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Thermo elastic Problem of a Thick Circular Plate due to Heat Generation

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Abstract- In this paper, an attempt has been made to study thermoelastic response of a thick circular plate occupying the space $D: 0 \le r \le a,\ -h \le z \le h,$ due to heat generation with radiation type boundary conditions. Here we apply integral transform techniques to find the thermoelastic solution.

Keywords: **Thermo elastic problem, Circular Plate, Thermal Stresses, integral transform.**

I. INTRODUCTION

Khobragade et al. [3 - 12] have derived temperature distribution, displacement function, thermal stresses and thermal deflection of a thick and thin circular plate. Further Khobragade et al. [13] have established displacement function, temperature distribution and stresses and deflection of a triangular plate.

This paper is concerned with transient thermoelastic problem of a thick circular plate occupying the space $D: 0 \le r \le a,\ -h \le z \le h,$ due to heat generation with radiation type boundary conditions.

II. STATEMENT OF THE PROBLEM

Consider thick circular plate of thickness $2h$ occupying the space $D: 0 \le r \le a,\ -h \le z \le h,$ the material is homogenous and isotropic. The differential equation governing the displacement potential function $\phi(r,z,t)$ as Nowacki [2] is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \left(\frac{1+\nu}{1-\nu}\right)\alpha_t T \tag{1}$$

Where ν and a_t are Poisson's ratio and linear coefficient of thermal expansion of the material of the plate and T is the temperature of the plate satisfying the differential equation as Noda [3] is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \chi(r,z,t) = \frac{1}{k}\frac{\partial T}{\partial t} \tag{2}$$

Subject to initial condition

$$M_t(T,1,0,0) = F(r,z) \qquad 0 \le r \le a, -h \le z \le h. \tag{3}$$

The boundary conditions are

$$\left.\begin{aligned} M_r(T,0,1,0) &= g_1(z,t) \\ M_r(T,0,1,a) &= g_2(z,t) \end{aligned}\right\}, \quad -h \le z \le h,\ t>0 \tag{4}$$

$$\left.\begin{aligned} M_z(T,1,k_1,h) &= f_1(r,t) \\ M_z(T,1,k_2,-h) &= -\frac{Q_0}{\lambda} f_2(r,t) \end{aligned}\right\}, \quad 0 \le r \le a,\ t>0 \tag{5}$$

Where k is thermal diffusivity of material of the plate.

The displacement function in the cylindrical coordinate system are represented by Love's function as Khobragade [4] are

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 L}{\partial r \partial z} \tag{6}$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial z^2} \tag{7}$$

The Love's function [14] must satisfy

$$\nabla^2 \nabla^2 L = 0 \tag{8}$$

Where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The component of stresses are represented by the thermoelastic displacement potential ϕ and Love's function L as Noda [3] are

$$\sigma_{rr} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial r^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\nu \nabla^2 L - \frac{\partial^2 L}{\partial r^2}\right)\right\} \tag{9}$$

$$\sigma_{\theta\theta} = 2G\left\{\left(\frac{1}{r}\frac{\partial \phi}{\partial r} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left(\nu \nabla^2 L - \frac{1}{r}\frac{\partial^2 L}{\partial r^2}\right)\right\} \tag{10}$$

$$\sigma_{zz} = 2G\left\{\left(\frac{\partial^2 \phi}{\partial z^2} - \nabla^2 \phi\right) + \frac{\partial}{\partial z}\left\{\left((z-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\}\right\} \tag{11}$$

$$\sigma_{rz} = 2G\left\{\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}\left\{\left((1-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right)\right\}\right\} \tag{12}$$

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For traction free surface stress function

$\sigma_z = \sigma_{r\theta} = 0$ at $z = \pm h$ for thick plate.

Equations (1) to (12) constitute the mathematical formulation of the problem under consideration.

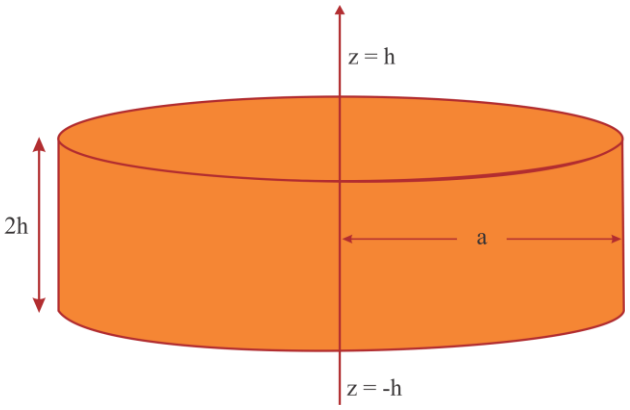

Fig. 1: The geometry of the problem

III. SOLUTION OF THE PROBLEM

Applying Hankel transform to the equation (2), we get

$$-\xi_m^2 \bar{T}(\xi_m,z,t) + \frac{d^2\bar{T}}{dz^2}(\xi_m,z,t) + \bar{\chi}(\xi_m,z,t) = \frac{1}{k}\frac{d\bar{T}}{dt} \qquad (13)$$

Again applying Marchi-Fasulo transform to above equation, we obtain

$$\frac{d\bar{\bar{T}}^*}{dt} + kP^2\bar{\bar{T}}^* = \Psi \qquad (14)$$

where

$$P^2 = \xi_m^2 + a_n^2$$

Equation (14) is a linear equation whose solution is given by

$$\bar{\bar{T}}^*(\xi_m,n,t) = e^{-kP^2 t}\int_0^t \Psi e^{kP^2 t'} dt^1 + Ce^{-kP^2 t} \qquad (15)$$

Using (3), we get

$$C = F^*(m,n)$$

Thus we have

$$\bar{\bar{T}}^*(\xi_m,n,t) = e^{-kP^2 t}\left[\int_0^t \Psi e^{kP^2 t^1} dt^1 + \bar{F}^*(m,n)\right] \qquad (16)$$

Applying inversion of Marchi-Fasulo transform and Hankel transform to the differential equation (16), we get

$$T(r,z,t) = \frac{2}{a^2}\sum_m \sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n} e^{-kP^2 t}$$

$$\times\left[\int_0^t \Psi e^{kP^2 t'} dt' + \bar{F}^*(m,n)\right] \qquad (17)$$

This is the desired solution of the given problem.

Temperature distribution (Cooling Process)

T' is the temperature(cooling) of the plate satisfying the differential equation as Noda [3] is

$$\frac{\partial^2 T'}{\partial r^2} + \frac{1}{r}\frac{\partial T'}{\partial r} + \frac{\partial^2 T'}{\partial z^2} = \frac{1}{k}\frac{\partial T'}{\partial t} \qquad (18)$$

Subject to initial condition

$$T'(r,z,t)\big|_{t=t_0} = T(r,z,t)\big|_{t=t_0} = T(r,z,t_0) = G(r,z) \qquad (19)$$

The boundary conditions are

$$\left.\begin{aligned} M_r(T',0,1,0) &= g_3(z,\ t) \\ M_r(T',0,1,a) &= g_4(z,\ t) \end{aligned}\right\}, \quad -h \le z \le h,\ t > 0 \qquad (20)$$

$$\left.\begin{aligned} M_z(T',1,k_1,h) &= f_3(r,\ t) \\ M_z(T',1,k_2,-h) &= f_4(r,\ t) \end{aligned}\right\}, \quad 0 \le r \le a,\ \ t > 0 \qquad (21)$$

Applying Hankel transform and Marchi- Fasulo transform one obtains

$$\frac{d\bar{\bar{T}}'^*}{dt} + kp^2\bar{\bar{T}}'^* = \Psi \qquad (22)$$

where

$$p^2 = \xi_m^2 + \lambda_n^2$$

and $\Psi = k\left(\frac{P_n(h)}{k_1}\bar{f}_3 - \frac{P_n(-h)}{k_2}\bar{f}_4\right)$

Equation (22) is a linear equation whose solution is given by

$$\bar{\bar{T}}'^*(\xi_m,n,t) = \phi + e^{-kP^2 t}\int_0^t \Psi e^{kP^2 t'} dt^1 \qquad (23)$$

where

$$\phi = \bar{G}^* - e^{-kP^2 t_0}\int_0^{t_0} \Psi e^{kP^2 t} dt$$

Applying inversion of Marchi-Fasulo transform and Hankel transform to the differential equation (16), we get

$$T'(r,z,t) = \frac{2}{a^2}\sum_m \sum_n \frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n} e^{-kP^2 t}$$

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$$\times\left(\phi+e^{-kP^2t}\int_0^t \Psi e^{kP^2t'}dt'\right) \quad (24)$$

This is the desired solution of the given problem.

Let us assume Love's function L, which satisfy condition (10) as

$$L(r,z)=\frac{2}{a^2}\sum_m\sum_n\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n}\Omega \quad (25)$$

where

$$\Omega=e^{-kP^2t}\left[\int_0^t \Psi e^{kP^2t'}dt'+\overline{F}^*(m,n)\right]$$

Using (1) and (17), we get displacement potential ϕ *as*

$$\phi=A\sum_m\sum_n\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n'(z)}{\lambda_n}\Omega+B(t)\right] \quad (26)$$

where

$$A=\left(\frac{1+\nu}{1-\nu}\right)\frac{2\alpha_t}{a^2},$$

$$B(t)=\int e^{-kP^2t}\left(\int_0^t \Psi\, e^{-kP^2t'}dt'+\overline{F}^*(m,n)\right)dt$$

IV. DETERMINATION OF DISPLACEMENT FUNCTION

Substituting equations (25) and (26) in equation (6), (7) we get

$$u_r=A\sum_m\sum_n\frac{\xi_mJ_1(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n'(z)}{\lambda_n}\psi+B(t)\right]$$

$$-\frac{2}{a^2}\sum_m\sum_n\frac{\xi_mJ_1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n'(z)}{\lambda_n}\Omega \quad (27)$$

$$u_z=A\sum_m\sum_n\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n''(z)}{\lambda_n}\Omega+B(t)\right]$$

$$+2(1-\nu)\left[\frac{2}{a^2}\sum_m\sum_n\frac{\xi_m^2[J_1'(r\xi_m)+J_1(r\xi_m)]}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n}\Omega\right]$$

$$+\frac{2(1-2\nu)}{a^2}\sum\sum\frac{J_0(r\xi_m)}{J_1(a\xi_m)^2}\frac{P_n''(z)}{\lambda_n}\Omega \quad (28)$$

Substituting equations (25) and (26) in equations (9) to (12), we obtain

$$\sigma_{rr}=2G\left\{\frac{2(\nu-1)}{a^2}\sum_m\sum_n\frac{\xi_m^2J_1'(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n'(z)}{\lambda_n}\Omega\right.$$

$$+\frac{2}{a^2r}\sum_m\sum_n\frac{\xi_mJ_1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n'(z)}{\lambda_n}\Omega$$

$$+\frac{2}{a^2}\sum_m\sum_n\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n'(z)}{\lambda_n}\Omega$$

$$-\frac{A}{r}\sum_m\sum_n\frac{\xi_mJ_1(r\xi_m)}{J_1(a\xi_m)^2}\left[\frac{P_n'(z)}{\lambda_n}\Omega+B(t)\right]$$

$$\left.-A\sum_m\sum_n\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n'(z)}{\lambda_n}\Omega+B(t)\right]\right\} \quad (29)$$

$$\sigma_{\theta\theta}=2G\left\{\frac{2\nu}{a^2}\sum_m\sum_n\frac{\xi_m^2J_1^1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^1(z)}{\lambda_n}\Omega\right.$$

$$+\frac{2(\nu-1)}{a^2}\sum\sum\frac{\xi_mJ_1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^1(z)}{\lambda_n}A\Omega$$

$$+\frac{2\nu}{a^2}\sum_m\sum_n\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^{111}(z)}{\lambda_n}\Omega$$

$$-A\sum_m\sum_n\frac{J_0(r\xi_m)}{J_1(a\xi_m)^2}\left[\frac{P_n^1(z)}{\lambda_n}\Omega+B(t)\right]$$

$$\left.-A\sum_m\sum_n\xi_m^2\frac{J_1^1(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n^1(z)}{\lambda_n}\Omega+B(t)\right]\right\} \quad (30)$$

$$\sigma_{zz}=2G\left\{\frac{(2-\nu)}{a^2}\sum\sum\frac{\xi_m^2J_1^1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^1(z)}{\lambda_n}\Omega\right.$$

$$+\frac{2(2-\nu)}{a^2r}\sum_m\sum_n\frac{\xi_mJ_1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^1(z)}{\lambda_n}\Omega$$

$$+\frac{2(1-\nu)}{a^2}\sum\sum\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^{111}(z)}{\lambda_n}\Omega$$

$$-A\sum\sum\frac{\xi_m^2J_1^1(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n^1(z)}{\lambda_n}\Omega+B(t)\right]$$

$$\left.-\frac{A}{r}\sum\sum\frac{\xi_mJ_1(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n^1(z)}{\lambda_n}\Omega+B(t)\right]\right\} \quad (31)$$

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$$\sigma_{rz}=2G\left\{\frac{2(1-\nu)}{a^2}\sum\sum\frac{\xi_m^3 J_1^{11}(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P^1(z)}{\lambda_n}\Omega\right.$$
$$+\frac{2(-\nu)}{a^2}\sum\sum\frac{\xi_m J_1(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n^{11}(z)}{\lambda_n}\Omega$$
$$+\frac{2(1-\nu)}{a^2}\sum\sum\frac{\xi_m}{[J_1(a\xi_m)]^2}\left\{\frac{\xi_m r J_1^1(r\xi_m)}{r_2}\right\}\frac{P_n(z)}{\lambda_n}\Omega$$
$$\left.+A\sum\sum\frac{\xi_m J_1(r\xi_m)}{[J_1(a\xi_m)]^2}\left[\frac{P_n^4(z)}{\lambda_n}\Omega+B(t)\right]\right\} \quad (32)$$

where

$$A=\left(\frac{1+\nu}{1-\nu}\right)\frac{2\alpha_t}{a^2},$$

$$\Omega=e^{-kp^2t}\left[\int_0^t \Psi e^{-kp^2t^1}dt^1+\overline{F}^*(m,n)\right],$$

$$B(t)=\int\Omega\, dt$$

V. SPECIAL CASE

Set $F(r,z)=z^2(1-r^2)$ (33)

Applying Marchi-Fasulo transform, are obtain

$$\overline{F}(r,n)=(1-r^2)\int_{-h}^{h} z^2 P_n(z)dz$$

$$\overline{F}(r,n)=(1-r^2)\Phi_n\left[\frac{2h^2\sin(a_n h)}{a_n}+\frac{4h\cos(a_n h)}{a_n{}^2}-\frac{4\sin(a_n h)}{a_n{}^3}\right]$$

Where

$$P_n(z)=Q_n\cos(a_n z)-W_n\sin(a_n z)\ ,$$

$$Q_n=a_n(\alpha_1+\alpha_2)\cos(a_n h)+(\beta_1-\beta_2)\sin(a_n h)$$

$$W_n=(\beta_1-\beta_2)\cos(a_n h)+a_n(\alpha_1-\alpha_2)\sin(a_n h)$$

Again on applying Hankel transform, we obtain

$$\overline{F}^*(m,n)=\Pi_n\left[\frac{a}{\xi_m}J_1(a\xi_m)-\frac{a(a^2\xi_m^2-4)}{\xi_m^3}J_1(a\xi_m)-\frac{2a^2}{\xi_m^2}J_0(a\xi_m)\right] \quad (35)$$

Where

$$\Pi_n=\Phi_n\left[\frac{2h^2\sin(a_n h)}{a_n}+\frac{4h\cos(a_n h)}{a_n{}^2}-\frac{4\sin(a_n h)}{a_n{}^3}\right]$$

And

$$\Phi_n=a_n(\alpha_1+\alpha_2)\cos(a_n h)+(\beta_1-\beta_2)\sin(a_n h).$$

Using equation (27) in equation (17), one obtains

$$T(r,z,t)=\frac{2}{a^2}\sum_m\sum_n\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n}e^{-kp^2t}$$
$$\times\left[\int_0^t \Psi e^{kp^2t}\, dt^1+\Pi_n\right]$$
$$\times\left(\frac{a}{\xi_m}J_1(a\xi_m)-\frac{a(a^2\xi_m^2-4)}{\xi_m{}^3}J_1(a\xi_m)-\frac{2a^2}{\xi_m^2}J_0(a\xi_m)\right) \quad (36)$$

VI. NUMERICAL RESULTS

Set

$a=2, k=15.9\times10^6, t=1$ second in equation (36), we get

$$T(r,z,t)=\frac{2}{4}\sum_m\sum_n\frac{J_0(r\xi_m)}{[J_1(a\xi_m)]^2}\frac{P_n(z)}{\lambda_n}e^{-(15.9\times10^6)P^2t}$$
$$\times\left[\int_0^1 \Psi e^{(15.9\times10^6)P^2t^1}dt^1+\Pi_n\left(\frac{2}{\xi_m}J_1(2\xi_m)-\frac{2(4\xi_m^2-4)}{\xi_m{}^3}J_1(2\xi_m)-\frac{2}{\xi_m^2}J_0(2\xi_m)\right)\right] \quad (37)$$

VII. CONCLUSION

In this article, the temperature distribution, displacement and thermal stresses of a thick circular plate are investigated with known boundary conditions. Finite integral transform techniques are used to obtain numerical results. The results are obtained in terms of Bessel's function in the form of infinite series.

Any particular cases of special interest can be assigned to the parameters and functions in expressions. The results that are obtained can be useful to the design of structure or machines in engineering applications.

(34)

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