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Numerical Study of the Variational Iteration Method for Special Non-Linear Partial Differential Equations

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Abstract*— In this paper, non-linear parabolic-hyperbolic partial differential equations are solved by means of Variational iteration method (VIM) and the results are compared with those of Homotopy perturbation method (HPM), and lesser computations will be expected. Parabolic-hyperbolic partial differential equations appear in mathematical modeling of many phenomena. Also, the convergence of the method is addressed briefly. For this purpose, Banach's fixed point theorem has been used. Some examples are presented to illustrate the ability and the simplicity of the method. The results reveal that this method is very effective and simple and can be applied for other problems in different fields of mathematics.

***Index Terms*—Convergence, Lagrange multiplier, Non-linear parabolic-hyperbolic partial differential equation, Variational iteration method.**

I. INTRODUCTION

Various methods have been presented to find the exact and approximate solutions of non-linear partial differential equations, for example, Exp-function method, $(G'/G)-$ function method, tanh-coth function method, sinh-function method, differential transformation method and so on [1]-[12]. In addition to these methods, several iterative methods for the solution of initial and boundary value problems in ordinary and partial differential equations were presented. These iterative procedures provide the solution or an approximation as a sequence of iterates. One of these methods is Variational iteration method (VIM) that has been proposed by Ji-Huan He, in 1998, and has been applied to solve many different linear or non-linear functional equations, such as autonomous ordinary differential equations [13]-[14], wave equations [15], non-linear mixed Volterra - Fredholm integral equations [16], nonlinear heat transfer equations [17] and many others. There are also some approaches to show that Variational iteration method is convergence such as [18]-[23]. For example, Ramos has used integration by parts, Green show that the Variational iteration method is nothing else by the Picard – Lindel of theory for first and second order initial value problems, in ordinary differential equations and Banach's fixed point theory for first and second order initial value problems in partial differential equations, and the convergence of these iterative procedures is insured provided that the resulting mapping is Lipchitz continuous and contractive [18]. However, there are different ways to show the convergence of VIM, like what was expressed for special cases in [18], but in present paper the convergence of method is studied and proved directly using Banach's fixed point theorem for third order partial differential equations respect to the time variable. In this paper, it is shown that the sequence of resulted approximate solutions of VIM will be converged to exact solution, if it is satisfied in Lipschitz condition, and this is a sufficient condition for convergence. The way of proving convergence of considerable method, which will be stated, will be the novelty of the present paper. Examples in this paper are more difficult than those in [18]. in this paper, Cauchy problem is considered for the non-linear parabolic - hyperbolic equation of the following form,

$$(\partial/\partial t-\Delta)(\partial^2/\partial t^2-\Delta)u=F(u), \qquad (1)$$

with the initial conditions,

$$\partial^k u(X,0)/\partial t^k = f_k(X),\;\; X=(x_1,x_2,\dots,x_n),\;\; k=0,1,2. \qquad (2)$$

Where $F(u)$ represents the non-linear term, and Δ is the Laplace operator in $\mathbb{R}^n$.

Let's consider the following non-linear functional equation:

$$L(u(X,t))+N(u(X,t))+g(X,t)=0, \qquad (3)$$

where $L, N,$ and $g(X,t)$ are a linear operator, a non-linear operator, and a known analytic function. In this method, a correction functional will be constructed, including a general Lagrange multiplier, as follows,

$$u_{n+1}(X,t)=u_n(X,t)+\int_0^t \lambda(s,t)\{L(u_n(X,s))+N(\tilde{u}_n(X,s))+g(X,s)\}ds,\;\; n\ge 0, \qquad (4)$$

where $\tilde{u}_n$ is a restricted variations, i.e., $\delta u_n=0$. Lagrange multiplier can be identified optimally via the variational theory. The iterative formula will be obtained as soon as the Lagrange multiplier is determined. The successive approximations $u_n(X,t)$, $n\ge 0$, of $u(X,t)$ will be obtained by selection an initial approximation of the

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solution, $u_0(X,t)$, usually called zero order approximation. u_0, may be chosen any function such that satisfies initial and boundary conditions. The exact solution may be obtained by;

$$u(X,t)=\lim_{n\to\infty} u_n(X,t). \qquad (5)$$

Iterative formula is considered as follows,

$$u_{n+1}(X,t)=u_n(X,t) + \int_0^t \lambda(s,t)\{L(u_n(X,s))+N(u_n(X,s))+g(X,s)\}ds,\ n\ge 0, \qquad (6)$$

Let´s consider the following operator,

$$A(u)=u(X,t)+\int_0^t \lambda(s,t)\{L(u(X,s))+N(u(X,s))+g(X,s)\}ds. \qquad (7)$$

The Variational iteration method is equivalent to determining the sequence $\{u_n\}$, defined as the following iterative formula,

$$u_{n+1}=A(u_n), \qquad (8)$$

associated with the functional equation,

$$u=A(u), \qquad (9)$$

If A is a contraction then the sequence $\{u_n\}$, defined by (8), converges to the unique solution u of (9). To pursue the theme let´s state the fixed point theorem.

Theorem. Suppose X and Y be Banach's space and $A:X\to Y$ be a contraction mapping, that is,

$$\forall v,\bar{v}\in X;\ \|A(v)-A(\bar{v})\|\le\gamma\|v-\bar{v}\|,\quad 0<\gamma<1.$$

Then,

$(i)\ \|u_n-u\|\le\gamma^n\|u_0-u\|$,

$(ii)\ \lim_{n\to\infty} u_n=u$,

which, u is the unique solution of $A(u)=u$.

Proof. (i) By the induction method on n, for $n=1$ one has,

$$\|u_1-u\|=\|A(u_0)-A(u)\|\le\gamma\|u_0-u\|.$$

Assume that $\|u_{n-1}-u\|\le\gamma^{n-1}\|u_0-u\|$ as an induction hypothesis, then

$$\|u_n-u\|=\|A(u_{n-1})-A(u)\|\le\gamma\|u_{n-1}-u\| \le\gamma\gamma^{n-1}\|u_0-u\|=\gamma^n\|u_0-u\|.$$

(ii) Using (i), one has,

$\|u_n-u\|\le\gamma^n\|u_0-u\|\to_{n\to\infty}0\quad(0<\gamma<1)$, it is derived $\lim_{n\to\infty}\|u_n-u\|=0$ that is, $\lim_{n\to\infty}u_n=u$.

II. EXAMPLES

To illustrate the ability and simplicity of the method three Examples are presented.

Example 1. Let's consider the following equation,

$$\left(\partial/\partial t-\partial^2/\partial x^2\right)\left(\partial^2/\partial t^2-\partial^2/\partial x^2\right)u=\left(\partial^2u/\partial t^2\right)^2 +\left(\partial^2u/\partial x^2\right)^2-2u^2,\quad 0\le x,t\le 1 \qquad (10)$$

or

$$\partial^3u/\partial t^3-\partial^3u/\partial t\,\partial x^2-\partial^4u/\partial t^2\,\partial x^2+\partial^4u/\partial x^4 =\left(\partial^2u/\partial t^2\right)^2+\left(\partial^2u/\partial x^2\right)^2-2u^2, \qquad (11)$$

with the initial conditions,

$$u(x,0)=\exp(x),\ \ \partial u(x,0)/\partial t=\exp(x),\ \ \partial^2u(x,0)/\partial t^2=\exp(x). \qquad (12)$$

The exact solution is $u(x,t)=\exp(x+t)$.

To solve the Equation (10) or (11), the following correction functional is constructed,

$$u_{n+1}(x,t)=u_n(x,t) +\int_0^t\lambda((u_n)_{sss}-(\tilde{u}_n)_{sxx}-(\tilde{u}_n)_{xxss}+(\tilde{u}_n)_{xxxx} -((\tilde{u}_n)_{ss})^2+((\tilde{u}_n)_{xx})^2+2(\tilde{u}_n)^2)ds, \qquad (13)$$

Where λ is general Lagrange multiplier, $\tilde{u}_n$ is considered as restricted variations, i.e. $\delta\tilde{u}_n=0$. To determine the value of λ the following procedure should be followed,

$$\delta u_{n+1}(x,t)=\delta u_n(x,t) +\delta\int_0^t\lambda((u_n)_{sss}-(\tilde{u}_n)_{sxx}-(\tilde{u}_n)_{xxss}+(\tilde{u}_n)_{xxxx} -((\tilde{u}_n)_{ss})^2+((\tilde{u}_n)_{xx})^2+2(\tilde{u}_n)^2)ds=0 \qquad (14)$$

which is equivalent to,

$$\delta u_{n+1}=(1+\lambda''(t))\delta u_n-\lambda'(t)\delta u_n'+\lambda(t)\delta u_n''-\int_0^t\lambda'''(s)\delta u_n\,ds. \qquad (15)$$

Stationary conditions on Equation (15) lead to,

$$1+\lambda''(t)=0,\quad \lambda'(t)=0,\quad \lambda(t)=0,\quad \lambda'''(s)=0. \qquad (16)$$

The Lagrange multiplier is obtained as, $\lambda(s,t)=-\frac{1}{2}(s-t)^2$. Having this multiplier the iteration formula (13) turns to;

$$u_{n+1}(x,t)=u_n(x,t) -\frac{1}{2}\int_0^t(s-t)^2((u_n)_{sss}-(u_n)_{sxx}-(u_n)_{xxss}+(u_n)_{xxxx} -((u_n)_{ss})^2+((u_n)_{xx})^2+2(u_n)^2)ds. \qquad (17)$$

Considering given initial conditions (12) a three-term approximation of $u_0(x,t)$ will be as follows,

$$u_0(x,t)=(1+t+t^2/2!)\exp(x).$$

Starting with $u_0(x,t)$, other terms are computed as follows,

$$u_1(x,t)=(1+t+t^2/2!+t^3/3!)\exp(x),$$

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$$u_2(x,t)=\left(1+t+t^2/2!+t^3/3!+t^4/4!\right)\exp(x),$$
$$u_3(x,t)=\left(1+t+t^2/2!+t^3/3!+t^4/4!+t^5/5!\right)\exp(x),$$
$$\vdots$$
$$u_n(x,t)=\left(\sum_{i=0}^{n+2}t^i/i!\right)\exp(x). \quad (19)$$

According to the Theorem for mapping A contraction of A is a sufficient condition for convergence of variational iteration method. Therefore, one has,

$$\|u_0-u\|=\left\|\left(1+t+t^2/2!\right)\exp(x)-\exp(x+t)\right\|$$
$$\|u_1-u\|=\left\|\left(1+t+t^2/2!+t^3/3!\right)\exp(x)-\exp(x+t)\right\|$$
$$\le\left\|\left(1+t+t^2/2!\right)\exp(x)-\exp(x+t)\right\|\times$$
$$\left\|1-\left(t^3/3!\exp(x)\right)/\left(\exp(x+t)-\left(1+t+t^2/2!\right)\exp(x)\right)\right\|.$$

Since for all $t,x\in[0,1]$, one has

$$\left\|1-\left(t^3/3!\exp(x)\right)/\left(\exp(x+t)-\left(1+t+t^2/2!\right)\exp(x)\right)\right\|\le\gamma=0.2364611104$$
$$<1$$

therefore ,

$$\|u_1-u\|\le\gamma\left\|\left(1+t+t^2/2!\right)\exp(x)-\exp(x+t)\right\|=\gamma\|u_0-u\|,$$
$$\|u_2-u\|=\left\|\left(1+t+t^2/2!+t^3/3!+t^4/4!\right)\exp(x)-\exp(x+t)\right\|$$
$$\le\left\|\left(1+t+t^2/2!\right)\exp(x)-\exp(x+t)\right\|\times$$
$$\left\|1-\left(\left(t^3/3!+t^4/4!\right)\exp(x)\right)/\left(\exp(x+t)-\left(1+t+t^2/2!\right)\exp(x)\right)\right\|$$

But, $\forall t,x\in[0,1]$ one has

$$\left\|1-\left(\left(t^3/3!+t^4/4!\right)\exp(x)\right)/\left(\exp(x+t)-\left(1+t+t^2/2!\right)\exp(x)\right)\right\|$$
$$\le 0.04557638802<\gamma,$$

thus,

$$\|u_2-u\|\le\gamma^2\|u_0-u\|.$$
$$\|u_3-u\|=\left\|\left(1+t+t^2/2!+t^3/3!+t^4/4!+t^5/5!\right)\exp(x)-\exp(x+t)\right\|$$
$$\le\left\|\left(1+t+t^2/2!\right)\exp(x)-\exp(x+t)\right\|\times$$
$$\left\|1-\left(\left(t^3/3!+t^4/4!+t^5/5!\right)\exp(x)\right)/\left(\exp(x+t)-\left(1+t+t^2/2!\right)\exp(x)\right)\right\|$$

But,

$\forall x,t\in[0,1]$,

$$\left\|1-\left(\left(t^3/3!+t^4/4!+t^5/5!\right)\exp(x)\right)/\left(\exp(x+t)-\left(1+t+t^2/2!\right)\exp(x)\right)\right\|$$
$$\le 0.0073994352<\gamma,$$

thus,

$$\|u_3-u\|\le\gamma^3\|u_0-u\|.$$
$$\vdots$$
$$\|u_n-u\|\le\gamma^n\|u_0-u\|.$$

Therefore, $\lim_{n\to\infty}\|u_n-u\|\le\lim_{n\to\infty}\gamma^n\|u_0-u\|=0$, that is,

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\left(\sum_{n=0}^{\infty}t^n/n!\right)\exp(x)=\exp(x+t), \quad (20)$$

which is an exact solution.

Example 2. Let's consider the following equation,

$$\left(\partial/\partial t-\partial^2/\partial x^2\right)\left(\partial^2/\partial t^2-\partial^2/\partial x^2\right)u=\left(u\ \partial u/\partial t\right)$$
$$+\left(\partial^2u/\partial t^2\right)\left(\partial u/\partial x\right),\quad 0\le x,t\le 1 \quad (21)$$

or

$$\partial^3u/\partial t^3-\partial^3u/\partial t\partial x^2-\partial^4u/\partial t^2\partial x^2+\partial^4u/\partial x^4$$
$$=\left(u\ \partial u/\partial t\right)+\left(\partial^2u/\partial t^2\right)\left(\partial u/\partial x\right), \quad (22)$$

with the initial conditions,

$$u(x,0)=\cos x,\qquad \partial u(x,0)/\partial t=-\sin x,$$
$$\partial^2u(x,0)/\partial t^2=-\cos x. \quad (23)$$

The exact solution is $u(x,t)=\cos(t+x)$.

To solve the Equation (21) or (22), the following correction functional is constructed,

$$u_{n+1}(x,t)=u_n(x,t)$$
$$+\int_0^t\lambda\left((u_n)_{sss}-(\tilde{u}_n)_{sxx}-(\tilde{u}_n)_{ssxx}+(\tilde{u}_n)_{xxxx}-\tilde{u}_n(\tilde{u}_n)_s-(\tilde{u}_n)_{ss}(\tilde{u}_n)_x\right)ds, \quad (24)$$

Similarly, the Lagrange multiplier is obtained as, $\lambda(s,t)=-1/2(s-t)^2$, and the iteration formula can be written as the following,

$$u_{n+1}(x,t)=u_n(x,t)$$
$$-1/2\int_0^t(s-t)^2\left((u_n)_{sss}-(u_n)_{sxx}-(u_n)_{ssxx}\right. \quad (25)$$
$$\left.+(u_n)_{xxxx}-u_n(u_n)_s-(u_n)_{ss}(u_n)_x\right)ds.$$

Considering given initial conditions (23) a three-term approximation of $u_0(x,t)$ will be as follows,

$$u_0(x,t)=\cos x-t\sin x-t^2/2\cos x, \quad (26)$$

Starting with $u_0(x,t)$, other terms are computed as follows,

$$u_1(x,t)=\left(1-t^2/2!\right)\cos x-\left(t-t^3/3!\right)\sin x,$$
$$u_2(x,t)=\left(1-t^2/2!+t^4/4!\right)\cos x-\left(t-t^3/3!+t^5/5!\right)\sin x,$$
$$u_3(x,t)=\left(1-t^2/2!+t^4/4!-t^6/6!\right)\cos x-\left(t-t^3/3!+t^5/5!-t^7/7!\right)\sin x,$$
$$\vdots$$
$$u_n(x,t)=\left(\sum_{i=0}^{n}(-1)^it^{2i}/(2i)!\right)\cos x-\left(\sum_{i=0}^{n}(-1)^it^{2i+1}/(2i+1)!\right)\sin x,\quad n\ge1. \quad (27)$$

Similarly, one has,

$$\|u_0-u\|=\left\|\cos x-t\sin x-t^2/2\cos x-\cos(x+t)\right\|$$

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$$\|u_1-u\|=\left\|\left(1-t^2/2!\right)\cos x-\left(t-t^3/3!\right)\sin x-\cos(x+t)\right\|$$
$$\le\left\|\cos x-t\sin x-t^2/2\cos x-\cos(x+t)\right\|\times$$
$$\left\|1-\left(t^3/3!\sin x\right)/\left(\cos(x+t)-\cos x+t\sin x+t^2/2\cos x\right)\right\|.$$

Since for all $t,x\in[0,1]$, one has,

$$\left\|1-\left(t^3/3!\sin x\right)/\left(\cos(x+t)-\cos x+t\sin x+t^2/2\cos x\right)\right\|\le\gamma=0.0962012177<1,$$

therefore,

$$\|u_1-u\|\le\gamma\left\|\cos x-t\sin x-t^2/2\cos x-\cos(x+t)\right\|=\gamma\|u_0-u\|,$$

$$\|u_2-u\|=\left\|\left(1-t^2/2!+t^4/4!\right)\cos x-\left(t-t^3/3!+t^5/5!\right)\sin x-\cos(x+t)\right\|$$
$$\le\left\|\cos x-t\sin x-t^2/2\cos x-\cos(x+t)\right\|$$
$$\left\|1-\left(t^4/4!\cos x+\left(t^3/3!-t^5/5!\right)\sin x\right)/\left(\cos(x+t)-\cos x+t\sin x+t^2/2\cos x\right)\right\|.$$

But, $\forall t,x\in[0,1]$, one has

$$\left\|1-\left(t^4/4!\cos x+\left(t^3/3!-t^5/5!\right)\sin x\right)/\left(\cos(x+t)-\cos x+t\sin x+t^2/2\cos x\right)\right\|\le0.003689474<\gamma,$$

thus,

$$\|u_2-u\|\le\gamma^2\|u_0-u\|.$$

$$\|u_3-u\|=\left\|\left(1-t^2/2!+t^4/4!-t^6/6!\right)\cos x-\left(t-t^3/3!+t^5/5!-t^7/7!\right)\sin x-\cos(x+t)\right\|$$
$$\le\left\|\cos x-t\sin x-t^2/2\cos x-\cos(x+t)\right\|\times$$
$$\left\|1-\left(\left(t^4/4!-t^6/6!\right)\cos x+\left(t^3/3!-t^5/5!+t^7/7!\right)\sin x\right)/\left(\cos(x+t)-\cos x+t\sin x+t^2/2\cos x\right)\right\|.$$

But, $\forall t,x\in[0,1]$,

$$\left\|1-\left(\left(t^4/4!-t^6/6!\right)\cos x+\left(t^3/3!-t^5/5!+t^7/7!\right)\sin x\right)/\left(\cos(x+t)-\cos x+t\sin x+t^2/2\cos x\right)\right\|\le0.000070596<\gamma,$$

thus,

$$\|u_3-u\|\le\gamma^3\|u_0-u\|.$$
$$\vdots$$
$$\|u_n-u\|\le\gamma^n\|u_0-u\|.$$

Therefore, $\lim_{n\to\infty}\|u_n-u\|\le\lim_{n\to\infty}\gamma^n\|u_0-u\|=0$, that is,

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)$$
$$=\left(\sum_{n=0}^{\infty}(-1)^n t^{2n}/(2n)!\right)\cos x-\left(\sum_{n=0}^{\infty}(-1)^n t^{2n+1}/(2n+1)!\right)\sin x$$
$$=\cos t\sin x-\sin t\cos x$$
$$=\cos(t+x),\tag{28}$$

which is an exact solution.

Example 3. Let's consider the following two-dimensional equation,

$$\left(\partial/\partial t-\partial^2/\partial x_1^{\,2}-\partial^2/\partial x_2^{\,2}\right)\left(\partial^2/\partial t^2-\partial^2/\partial x_1^{\,2}-\partial^2/\partial x_2^{\,2}\right)u=\partial u/\partial t-2u,$$
$$0\le t\le1/2,\quad x_1,x_2\in\mathbb{R}\tag{29}$$

or

$$\partial^3u/\partial t^3-\partial^3u/\partial t\,\partial x_1^{\,2}-\partial^3u/\partial t\,\partial x_2^{\,2}-\partial^4u/\partial x_1^{\,2}\,\partial t^2$$
$$+\partial^4u/\partial x_1^{\,4}+2\partial^4u/\partial x_1^{\,2}\partial x_2^{\,2}-\partial^4u/\partial x_2^{\,2}\,\partial t^2\tag{30}$$
$$+\partial^4u/\partial x_2^{\,4}=\partial u/\partial t-2u,$$

with the initial conditions,

$$u(x,0)=\sinh(x_1+x_2),\ \partial u(x,0)/\partial t=2\sinh(x_1+x_2),$$
$$\partial^2u(x,0)/\partial t^2=4\sinh(x_1+x_2).\tag{31}$$

The exact solution is $u(x,t)=\exp(2t)\sinh(x_1+x_2)$.

To solve the Equation (29) or (30), the following correction functional is constructed,

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda\Big((u_n)_{sss}-(\tilde u_n)_{sx_1x_1}-(\tilde u_n)_{sx_2x_2}-(\tilde u_n)_{x_1x_1ss}+(\tilde u_n)_{x_1x_1x_1x_1}$$
$$+2(\tilde u_n)_{x_1x_1x_2x_2}-(\tilde u_n)_{x_2x_2ss}+(\tilde u_n)_{x_2x_2x_2x_2}-(\tilde u_n)_s+2\tilde u_n\Big)ds,\tag{32}$$

Similarly, the Lagrange multiplier is obtained as, $\lambda(s,t)=-1/2(s-t)^2$, and the iteration formula can be written as the following,

$$u_{n+1}(x,t)=u_n(x,t)-1/2\int_0^t(s-t)^2\Big((u_n)_{sss}-(u_n)_{sx_1x_1}-(u_n)_{sx_2x_2}-(u_n)_{x_1x_1ss}+(u_n)_{x_1x_1x_1x_1}$$
$$+2(u_n)_{x_1x_1x_2x_2}-(u_n)_{x_2x_2ss}+(u_n)_{x_2x_2x_2x_2}-(u_n)_s+2u_n\Big)ds,\tag{33}$$

Considering given initial conditions (31) a three-term approximation of $u_0(x,t)$ will be as follows,

$$u_0(x,t)=\left(1+2t+(2t)^2/2!\right)\sinh(x_1+x_2),\tag{34}$$

Starting with $u_0(x,t)$, other terms are computed as follows,

$$u_1(x,t)=\left(1+2t+(2t)^2/2!+(2t)^3/3!\right)\sinh(x_1+x_2),$$
$$u_2(x,t)=\left(1+2t+(2t)^2/2!+(2t)^3/3!+(2t)^4/4!\right)\sinh(x_1+x_2),$$
$$u_3(x,t)=\left(1+2t+(2t)^2/2!+(2t)^3/3!+(2t)^4/4!+(2t)^5/5!\right)\sinh(x_1+x_2),$$
$$\vdots$$
$$u_n(x,t)=\left(\sum_{i=0}^{n+2}(2t)^i/i!\right)\sinh(x_1+x_2),\tag{35}$$

Similarly, one has,

$$\|u_0-u\|=\left\|\left(1+2t+(2t)^2/2!-\exp(2t)\right)\sinh(x_1+x_2)\right\|,$$
$$\|u_1-u\|=\left\|\left(1+2t+(2t)^2/2!+(2t)^3/3!-\exp(2t)\right)\sinh(x_1+x_2)\right\|$$
$$\le\left\|\left(1+2t+(2t)^2/2!-\exp(2t)\right)\sinh(x_1+x_2)\right\|\times$$
$$\left\|1-\left((2t)^3/3!\right)/\left(\exp(2t)-\left(1+2t+(2t)^2/2!\right)\right)\right\|.$$

Since for all $t\in[0,1/2]$, one has,

$$\left\|1-\left((2t)^3/3!\right)/\left(\exp(2t)-\left(1+2t+(2t)^2/2!\right)\right)\right\|\le\gamma=0.2364611006<1,$$

therefore,

$$\|u_1-u\|\le\gamma\left\|\left(1+2t+(2t)^2/2!-\exp(2t)\right)\sinh(x_1+x_2)\right\|=\gamma\|u_0-u\|,$$

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$$\begin{aligned}\|u_2-u\| &= \left\|\left(1+2t+(2t)^2/2!+(2t)^3/3!+(2t)^4/4!-\exp(2t)\right)\sinh(x_1+x_2)\right\| \\ &\le \left\|\left(1+2t+(2t)^2/2!-\exp(2t)\right)\sinh(x_1+x_2)\right\|\times \\ &\quad \left\|1-\left((2t)^3/3!+(2t)^4/4!\right)/\left(\exp(2t)-\left(1+2t+(2t)^2/2!\right)\right)\right\|.\end{aligned}$$

But, $\forall t\in[0,1/2]$ one has

$$\left\|1-\left((2t)^3/3!+(2t)^4/4!\right)/\left(\exp(2t)-\left(1+2t+(2t)^2/2!\right)\right)\right\| \le 0.0455763762<\gamma,$$

thus,

$$\|u_2-u\|\le\gamma^2\|u_0-u\|.$$

$$\begin{aligned}\|u_3-u\| &= \left\|\left(1+2t+(2t)^2/2!+(2t)^3/3!+(2t)^4/4!+(2t)^5/5!-\exp(2t)\right)\sinh(x_1+x_2)\right\| \\ &\le \left\|\left(1+2t+(2t)^2/2!-\exp(2t)\right)\sinh(x_1+x_2)\right\|\times \\ &\quad \left\|1-\left((2t)^3/3!+(2t)^4/4!+(2t)^5/5!\right)/\left(\exp(2t)-\left(1+2t+(2t)^2/2!\right)\right)\right\|.\end{aligned}$$

But, $\forall t\in[0,1/2]$,

$$\left\|1-\left((2t)^3/!+(2t)^4/4!+(2t)^5/5!\right)/\left(\exp(2t)-\left(1+2t+(2t)^2/2!\right)\right)\right\| \le 0.0073994309<\gamma,$$

thus,

$$\|u_3-u\|\le\gamma^3\|u_0-u\|,$$

$$\vdots$$

$$\|u_n-u\|\le\gamma^n\|u_0-u\|.$$

Therefore, $\lim_{n\to\infty}\|u_n-u\|\le\lim_{n\to\infty}\gamma^n\|u_0-u\|=0,$ that is,

$$\begin{aligned}u(x,t)&=\lim_{n\to\infty}u_n(x,t)=\left(\sum_{n=0}^{\infty}(2t)^n/n!\right)\sinh(x_1+x_2) \\ &=\exp(2t)\sinh(x_1+x_2),\end{aligned} \tag{36}$$

which is an exact solution.

III. CONCLUSION

In this paper, the problem of convergence of the Variational iteration method has been studied. Unlike Ramos, [18], that proved the convergence of VIM using the integration by parts, Green functions, adjoint operators, variation of parameters, and method of weighted residuals, in this paper, Banach's fixed point theorem was applied directly to investigate the convergence of the method in a different way. The sufficient condition for the convergence of the method has been presented, and by applying VIM, we have been able to derive exact solutions, where they are the same results obtained in [24]. This can be pointed as an advantage of the method in comparison with analytical approaches, and other common methods. It is expected that VIM, be a strong tool for solving any non-linearity. Computations are performed by using package Maple 13.

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