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Extension of Euler's Theorem on Homogeneous Functions for Finite variables and Higher Derivatives

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***Abstract*— – In this paper we are extending Euler's Theorem on Homogeneous functions from the functions of two variables to the functions of "n" variables. We have extended the result from second order derivatives to higher order derivatives. We have also generalized this statement on composite functions. This work is applicable to Thermodynamics like study of extensive and intensive variable. This result is also applicable to certain area of Financial Mathematics.**

***Index Terms*— Homogeneous Function, Euler's Theorem.**

I. INTRODUCTION

The Euler's theorem on Homogeneous functions is used to solve many problems in engineering, science and finance. Hiwarekar [1] discussed extension and applications of Euler's theorem for finding the values of higher order expression for two variables. In this paper we have extended the result from function of two variables to "n" variables. We have also obtained the results for higher order derivatives. We have also corrected the result on composite function of Hiwarekar [1].

If u is homogeneous function of n variables, $x_1, x_2, \dots, x_n$ of degree K then it is useful to find the value of expressions like $\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}, \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}$ and finding the value of the expression of higher order. In this paper we calculate the higher order expressions and extend it for composite functions. We have also discussed some examples based on these results.

II. PRELIMANARIES

Definition 2.1 Scalar Function: [4] A scalar function f on R^n is rule that it assigns each vector $\overline{x} = (x_1, x_2, \dots, x_n)$ in R^n a unique real number $f(x_1, x_2, \dots, x_n)$.

Definition 2.2 Differentiability of Scalar Function: [4] A scalar function on R^n is differentiable on R^n if all partial derivatives $\frac{\partial f}{\partial x_i}$ $i = 1,2,\dots n.$ exist and are continuous on R^n, a scalar function on R^n is m times differentiable on R^n if all m^{th} order partial derivative exist and are continuous on R^n.

Definition 2.3 Chain Rule: [4] Let f be differentiable function of vector $\overline{x} = (x_1, x_2, \dots, x_n)$ in R^n and all x_i's are functions of $(t_1, t_2, \dots, t_m)$ then,

$$\frac{\partial f}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial t_i}.$$

Definition 2.3 Homogeneous Function: [2] A scalar differential function f is homogeneous of degree p on R^n if $f(t\overline{x}) = t^p f(\overline{x})$ for each $\overline{x}$ in R^n and for any scalar $t > 0$.

Lemma 2.4: A scalar function f on R^n is homogeneous of degree M if and only if

$$f(x_1, x_2, \dots, x_n) = x_1^M \phi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_n}{x_1}\right), \quad if\ x_1 > 0.$$

Proof: $(\Rightarrow)$ Suppose f is homogeneous function of degree M .Therefore $f(tx_1, tx_2, \dots, tx_n) = t^M f(x_1, x_2, \dots, x_n)$ for $t > 0$.

Without loss of generality we can assume $x_1 > 0$. then $f(x_1, x_2, \dots, x_n) = x_1^M f\left(1, \frac{x_2}{x_1}, \frac{x_3}{x_1} \dots, \frac{x_n}{x_1}\right)$

$$\Rightarrow f(x_1, x_2, \dots, x_n) = x_1^M \phi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_n}{x_1}\right).$$

$(\Leftarrow)$

Suppose $f(x_1, x_2, \dots, x_n) = x_1^M \phi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_n}{x_1}\right)$, $x_1 > 0$.

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$$\Rightarrow f(tx_1, tx_2, \ldots, tx_n) = (tx_1)^M \varphi\left(\frac{tx_2}{tx_1}, \frac{tx_3}{tx_1}, \ldots, \frac{tx_n}{tx_1}\right)$$
$$= t^M x_1^M \varphi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_n}{x_1}\right)$$
$$= t^M f(x_1, x_2, \ldots, x_n).$$

Therefore f is homogeneous function of degree M .

Notations 2.5: If $f : R^n \to R$ is M times differentiable then,

1. $$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \ldots + x_n\frac{\partial}{\partial x_n}\right) f = \left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\right) f = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}.$$

2. $$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \ldots + x_n\frac{\partial}{\partial x_n}\right)^2 f = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right)^2 f = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right)^2 f$$

3. $$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \ldots + x_n\frac{\partial}{\partial x_n}\right)^k f = \left(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n} x_{i_1} x_{i_2} \ldots x_{i_k} \frac{\partial}{\partial x_{i_1}}\frac{\partial}{\partial x_{i_2}}\cdots\frac{\partial}{\partial x_{i_k}}\right) f = \left(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_k=1}^{n} x_{i_1} x_{i_2} \ldots x_{i_k} \frac{\partial^k f}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_k}}\right)$$

We use following notation in the result for simplicity. These will help to prove Extension of Euler theorem on homogeneous function.

III. MAIN RESULTS

Theorem 3.1: EXTENSION OF EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

If $f : R^n \to R$ is homogeneous function of degree M and all partial derivatives of up to order K exist and continuous then,

$$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \ldots + x_n\frac{\partial}{\partial x_n}\right)^k f = M(M-1)(M-2)\ldots(M-k+1) f$$

For any positive integer k.

Proof: We shall prove this result by Principle of Mathematical Induction on k.

Let $k = 1$. Then, we shall show that

$$\left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\right) f = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = Mf.$$

Without loss of generality we can assume that $x_1 > 0$.

Since, f is homogeneous function of degree M then by lemma 2.4,

$$f(x_1, x_2, \ldots, x_n) = x^M \varphi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_n}{x_1}\right) \ldots\ldots\ldots\ldots(1)$$

Take $\frac{x_i}{x_1} = y_i$ *for all* $i = 2,3\ldots n$, then (1) becomes

$$f(x_1, x_2, \ldots, x_n) = x_1^M \varphi(y_2, y_3, \ldots, y_n) \ldots\ldots\ldots\ldots(2)$$

Differentiating (2) partially with respect to x_1 then multiply it by x_1 we get,

$$x_1\frac{\partial f}{\partial x_1} = Mx^M \varphi(y_2, y_3, \ldots, y_n) - x_1^{M-1}\sum_{i=2}^{n} x_2 \frac{\partial \varphi}{\partial y_i} .(3)$$

Differentiating (2) partially with respect to x_i then multiply it by x_i *for* $i = 2,3\ldots n$, we get,

$$x_i \frac{\partial f}{\partial x_i} = x_1^{M-1} x_i \frac{\partial \varphi}{\partial y_i} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Adding equation (3) and (n-1) equations in (4) we get,

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = Mx^M \varphi(y_2, y_3, \ldots, y_n) - x_1^{M-1}\sum_{i=2}^{n} x_2 \frac{\partial \varphi}{\partial y_i} + x_1^{M-1}\sum_{i=2}^{n} x_2 \frac{\partial \varphi}{\partial y_i}$$
$$= Mx_1^M \phi(y_2, y_3, \ldots, y_n)$$
$$= Mf(x_1, x_2, \ldots, x_n)$$

Therefore,

$$\left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\right) f = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = Mf \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Therefore this result is true for k=1.

Take k=2,

Differentiating (5) partially with respect to x_i then multiply it by x_i *for* $i = 1,2,3\ldots, n$ we get,

$$\sum_{j=1}^{n} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} + x_i \frac{\partial f}{\partial x_i} = Mx_i \frac{\partial f}{\partial x_i}$$

Adding all equations in (6), we get,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = M\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}$$

Solving we get,

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$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} = M(M-1)f$$

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right)^2 f = M(M-1)f \ldots\ldots\ldots(6)$$

Therefore this result is true for $k=2$.

Assume that this result is true for $k=m$, that is,

$$\left(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_m=1}^{n} x_{i_1}x_{i_2}\ldots x_{i_m}\frac{\partial}{\partial x_{i_1}}\frac{\partial}{\partial x_{i_2}}\cdots\frac{\partial}{\partial x_{i_m}}\right)f$$

$$=\left(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_m=1}^{n} x_{i_1}x_{i_2}\ldots x_{i_m}\frac{\partial^m f}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_m}}\right)$$

$$= M(M-1)\ldots(M-m+1)f \ldots\ldots\ldots\ldots(7)$$

Let $k=m+1$,

Differentiating (5) partially with respect to x_i then multiply it by x_i $for\ i=1,2,3 \ldots n$ then adding we get,

$$\left(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_{m+1}=1}^{n} x_{i_1}x_{i_2}\ldots x_{i_{m+1}}\frac{\partial^{m+1} f}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_{m+1}}}\right)$$

$$+m\left(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_m=1}^{n} x_{i_1}x_{i_2}\ldots x_{i_m}\frac{\partial^m f}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_m}}\right)$$

$$= M(M-1)\ldots(M-m+1)\sum_{i=1}^{n} x_i\frac{\partial f}{\partial x_i}$$

Solving we get,

$$\left(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_{m+1}=1}^{n} x_{i_1}x_{i_2}\ldots x_{i_{m+1}}\frac{\partial^{m+1} f}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_{m+1}}}\right)$$

$$= M(M-1)\ldots(M-m)f$$

Therefore,

$$\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+\ldots+x_n\frac{\partial}{\partial x_n}\right)^k f$$

$$= M(M-1)(M-2)\ldots(M-k+1)f \ldots\ldots(8)$$

Thus, this result is true for $k=m+1$.

Therefore by Principle of Mathematical induction this result is true for any positive integer k.

This completes the proof of the theorem.

Now we are extending above theorem for composite functions.

Corollary 3.2:

If $z=f(u)$ is homogeneous function of $(x_1, x_2, \ldots, x_n)$ with degree of homogeneity M and all partial derivatives of u up to k^{th} order exist and continuous then,

$$\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+\ldots+x_n\frac{\partial}{\partial x_n}\right)^k u = g(u)H_{k-1}(u)$$

For any positive integer k.

Where, $H_0(u)=1,\ H_1(u)=(g'(u)-1)$ and for $m\neq 1$

$$H_m(u)=g(u)H'_{m-1}(u)+(g'(u)-m+1)H_{m-1}(u)$$

$$g(u)=M\frac{f(u)}{f'(u)}.$$

Proof: We shall prove this result by principle of mathematical induction on k.

Take $k=1$,

Since $z=f(u)$ is homogeneous function of degree M therefore by theorem 3.1 we have,

$$\sum_{i=1}^{n} x_i\frac{\partial z}{\partial x_i} = Mz \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

But $z=f(u)$

Therefore,

$$\frac{\partial z}{\partial x_i}=\frac{dz}{du}\frac{\partial u}{\partial x_i}= f'(u) \text{ For } i=1,2..n \quad \text{(by chain rule)}$$

Putting these values in (1) we get,

$$\left(\sum_{i=1}^{n} x_i\frac{\partial u}{\partial x_i}\right)f'(u)=f(u)$$

$$\Rightarrow\left(\sum_{i=1}^{n} x_i\frac{\partial u}{\partial x_i}\right)=M\frac{f(u)}{f'(u)}=g(u)\ldots\ldots\ldots\ldots\ldots(2)$$

Provided, $f'(u)\neq 0$.

Therefore the result is true for $k=1$.

Take $k=2$,

Differentiating (2) partially with respect to x_i then multiply it by x_i $for\ i=1,2,3\ldots n$ and adding it we get,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} x_i\frac{\partial u}{\partial x_i}=g'(u)\left(\sum_{i=1}^{n} x_i\frac{\partial f}{\partial x_i}\right)$$

$$\Rightarrow\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\frac{\partial^2 u}{\partial x_i\partial x_j}=g(u)\big(g'(u)-1\big)$$

$$=g(u)H_1(u)\ldots\ldots\ldots(3)$$

Therefore this result is true of $k=2$

Assume that this result is true for $k=m$ that is,

$$\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+\ldots+x_n\frac{\partial}{\partial x_n}\right)^m u=g(u)H_{m-1}(u)$$

Now take $k=m+1$,

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Differentiating above equation partially with respect to x_i then multiply it by x_i *for* $i=1,2,3...n$ and adding it we get,

$$\left(x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+\ldots+x_n\frac{\partial}{\partial x_n}\right)^{m+1}u$$

$$=g(u)[g(u)H_{m-1}'(u)+H_{m-1}(u)g'(u)]-mg(u)H_{m-1}(u)$$
$$=g(u)[g(u)H_{m-1}'(u)+H_{m-1}(u)(g'(u)-m)]$$
$$=g(u)H_m(u)\text{.......... ....}(4)$$

Therefore this result is true for $k=m+1$. Hence this result is true for any positive integer k.
This completes the proof of theorem.

IV. NUMERICAL ILLUSTRATION

Example 4.1: If $f(x_1,x_2,x_3)=x_1^3x_2^2+x_3^5$ is homogeneous function of degree 5 therefore,

$$\sum_{i=1}^{3}x_i\frac{\partial f}{\partial x_i}=5f,\ \left(\sum_{i=1}^{3}\sum_{j=1}^{3}x_ix_j\frac{\partial^2 f}{\partial x_i\partial x_j}\right)=20f\ .$$

Example 4.2: If $u=tan^{-1}(x_1^3x_2^2+4x_3^5)$ then,

$z=\tan u=x_1^3x_2^2+x_3^5$ is homogeneous function of degree 5. Therefore by Euler's theorem on composite function we get

$$\sum_{i=1}^{3}x_i\frac{\partial u}{\partial x_i}=\frac{5}{2}\sin 2u$$

$$\left(\sum_{i=1}^{3}\sum_{j=1}^{3}x_ix_j\frac{\partial^2 f}{\partial x_i\partial x_j}\right)=5\sin 2u\ (\cos 2u-1).$$

V. CONCLUSION

This theorem is useful to find expressions like $\sum_{i=1}^{n}x_i\frac{\partial f}{\partial x_i}$

$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j\frac{\partial^2 f}{\partial x_i\partial x_j}\right)$ Which can be used in Thermodynamics and various areas of finance where function is depend on more than three variables.

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