M such that
i) $\mu(0) = 1$
ii) $\mu(rx) \geq \mu(x)$, $\forall$ $r \in R$ and $\forall x \in M$
iii) $\mu(x+y) \geq \min(\mu(x),\mu(y))$, $\forall x, y \in M$

Definition1.5

A fuzzy R-map $\tilde{f} \in Hom_R(\lambda_M,\eta_P)$ is called fuzzy split iff there exists some
$\tilde{g} \in Hom_R(\eta_P,\lambda_M)$ such that
$\tilde{f}\tilde{g} = \tilde{I_P}$ and $\tilde{g}\tilde{f} = \tilde{I_M}$,

Definition 1.6

A category C consists of
(a) a class of objects X,Y ,Z,....,denoted by Ob(C);
(b) For each ordered pair of objects X,Y a set of morphisms with domain X and range
Y denoted by C(X,Y);
(c) For each order triple of objects X, Y and Z and a pair of morphisms $f : X \to Y$ and
$g : Y \to Z$ their composite is denoted by $g\ f : X \to Z$, satisfying the Following two axioms:
i) associativity : if $f \in C(X,Y)$ and $g \in C(Y,Z)$ and $h \in C(Z,W)$,then
$h(gf) = (hg)f \in C(X,W)$
ii)identity : for each object Y in C there is a morphism $I_Y \in C(Y,Y)$ such that if
$f \in C(X,Y)$ then $I_Y\ f = f$ and if $h \in C(Y,Z)$, then $h\ I_Y = h$.

Definition 1.7

Let C and D be categories. A covariant functor T from C to D consists of
i) an object function which assigns to every object X of C an object T(X) of D; and
ii) a morphism function which assigns to every morphism $f : X \to Y$ in C a morphism
$T(f) : T(X) \to T(Y)$ in D such that
a) $T(I_X) = I_{T(X)}$
b) $T(gf) = T(g).T(f)$, for $g : Y \to W$ in C.

Definition 1.8

Let **A** and B be topologically enriched categories. A functor $F : A \to B$ is called continuous if
$F : A(A,B) \to B\ (FA, FB)$ is continuous for all A and B in A.
A natural transformation $\alpha : F \Rightarrow G$ of continuous functors $F,G : \mathcal{A} \to \mathcal{B}$ is called continuous if is a

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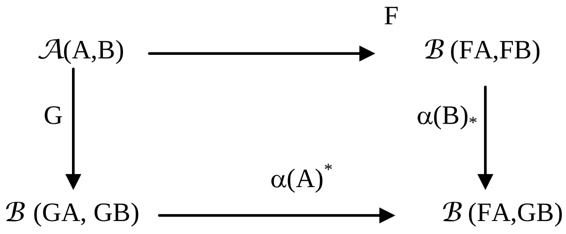
commutative diagram of continuous maps for all A and B in $\mathcal{A}$. A collection of morphisms $\{\beta(A) : FA \to GA;\ A \in ob\mathcal{A}\}$ is called a continuous natural transformation up to homotopy if the above diagram is homotopy commutative.



Definition 1.9

A base point preserving continuous map f: M→N is a homotopy equivalence if there is a base point preserving continuous map

$g: N \to M$ with $g \circ f \simeq I_M$ and $f \circ g \simeq I_N$, where M and N are two R-modules.

Lemma1.10

Let $Hom_R(\lambda_M, \eta_N)$ denotes the set of all fuzzy R-maps from λ_M to η_N, then $Hom_R(\lambda_M, \eta_N)$ is an additive group. Moreover, if R is a commutative ring, then $Hom_R(\lambda_M, \eta_N)$ is a left R-modules.
Proof: Using [10], it follows.

Lemma1.11

For a commutative ring R , R-modules M and N, $Hom_R(M,N)$ is an R-module
Proof: Using [2], it follows.

Lemma1.12

Let $Hom_R(M,N)$ denotes the set of all fuzzy R-maps from R-modules M to R-modules N, then $Hom_R(M,N)$ is an R –module, if R is a commutative ring
Proof
Using definition 1.1 and [7], it follows

Lemma 1.13

Given a fixed R-module M, the R-homomorphism f:N→P induces

a) an R-homomorphism $f_*: Hom_R(M,N) \to Hom_R(M,P)$ defined by $f_*(\alpha) = f \circ \alpha$, $\forall \alpha \in Hom_R(M,N)$
b) an R-homomorphism
c) $f^*: Hom_R(P,M) \to Hom_R(N,M)$ defined by $f^*(\beta) = \beta \circ f, \forall \beta \in Hom_R(P,M)$

Proof:
Using [6,7], it follows

Lemma1.14

Let M,N,P be R-modules and f:M→N and g:N→P be R-homomorphisms.
Then for any R- module A

i) $(g \circ f)_* : Hom_R(A,M) \to Hom_R(A,P)$ is an R-homomorphism such that
$(g \circ f)_* = g_* \circ f_*$;
ii) $(g \circ f)^* : Hom_R(P,A) \to Hom_R(M,A)$ is an R-homomorphism such that
$(g \circ f)^* = f^* \circ g^*$;

Proof:
Using [6,7], it follows

II. CATEGORIES OF FUZZY MODULES

In this section we construct some categories associated with fuzzy modules.

Proposition2.1

Let R be a ring and M be left or right R-module. Then R-modules and R-homomrphisms form a category. This category is denoted by 'CM'

Proof:
We take all left R-modules of '**C*M*'** as the set of objects and the set of their R- homeomorphisms, the set of morphisms $Hom_R(M,N)$ and for every pair of objects (M,N) and (N,P), the compositions $Hom_R(M,N) \times Hom_R(N,P)$, denoted by
$(f , g) = g \circ f$, where $f \in Hom_R(M,N)$ and $g \in Hom_R(N,P)$,satisfying the following axioms:

a) for any object M ∈ obj.'CM', there exists an identity morphism $I_M \in Hom_R(M,M)$;
b) Associativity of the composition holds.

Proposition 2.2

Let R be a ring and M be left or right R-module. Then fuzzy left R-module and fuzzy R-map forms a category. This category is denoted by 'CF'

Proof:
We take all elements of '**C*F*'** as the set of objects and the set of their fuzzy R- maps, the set of morphisms $Hom_R(\lambda_M, \eta_N)$ and for every pair of objects (λ_M, η_N) and (λ_N, η_P) ,the compositions $Hom_R(\lambda_M, \eta_N) \times Hom_R(\lambda_N, \eta_P)$,denoted by $(\tilde{f}, \tilde{g}) = \widetilde{g.f} = \tilde{g} \circ \tilde{f}$, where $\tilde{f} \in Hom_R(\lambda_M, \eta_N)$ and $\tilde{g} \in Hom_R(\lambda_N, \eta_P)$, satisfying the following axioms:

a) for any object $\lambda_M \in$ obj.'CM', there exists an identity morphism $\tilde{I}_M \in Hom_R(\lambda_M, \lambda_M)$
b) Associativity of the composition holds.

Proposition2.3

Let M and N be arbitrary fuzzy left R-modules. A fuzzy R-map f:M→N is an fuzzy R-homomorphism, then for any fuzzy sub-module S of N ,then

i) $f(O_M) = O_N$ and $f(-x) = -f(x)$, $\forall x \in M$; and
ii) **The set $f^{-1}(S) = \{x \in M: f(x) \in S\}$is a fuzzy sub-module of M.**

Proof:

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i) Now $f(O_M) = f(O_M + O_M) = f(O_M) + f(O_M) \Rightarrow f(O_M) = O_N$.

Again $f(O_M) = f(x + (-x)) = f(x) + f(-x) = O_N \Rightarrow f(-x) = -f(x)$.

ii) Since $f(O_M) = O_N \in S \Rightarrow O_M \in f^{-1}(S) \Rightarrow f^{-1}(S) \neq \Phi$. Let $x,y \in f^{-1}(S) \Rightarrow f(x), f(y) \in S \Rightarrow$ $f(x-y) = f(x) - f(y) \in S$, since S Iis a sub module of N $\Rightarrow x\text{-}y \in f^{-1}(S)$. Similarly, for $r \in R$ and $x \in f^{-1}(S)$, $rx \in f^{-1}(S) \Rightarrow f^{-1}(S)$ is a sub module of M. Since M is a fuzzy module and hence by **definition 1.1 and 1.4**, it follows that $f^{-1}(S)$ is a sub module of M.

Proposition 2.4

Let CF denotes the category of fuzzy R-modules and fuzzy R-maps and CM denotes the category of R-modules and R-homomorphisms, then there exists a covariant functor

λ: CM $\rightarrow$ CF

Proof:

Defineλ: **CM** $\rightarrow$ **CF** by

$\lambda(M) = (M,\lambda) = \lambda_M$, which is the object of **CF**

Let M.N are two R-modules in CM and f: M$\rightarrow$N be R-homomorphisms in CM, then

$\lambda(f) : \lambda(M) \rightarrow \lambda(N)$ in CF.

$\lambda(f)(\alpha) = \alpha \cdot f^{-1}$, $\forall$ α in $\lambda(M)$

i) $\alpha(a+b) \geq \min\{\alpha(a), \alpha(b)\}, (\forall\ a,b \in M)$
ii) $\alpha(-a) = \alpha(a), \forall\ a \in M$
iii) $\alpha(0) = 1$
iv) $\alpha(ra) = \alpha(a)\ (\forall\ a \in M, r \in R)$

Let μ in N, then $f^{-1}(\mu)(x) = \mu(f(x)) \Rightarrow$

$\alpha_1(f^{-1}(\mu))(x) = \alpha_1(\mu f(x)) = (\alpha_1\mu)(f(x))$

$\Rightarrow \alpha_2(f^{-1}(\mu))(x) = \alpha_2(\mu f(x)) = \alpha_2\mu(f(x))$

Thus $\alpha_1 = \alpha_2 \Rightarrow \alpha_1 \cdot f^{-1} = \alpha_2 \cdot f^{-1}$

$\Rightarrow \lambda(f)(\alpha_1) = \lambda(f)(\alpha_2)$

Let f:M$\rightarrow$N and g: N$\rightarrow$P are in CM, then $\lambda(f) : \lambda(M) \rightarrow \lambda(N)$,

$\lambda(g) : \lambda(N) \rightarrow \lambda(P)$ and gof : M $\rightarrow$ P are in **CM**.

Now $\lambda(g \text{ o } f) : \lambda(M) \rightarrow \lambda(P)$ by $\lambda(g.f)(\alpha) = \alpha(gf)^{-1} = \alpha(f^{-1}.\, g^{-1}) = (\alpha f^{-1})\, g^{-1} = (\lambda(f)(\alpha))\, g^{-1} = \lambda(f)(\alpha g^{-1}) = (\lambda(f)\, \lambda(g))(\alpha)$ are in **CF**, also $\lambda(g)\, \lambda(f) : \lambda(M) \rightarrow \lambda(P)$ in **CF**

$\Rightarrow \lambda(g \text{ o } f) = \lambda(g)\, \lambda(f)$.

Also $\lambda(I_M) = I_{\lambda(M)}$

$\Rightarrow$ λ: **CM** $\rightarrow$ **CF** is a covariant functor

Proposition 2.5

Let R be a ring and M be a fixed R- module, then $Hom_R(M,N)$ is a fuzzy R-module, for any R-module N.

Proof

Using Definition 1.1 and [7], it follows.

Proposition2.6

Let R be a ring and M be a fixed R- module, the R-homomorphism f:N$\rightarrow$P induces

i) an fuzzy R-homomorphism $f_* : Hom_R(M,N) \rightarrow Hom_R(M,P)$ and
ii) an fuzzy R-homomorphism $f^* : Hom_R(P,M) \rightarrow Hom_R(N,M)$

Proof: Using Definition 1.1 and Lemma1.12, it follows.

Corollary 2.7

For any fixed R-module M, the fuzzy R module $Hom_R(M,N)$ and their fuzzy R-homomorphisms forms category, for any R-module N; this category is denoted by 'CF'

Proof

Using Proposition 2.6 and Lemma1.12(a) , it follows

Proposition 2.8

'Hom_R' is a invariant functor in the sense that it is both a covariant and a contravariant functor

Hom_R : CM$\rightarrow$ CM is an invariant functor, for any fixed R-module M.

Proof:

Define Hom_R : CM$\rightarrow$ CF by

$Hom_R(N) = Hom_R(M,N)$, for any fixed R-module M, which is the object of CF.

Let N,P are two R-modules in CM and f:N$\rightarrow$ P be R-homomorphisms in **C**M, then

$Hom_R(f) = f_* : Hom_R(M,N) \rightarrow Hom_R(M,P)$ in CF and $f^* : Hom_R(P,M) \rightarrow Hom_R(N,M)$ are well defined mapping and so by **Definition1.1, Lemma1.12** and **Theorems 2.6**,the theorem follows.

Proposition 2.9

'Hom_R' is a Homotopy type functor in the sense that if f is a homotopy equivalence for any two R-modules M and N, then $Hom_R(f)$ is a isomorphism

Proof:

Since f is a homotopy equivalence for any two R-modules M and N, there exists f:M $\rightarrow$N and

g: N $\rightarrow$M such that $g.f \simeq I_M$ and $f.\,g \simeq I_N$, then $Hom_R(f): Hom_R(P,M) \rightarrow Hom_R(P,N)$ and

$Hom_R(f): Hom_R(N,P) \rightarrow Hom_R(M,P)$ are fuzzy R-homomorphisms, then Hom_R satisfies the following conditions:

i) **f** $\simeq$ **g** $\Rightarrow Hom_R(f) = Hom_R(g)$
ii) $g.f \simeq I_M \Rightarrow Hom_R(g\ f) = Hom_R(I_M) = Id. \Rightarrow Hom_R(g).Hom_R(f) = Id$
iii) $f.\,g \simeq I_N \Rightarrow Hom_R(f\ g) = Hom_R(I_N) = Id. \Rightarrow Hom_R(f).Hom_R(g) = Id$

Thus $Hom_R(f)$ is isomorphic to $Hom_R(g)$

Corollary 2.10.

Hom_R is also a Homotopy type invariant functor.

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Proposition 2.11

A fuzzy R-map $\tilde{f} \in Hom_R(\lambda_M,\eta_P)$ is called fuzzy split iff $\tilde{f}$ is an isomorphism

Proof: By the definition of a fuzzy split, it follows that there exists $\tilde{g} \in Hom(\eta_P,\lambda_M)$ such that $\tilde{f}\tilde{g} = \tilde{I_P}$ and $\tilde{g}\tilde{f} = \widetilde{I_M}$.

$\Rightarrow$ $\tilde{f}$ is a epimorphism and $\tilde{f}$ is also a monomorphism.

$\Rightarrow \tilde{f}$ is an isomorphism.

Conversely suppose that a fuzzy R-map $\tilde{f}$ is an isomorphism, then $\tilde{f}$ is an epimorphism and monomorphism,i.e.,there exists $\tilde{g} \in Hom(\eta_P,\lambda_M)$ such that $\tilde{f}\tilde{g} = \tilde{I_P}$ and $\tilde{g}\tilde{f} = \widetilde{I_M}$,

$\Rightarrow \tilde{f}$ is fuzzy split.

Proposition 2.12

All homotopy type invariant functors form a function spaces from category CM to the category CF, it is denoted by F^M.

Proof:

Using the **Definition1.8**, **Theorems 2.8**, **Theorem 2.9** and **Theorem 2.10**, it follows.

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