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Holder Valuation for Z-Module

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Abstract. ***The purpose of this article is to introduce the notion (C_1, C_2)-H¨older valuation on Z-module G and it is proved that a (C_1,C_2)-H¨older valuation equivalent to classical valuation on Z-module G . These results provide an extension of the Garsia theorem [2] for Z -module.***

Key words **.Valuation, H¨older valuation, H¨older equivalent,**

I. INTRODUCTION

The theory of valuations may be viewed as a branch of topological algebra. The development of valuation theory has spanned over more than a hundred years. First introduced the notion of valuations on fields. Details on valuations on fields can be found in many monographs, e.g Endler [1], Ribenboim [8], and Schilling [9]. Then Manis introduced the notion of valuations in the category of commutative rings can be found in Manis[6], Huckaba[5], and Knebusch and Zhang[6].Then Zeng Guangxing[4] introduced the notion of valuations on a module.

Definition 1.1. Let M be an R-module where R **is** a ring and Δ be and ordered set with maximum element ∞.
A mapping ν of M onto Δ is called a valuation on M, if the following conditions satisfied:

(i) For any x,y$\in$ M, $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$;

(ii) If $\nu(x) \leq \nu(y)$, x, y $\in$ M, then $\nu(ax) \leq \nu(ay)$ for all a $\in$ R;

(iii) Put $\nu^{-1}(\infty) = \{x \in M \mid \nu(x) = \infty\}$. If $\nu(az) \leq \nu(bz)$, where a, b $\in$ R, and z$\in$ $M\setminus \nu^{-1}(\infty)$, then $\nu(ax) \leq \nu(bx)$ for all x $\in$ M;

(iv) For every a $\in$ $R\setminus(\nu^{-1}(\infty) : M)$, there is an $a' \in R$ such that $\nu((a'a)x) = \nu(x)$.

In this case Δ is called the value set of ν, and $\nu^{-1}(\infty)$ is called the core of ν.

Remark 1.2. In Fuchs [2] for an ordered abelian group G and ordered set Δ With maximum element ∞, a mapping ν of G onto Δ is defined to be a valuation G if the following conditions satisfied:

V1) For x $\in$ G, $\nu(x) = \infty$ if and only if x = 0;

(V2) For any x, y $\in$ G
$\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$

(V3) For every nonzero integers n, $\nu(nx) = \nu(x)$.

Viewing G as a Z-module, it is easy to see that a mapping ν is a valuation on G in the sence of the Definition 1.1. In this case, the core of ν is $\{0\}$, and the induced valuation pair is
$(G, \{0\})$.

Definition 2.1. Let G be Z-module, and Δ be an ordered set with maximum element ∞, and $C_1 \geq 1$, $C_2 \geq 1$. A (C_1, C_2) - H¨older valuation on Z-module G, **is** a mapping $\|.\| : G \to \Delta$ satisfying:

(H i) For x $\in$ G , $\|x\| = \infty$ if and only if x = 0;

(H ii) For any x, y $\in$ G-{0}, $\|x + y\| \geq C_2 \min\{\|x\|, \|y\|\}$;

(H iii) For every nonzero integers n and x $\in$ G, $C_1^{-1}\|x\| \leq \|nx\| \leq C_1\|x\|$.

Remark 2.2. We note a (1, 1)-H¨older valuation on Z-module G is a classical valuation on Z-module G.

Definition 2.3. Let $|.|_1$, $|.|_2$ be two valuations on Z-module G. Then we say that they are (C_0, α)-H¨older equivalent, if for all x$\in$ G,
$C_0 \geq 1$, $\alpha > 0$, $C_0^{-1}|x|_1^{\alpha} \leq |x|_2 \leq C_0|x|_1^{\alpha}$ where $\alpha' = \alpha$ or $\alpha = \alpha^{-1}$.

Theorem 2.4. let $|.|_1 : G \to \mathbf{R}_+ \cup \{\infty\}$ be a (C_1, C_2)-H¨older valuation on Z-module G, where $C_1 \geq 1$, $C_2^{\alpha} \geq 4$. Then there exists a classical valuation $|.|_2$ on Z-module G that is (2, α)-H¨older equivalent to the (C_1, C_2)-H¨older valuation $|.|_1$, where $\alpha = \log^2_{C1}$.

For the proof of the theorem we need several lemmas and

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propositions.

Definition 2.5. For $x \in G$, we define $|x|_3 = |x|^{\alpha}_1$.

Lemma 2.6. The mapping $|.|_3 : G \to \mathbf{R}_+ \cup \{\infty\}$ is a $(2, C_2^{\alpha})$-H¨older valuation on Z-module G.

Proof. (1): For $x \in G$, $|x|_3 = |x|^{\alpha}_1 = \infty$, if and only if

$|x|_1 = \infty$, if and only if $x = 0$;

(2): If $x, y \in G-\{0\}$, then

$|x + y|_3 = |x + y|^{\alpha}_1 \geq (C_2 \min\{|x|_1, |y|_1\})^{\alpha}$.

Therefore

$|x + y|_3 \geq (C_2^{\alpha} \min\{|x|^{\alpha}_1, |y|^{\alpha}_1\} = C_2^{\alpha} \min\{|x|_3, |y|_3\})$.

(3): For any nonzero integers n, we have

$$C_1^{-1}|x|_1 \leq |nx|_1 \leq C_1|x|_1.$$

Hence

$$C_1^{-\alpha}|x|^{\alpha}_1 \leq (|nx|_3 = |nx|^{\alpha}_1) \leq C_1^{\alpha}|x|^{\alpha}_1.$$

Therefore

$$2^{-1}|x|_3 \leq |nx|_3 \leq 2|x|_3. \quad \blacksquare$$

Lemma 2.7. Let $x \in G-\{0\}$ and $\{a_n\} = \{|nx|_3\}$.

Then the sequenc $\{a_n\}$ is bounded.

Proof. By part (3) of lemma 2.6, we have
$2^{-1}|x|_3 \leq |nx|_3 \leq 2|x|_3$.

Therefore the sequence $\{a_n\}$ is bounded. $\blacksquare$

Now we prove that the sequence $\{a_n\}$ is increasing.

$a_{n+1} = |(n + 1)x|_3 = |nx + x|_3$
$\geq C_2^{\alpha} \min\{|nx|_3, |x|_3\} \geq C_2^{\alpha} \min\{|nx|_3, 2^{-1}|nx|_3\}$

(by part (2),(3) of lemma 2.6). Therefore

$a_{n+1} \geq C_2^{\alpha} 2^{-1}|nx|_3$. Since $C_2^{\alpha} \geq 4 \geq 2$ and so $C_2^{\alpha} 2^{-1} \geq 1$.

Consequently $a_{n+1} \geq (|nx|_3 = a_n)$

Therefore $\{a_n\}$ is increasing. Hence we have the following results

Corollary 2.8. The sequence $\{a_n\} = \{|nx|_3\}$ is convergent sequence.

Proof. The sequence $\{a_n\} = \{|nx|_3\}$ is bounded and increasing on $\mathbf{R}_+$

since $\mathbf{R}_+$ is metric space. Therefore the sequence $\{a_n\} = \{|nx|_3\}$ is

convergent sequence. $\blacksquare$

Proposition 2.9. Let $|x|_2 = \lim_{n\to\infty} |nx|_3$. Then $2^{-1}|x|_3 \leq |x|_2 \leq 2|x|_3$.

Proof. By part (3) of lemma 2.6, we have

$$2^{-1}|x|_3 \leq |nx|_3 \leq 2|x|_3.$$

Therefoe

$$\lim_{n\to\infty} 2^{-1}|x|_3 \leq \lim_{n\to\infty} |nx|_3 \leq \lim_{n\to\infty} 2|x|_3.$$

Hence

$$2^{-1}|x|_3 \leq |x|_2 \leq 2|x|_3. \quad \blacksquare$$

Proposition 2.10. The mapping

$|.|_2 : G \to \mathbf{R}_+ \cup \{\infty\}$ by equation

$$|x|_2 = \lim_{n\to\infty} |nx|_3$$

is classical valuation on Z-module G.

Proof. (1) By proposition 2.9, for $x \in G$, we have $|x|_2 = \infty$, if and only if

$|x|_3 = \infty$, if and only if $x \in \{a_n\}$ is

(by part (1) of lemma 2.6).

(2) If $x, y \in G - \{0\}$, then

$|x + y|_3 \geq C_2^{\alpha} \min\{|x|_3, |y|_3\} \geq 2\min\{2|x|_3, 2|y|_3\}$.

Thus by proposition 2.9, we have
$|x + y|_3 \geq 2\min\{|x|_2, |y|_2\}$. On the other hand
$|x + y|_2 \geq 2^{-1}|x + y|_3 \geq 2^{-1}(2\min\{|x|_2, |y|_2\})$.
$= \min\{|x|_2, |y|_2\}$.
If x=o or y=0, then it is clear that
$|x + y|_2 = \min\{|x|_2, |y|_2\}$.

Therefore for all $x,y \in G$, we have
$|x + y|_2 \geq \min\{|x|_2, |y|_2\}$.

3) For any nonzero integers m and $x \in G$, we have
$|mx|_2 = \lim_{n\to\infty} |nmx|_3 = \lim_{n|m|\to\infty} |n\,mx|_3 = |x|_2$.

Therefore

$|.|_2$ is classical valuation on Z-module G.

Proof of Theorem 2.4. By proposition 2.10, the mapping

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$|.|_2 : G \to \mathbf{R}_+ \cup \{\infty\}$ is classical valuation on Z-module G , and by proposition 2.9, we have

$$2^{-1}|x|_3 \leq |x|_2 \leq 2|x|_3.$$

Therefore

$$2^{-1}|x|^{\alpha}_1 \leq |x|_2 \leq 2|x|^{\alpha}_1.$$

Thus classical valuation $|\ .\ |_2$ on Z-module G which is (2, α)-H¨older equivalent to the

(C_1, C_2)-H¨older valuation $|.|_1$ on Z-module G.

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