Hölder Valuation for Z-Module

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Abstract. The purpose of this article is to introduce the notion (C₀, C₂)-Hölder valuation on Z-module G and it is proved that a (C₀, C₂)-Hölder valuation equivalent to classical valuation on Z-module G. These results provide an extension of the Garsia theorem [2] for Z-module.

Key words. Valuation, Hölder valuation, Hölder equivalent.

I. INTRODUCTION

The theory of valuations may be viewed as a branch of topological algebra. The development of valuation theory has spanned over more than a hundred years. First introduced the notion of valuations on fields. Details on valuations on fields can be found in many monographs, e.g. Endler [1], Ribenboim [8], and Schilling [9]. Then Manis introduced the notion of valuations in the category of commutative rings can be found in Manis[6], Huckaba[5], and Knebusch and Zhang[6]. Then Zeng Guangxing[4] introduced the notion of valuations on a module.

Definition 1.1. Let M be an R-module where R is a ring and ∆ be an ordered set with maximum element ∞. A mapping v of M onto ∆ is called a valuation on M, if the following conditions satisfied:

(i) For any x, y ∈ M, v(x + y) ≥ min{v(x), v(y)};
(ii) If v(x) ≤ v(y), x, y ∈ M, then v(ax) ≤ v(ay) for all a ∈ R;
(iii) Put v^(−1)(∞) = {x ∈ M | v(x) = ∞}. If v(az) ≤ v(bz), where a, b ∈ R, and z ∈ M, v(−1)(∞), then v(ax) ≤ v(bx) for all x ∈ M;
(iv) For every a ∈ R, v^(−1)(∞) : M, there is an a' ∈ R such that v((a'a)x) = v(x).

In this case ∆ is called the core of v, and v^(−1)(∞) is called the core of v.

Remark 1.2. In Fuchs [2] for an ordered abelian group G and ordered set ∆ With maximum element ∞, a mapping v of G onto ∆ is defined to be a valuation G if the following conditions satisfied:

V1) For x ∈ G, v(x) = ∞ if and only if x = 0;
V2) For any x, y ∈ G
\[ v(x+y) \geq \min\{v(x),v(y)\} \]
V3) For every nonzero integers n, v(nx) = v(x).

Viewing G as a Z-module, it is easy to see that a mapping v is a valuation on G in the sense of the

Definition 1.1. In this case, the core of v is {0}, and the induced valuation pair is (G, {0}).

Definition 2.1. Let G be a Z-module, and ∆ be an ordered set with maximum element ∞, and C₁ ≥ 1, C₂ ≥ 1. A (C₁, C₂)-Hölder valuation on Z-module G, is a mapping v : G → ∆ satisfying:

(H i) For x ∈ G, ∥x∥ = ∞ if and only if x = 0;
(H ii) For any x, y ∈ G-{0}, ∥x + y∥ ≥ C₂ min ∥x∥, ∥y∥,
(H iii) For every nonzero integers n and x ∈ G, C₁∥nx∥ ≤ C₁∥x∥.

Remark 2.2. We note a (1, 1)-Hölder valuation on Z-module G is a classical valuation on Z-module G.

Definition 2.3. Let | . |, | | : be two valuations on Z-module G. Then we say that they are (C₀, α)-Hölder equivalent, if for all x ∈ G, C₀ ≥ 1, α > 0, C₀α|x|α ≤ |x| ≤ C₀α|x|α, where α = α.α.

Theorem 2.4. Let | . | : G → R, U = ∞ be a (C₁, C₂)-Hölder valuation on Z-module G, where C₁ ≥ 1, C₂ ≥ 4. Then there exists a classical valuation | . | on Z-module G that is (2, α)-Hölder equivalent to the (C₁, C₂)-Hölder valuation on Z-module G, where α = log² C₁.

For the proof of the theorem we need several lemmas and
propositions.

**Definition 2.5.** For \( x \in G \), we define \( |x|_3 = |x|^3 \).

**Lemma 2.6.** The mapping \( |.| : G \rightarrow |.| \) is a \( (2, \, C_2^3) \)-Hölder valuation on \( G \).

Proof. (1): For \( x \in G \), \( |x|_3 = |x|^3 \), if and only if \( |x|_3 = \infty \), if and only if \( |x| = \infty \).

(2): If \( x, y \in G \), then
\[
|x + y|_3 = |x + y|^3 \geq \left( C_2^3 \min \{ |x|^3, |y|^3 \} \right)^3.
\]

(3): For any nonzero integers \( n \), we have
\[
C_1^{-1} |x|_3 \leq |nx|_3 \leq C_1 |x|_3.
\]

Hence
\[
C_1^{-1} |x|^3 \leq |nx|^3 \leq C_1 |x|^3.
\]

Therefore
\[
2^{-1} |x|_3 \leq |nx|_3 \leq 2 |x|_3.
\]

**Lemma 2.7.** Let \( x \in G \) and \( \{a_n\} = \{ |nx|_3 \} \). Then the sequence \( \{a_n\} \) is bounded.

Proof. By part (3) of Lemma 2.6, we have
\[
2^{-1} |x|_3 \leq |nx|_3 \leq 2 |x|_3.
\]

Therefore the sequence \( \{a_n\} \) is bounded.

\[
\text{Now we prove that the sequence } \{a_n\} \text{ is increasing.}
\]

\[
a_{n+1} = |(n + 1)x|_3 = |nx + x|_3 \\
\geq C_2^3 \min \{ |nx|_3, |x|_3 \} \geq C_2^3 \min \{ |nx|_3, 2^{-1} |x|_3 \}
\]

(by part (2),(3) ofLemma 2.6). Therefore
\[
a_{n+1} \geq C_2^3 2^{-1} |x|_3.
\]

Consequently \( a_{n+1} \geq ( |nx|_3 - a_n ) \)

Therefore \( \{a_n\} \) is increasing. Hence we have the following results.

**Corollary 2.8.** The sequence \( \{a_n\} = \{ |nx|_3 \} \) is convergent sequence.

Proof. The sequence \( \{a_n\} = \{ |nx|_3 \} \) is bounded and increasing on \( G \), since \( G \) is metric space. Therefore the sequence \( \{a_n\} = \{ |nx|_3 \} \) is convergent sequence.

**Proposition 2.9.** Let \( |x|_2 = \lim_{n \rightarrow \infty} |nx|_3 \). Then \( 2^{-1} |x|_3 \leq |x|_2 \leq 2 |x|_3 \).

Proof. By part (3) of Lemma 2.6, we have
\[
2^{-1} |x|_3 \leq |nx|_3 \leq 2 |x|_3.
\]

Therefore
\[
\lim_{n \rightarrow \infty} 2^{-1} |x|_3 \leq \lim_{n \rightarrow \infty} |nx|_3 \leq \lim_{n \rightarrow \infty} 2 |x|_3.
\]

Hence
\[
2^{-1} |x|_3 \leq |x|_2 \leq 2 |x|_3.
\]

**Proposition 2.10.** The mapping \( |.| : G \rightarrow |.| \) by equation
\[
|x|_2 = \lim_{n \rightarrow \infty} |nx|_3
\]

is classical valuation on \( G \).

Proof. (1) By Proposition 2.9, for \( x \in G \), we have \( |x|_2 = \infty \), if and only if
\[
|x|_2 = \infty, \text{ if and only if } x = 0
\]

(by part (1) of Lemma 2.6).

(2) If \( x, y \in G \), then
\[
|x + y|_2 \geq C_2^3 \min \{ |x|_2, |y|_2 \} \geq 2^{-1} \min \{ 2 |x|_2, 2 |y|_2 \}.
\]

Thus by Proposition 2.9, we have
\[
|x + y|_2 \geq 2^{-1} \min \{ |x|_2, |y|_2 \}.
\]

If \( x = 0 \) or \( y = 0 \), then it is clear that
\[
|x + y|_2 = \min \{ |x|_2, |y|_2 \}.
\]

Therefore for all \( x, y \in G \), we have
\[
|x + y|_2 \geq \min \{ |x|_2, |y|_2 \}.
\]

3) For any nonzero integers \( m \) and \( x \in G \), we have
\[
|mx|_2 = \lim_{n \rightarrow \infty} |nx|_3 = \lim_{n \rightarrow \infty} |nx|_3 = |x|_2.
\]

Therefore
\[
|.|_2 \text{ is classical valuation on } G.
\]

Proof of Theorem 2.4. By Proposition 2.10, the mapping
\[ |x|_2 : \mathbb{G} \rightarrow \mathbb{R} \cup \{\infty\} \text{ is classical valuation on } \mathbb{G}, \text{ and by proposition 2.9, we have }\]
\[ 2^{-1}|x|_1 \leq |x|_2 \leq 2|x|_1.\]
Therefore
\[ 2^{-\alpha}|x|_1^\alpha \leq |x|_2 \leq 2|x|_1^\alpha.\]

Thus classical valuation \( |\cdot|_2 \) on \( \mathbb{G} \) which is \((2, \alpha)\) H\-\older equivalent to the \((C_1, C_2)\) H\-\older valuation \( |\cdot|_1 \) on \( \mathbb{G} \).

REFERENCES